Code-based Cryptography

Irene Márquez Corbella
Universidad de La Laguna
Secure CAT - Kick off Meeting

Prerrequisites on error correcting codes

A **linear code** is a vector subspace $\mathcal{C} \subseteq \mathbb{F}_q^n$ where:

- $n(\mathcal{C}) = n$ is its **length**,
- $k(\mathcal{C}) = k$ is its **dimension** as $\mathbb{F}_q$-vector space.
- $d(\mathcal{C}) = d$ is its **minimum Hamming distance**.

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The **Hamming distance** on $\mathbb{F}_q^n$ is defined by:

$$d_H(\mathbf{x}, \mathbf{y}) = |\, \{i \in \{1, \ldots, n\} \mid x_i \neq y_i\} \,|$$

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A **t-decoder** for $\mathcal{C}$ is an algorithm $\mathcal{D}_{\mathcal{C}}$ taking as input $\mathbf{x} \in \mathbb{F}_q^n$ and returning:

- $\mathbf{c} \in \mathcal{C}$ such that $d_H(\mathbf{x}, \mathbf{c}) \leq t$ it exists.
- ? or *FAILURE* else.

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Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be a code. Its **subfield subcode** is defined by

$$\mathcal{C} \cap \mathbb{F}_q^n$$

Many algebraic codes derive from GRS codes using this operation:
Goppa Codes, BCH codes, Srivastava codes, etc

Outline

1. History of code-based cryptography
2. Algebraic cryptanalysis in code-based cryptography
3. How to design secure schemes with codes?

Public Key Cryptography

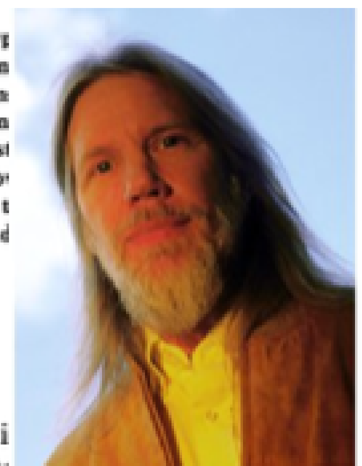
644 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. IT-22, NO. 6, NOVEMBER 1976

New Directions in Cryptography

Invited Paper

WHITFIELD DIFFIE AND MARTIN E. HELLMAN, MEMBER, IEEE

Abstract—Two kinds of contemporary developments in cryp tography are examined. Widening applications of teleprocessin have given rise to a need for new types of cryptographic system which minimize the need for secure key distribution channels an supply the equivalent of a written signature. This paper suggest ways to solve these currently open problems. It also discusses ho the theories of communication and computation are beginning t provide the tools to solve cryptographic problems of long stand ing.



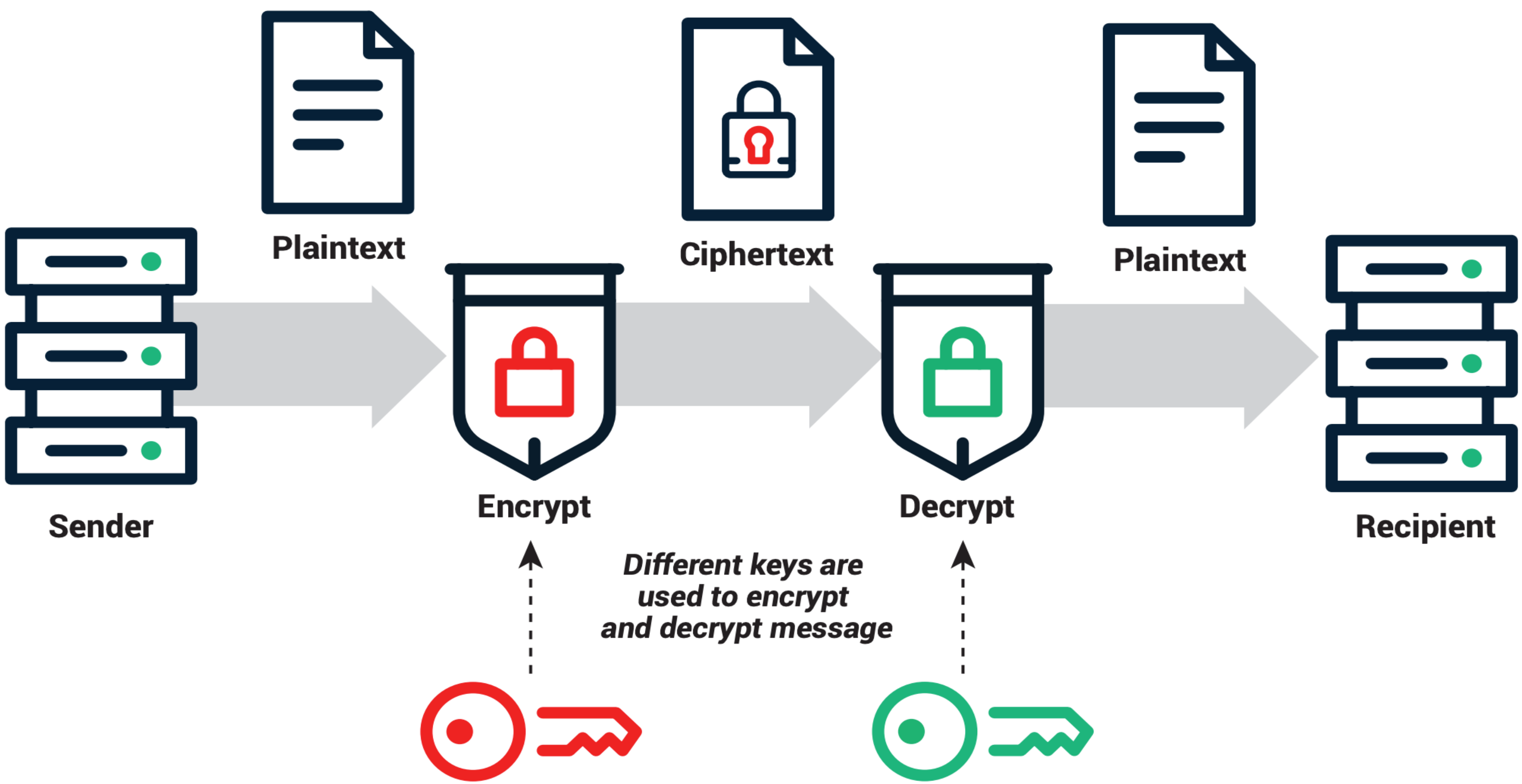
I. Introduction

WE STAND TODAY on the brink of a revolution i cryptography. The development of cheap digit hardware has freed it from the design limitations of mechanical computing and brought the cost of high grade cryptographic devices down to where they can be used in such commercial applications as remote cash dispensers and computer terminals. In turn, such applications create a need for new types of cryptographic systems which minimize the necessity of secure key distribution channels and supply the equivalent of a written signature. At the same time, theoretical developments in information theory

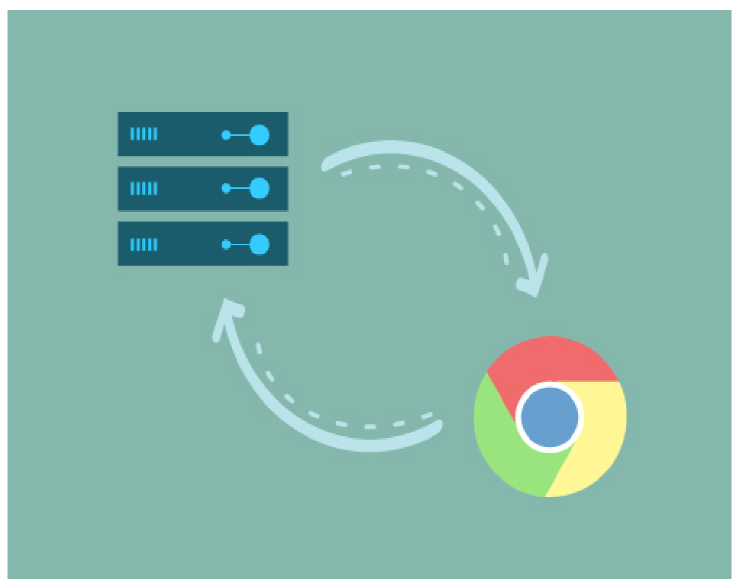
communications to large teleprocessing networks.

Section III proposes two approaches to transmitting keying information over public (i.e., insecure) channels without compromising the security of the system. In a *public key cryptosystem* enciphering and deciphering are governed by distinct keys, E and D, such that computing D from E is computationally infeasible (e.g., requiring 10^{100} instructions). The enciphering key E can thus be publicly disclosed without compromising the deciphering

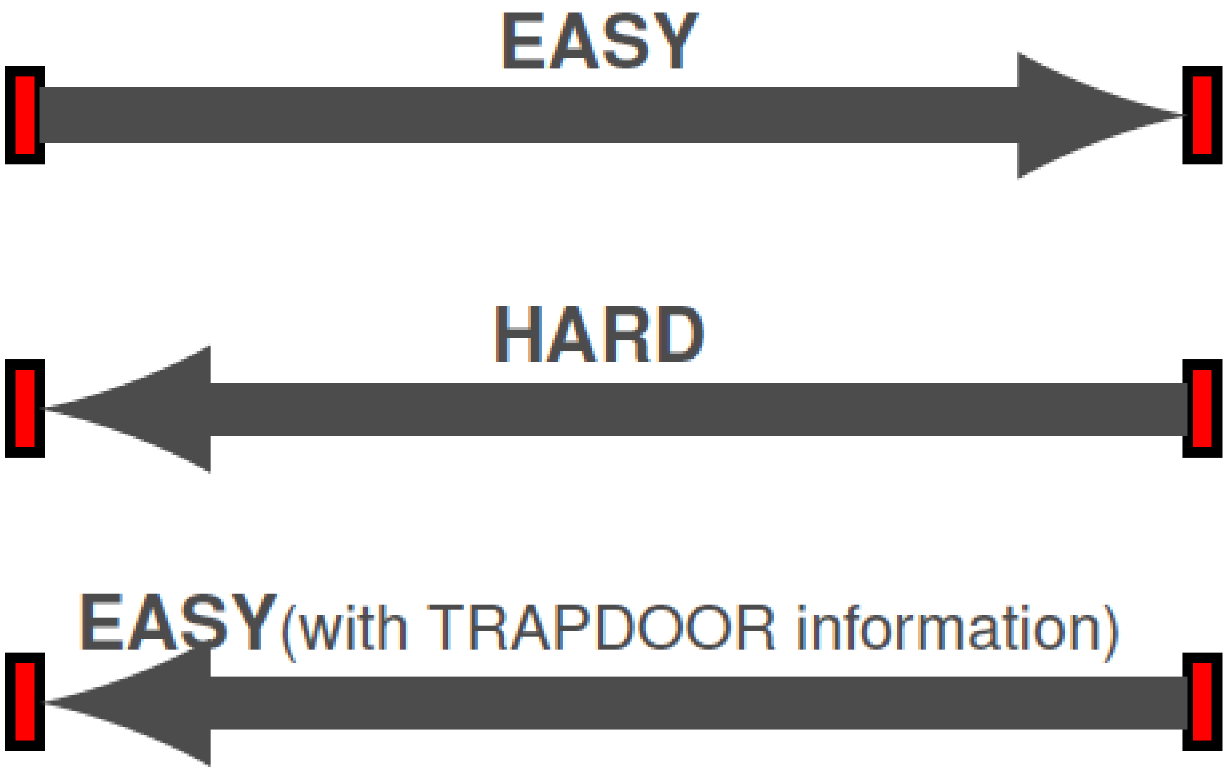
Public Key Cryptography vs Secret Key Cryptography



Do we need PKC?



Public Key Cryptography (PKC)



PKC - Easy example

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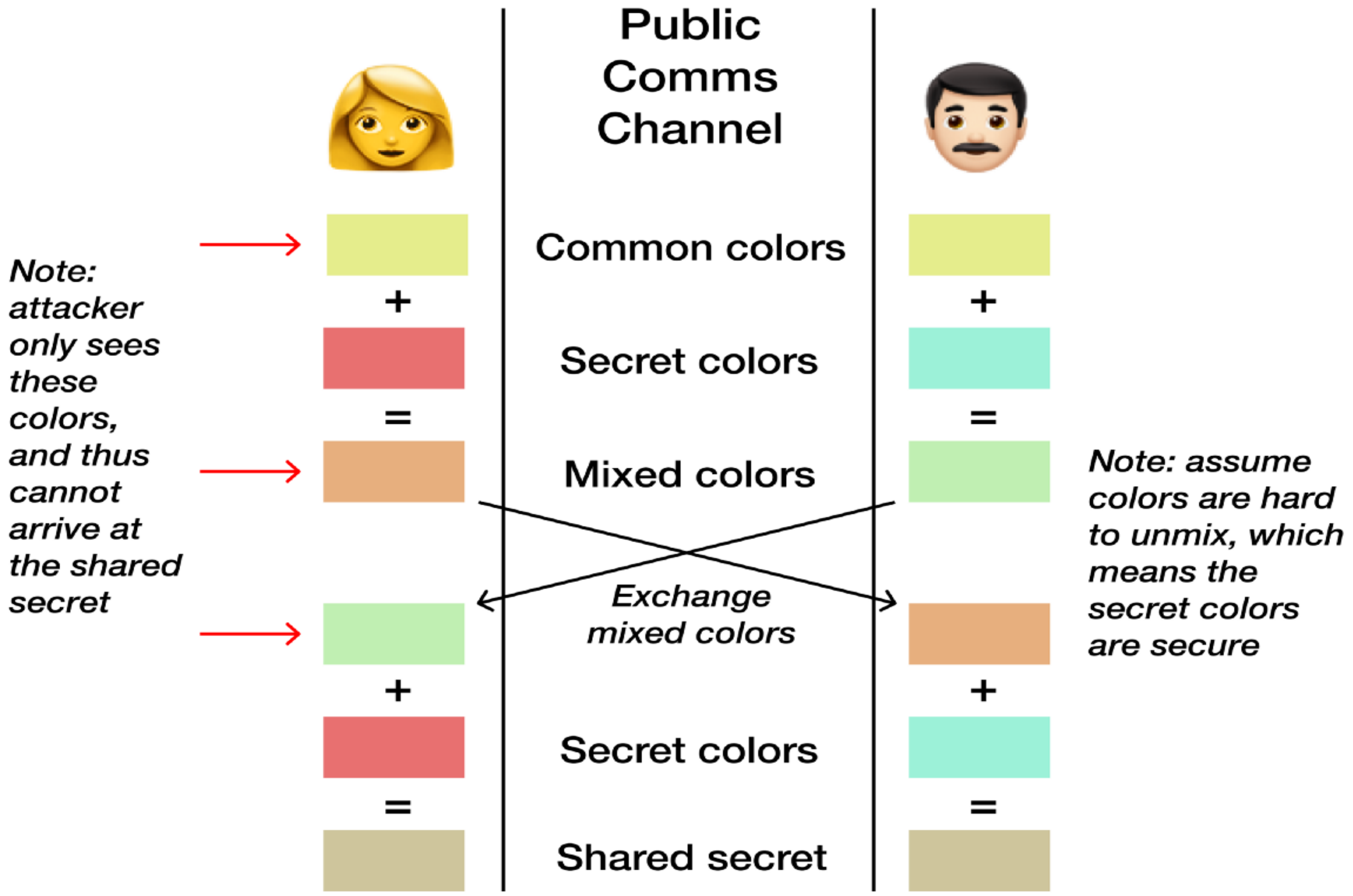

PKC - Famous example

Trapdoor one-way functions - Integer Factorization

First Challenge: Find prime factors of 4757.

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Trapdoor one-way functions - Integer Factorization

First Challenge: Find prime factors of 4757.

Seconde Challenge: Multiply 67 and 71.

PKC - Famous example

Trapdoor one-way functions - Integer Factorization

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4673331833592310999883355855611155212513211028177144957985823385935679234805211772074843110997402088
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PKC - Famous example

Trapdoor one-way functions - Integer Factorization

The largest known prime number (in May 2019) is

$$2^{82.589.933} - 1$$

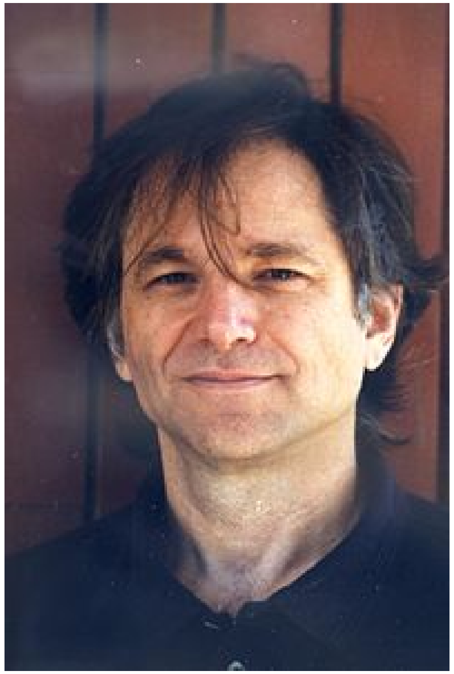
this number has 24.862.048 digits and was found by Patrick Laroche in 2018 as part of the proyect Great Internet Mersenne Prime Search (GIMPS)

3000\$ rewards

RSA

Ronald Rivest (1947)

Adi Shamir (1952)



Leonard Adleman (1945)

RSA is a public key cryptographic system developed in 1979 at the Massachusetts Institute of Technology (MIT). It is the algorithm of PKC most widely used today.
The security of the **RSA** relies on the **Integer Factorization**

Post-quantum Cryptography

There are **2** quantum algorithms that affect cryptography:

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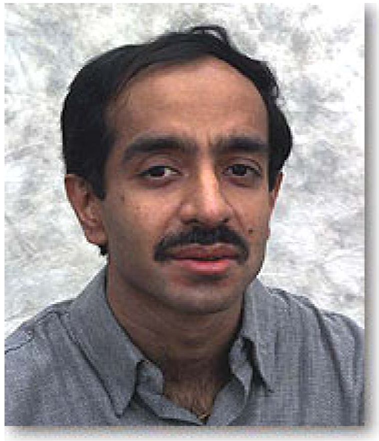
There are **2** quantum algorithms that affect cryptography:

- Grover's Algorithm: Finds b-bit preimages in $2^{\frac{b}{2}}$ quantum operations. It requieres:
  - $2\times$ key size in Symmetric ciphers
  - Longer output sizes in Hash Functions



Lov Grover

Post-quantum Cryptography

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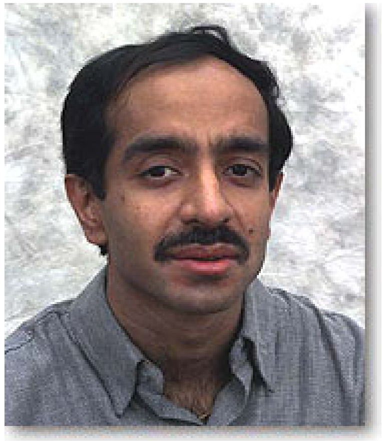
  - $2\times$ key size in Symmetric ciphers
  - Longer output sizes in Hash Functions
- SHOR'S ALGORITHM: Has dramatic effects on PKC, it breaks:
  - RSA cryptosystem.
  - Cryptosystems based on Discret log in finite fields and elliptic curves.



Lov Grover

Peter Shor

Preparing for the Cryptopocalypse

IBM Q system One - 2019

Preparing for the Cryptoapocalypse

Crypto-apocalypse

Preparing for the Cryptoapocalypse

Crypto-apocalypse

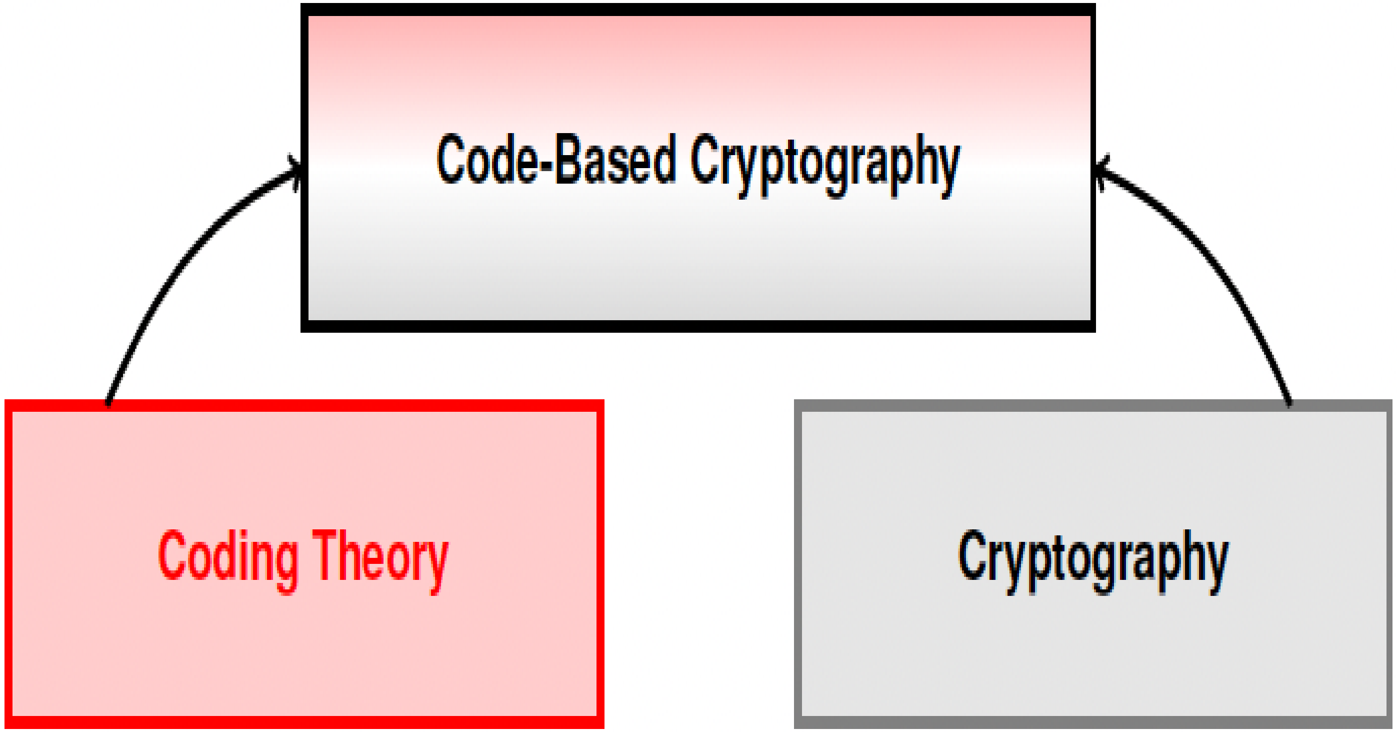
NIST (National Institute of Standards and Technology)

starts a proyect entitled:

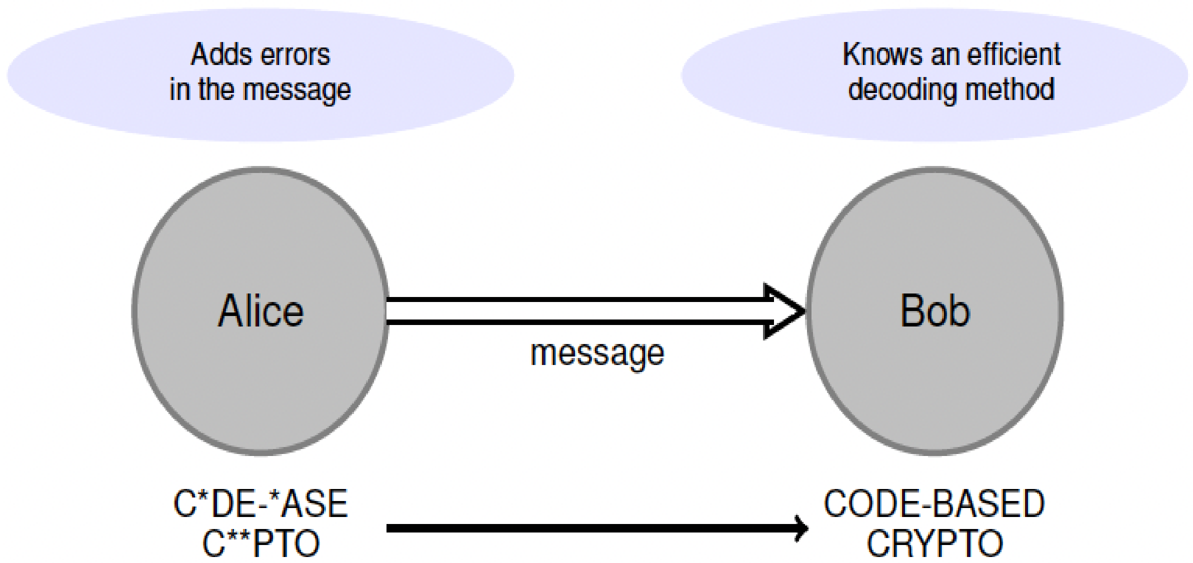
Post-Quantum Cryptography standardization.

- It start in November 2017: 69 proposals where sent.
- July 2022: 4 submissions were announced

Coding Theory vs. Cryptography



How to use Coding Theory in Cryptography?



It starts with two articles

[1] E.R. Berlekamp, R.J. McEliece and H.C.A. Van Tilborg. On the inherent intractability of certain coding problems. IEEE Trans. Inform. Theory 24(2), 1978.

[2] R.J. McEliece. A public key cryptosystem based on algebraic coding theory. DSN Progress Report 44; 1978.

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In the article **[1]**
The proof that the following problem is NP-complete:
Bounded decoding problem. Given $\mathcal{C} \subseteq \mathbb{F}_q^n$, $\mathbf{y} \in \mathbb{F}_q^n$ and $t \geq 0$.
Does there exists $\mathbf{c} \in \mathcal{C}$ such that:

$$d_H(\mathbf{c}, \mathbf{y}) \leq t?$$

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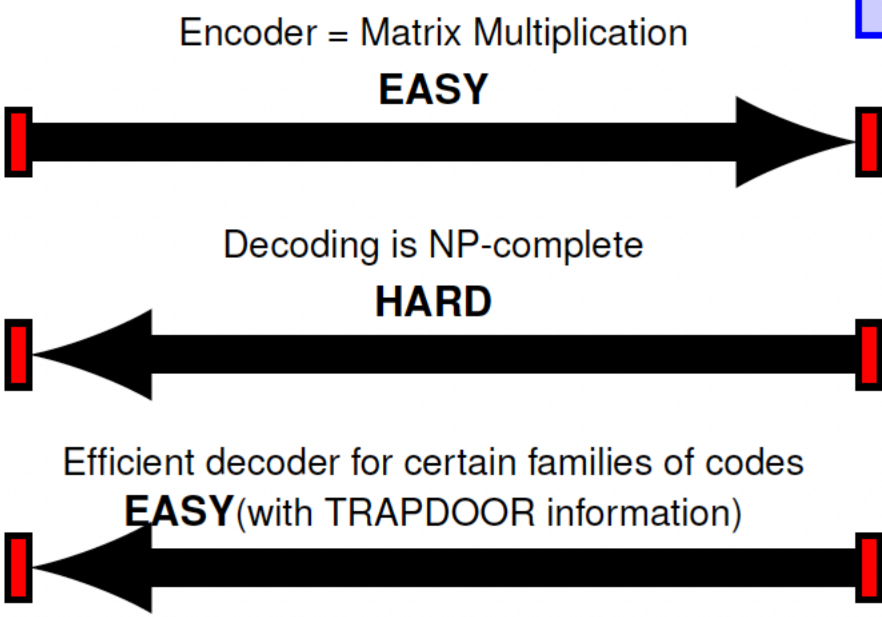
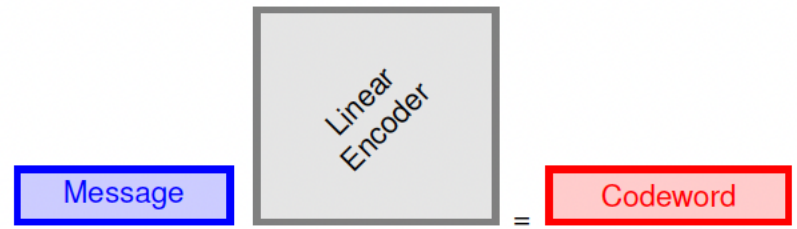
[2] R.J. McEliece. A public key cryptosystem based on algebraic coding theory. DSN Progress Report 44; 1978.

In article **[2]** McEliece proposes a new PKC encryption scheme.

Trapdoor One-Way Functions - McEliece





E. R. Berlekamp, R. J. McEliece and H. C. A. van Tilborg.
On the Inherent Intractability of Certain Coding Problems.
IEEE Trans. Inf. Theory. Vol. 24, pp. 384-386, 1978.

A. Barg.
Complexity Issues in Coding Theory.
Chapter 7, in Handbock of Coding Theory, 1998.

McEliece presented in the literature

Secret Key:

- G a $k \times n$ generator matrix of a code $\mathcal{C}$.
- S a $k \times k$ non-singular matrix.
- P a $n \times n$ permutation matrix.

Public Key: $(G' = SGP, t)$

Encryption: $\mathrm{Enc}(\mathbf{m}) = \mathbf{m}G' + \mathbf{e}$ with $\mathbf{e} \in \mathbb{F}_q^n$ uniformly random of weight t.

Decryption:

1. Right multiply by P^{-1}: $(\mathbf{m}G' + \mathbf{e}) \times P^{-1} = \mathbf{m}SG + \mathbf{e}P^{-1}$
2. Decode to get $\mathbf{m}S$
3. Right multiply by S^{-1} to get $\mathbf{m}$

This is what McEliece said:

A Public-Key Cryptosystem Based On Algebraic Coding Theory

R. J. McEliece
Communications Systems Research Section

Using the fact that a fast decoding algorithm exists for a general Goppa code, while no such exists for a general linear code, we construct a public-key cryptosystem which appears quite secure while at the same time allowing extremely rapid data rates. This kind of cryptosystem is ideal for use in multi-user communication networks, such as those envisioned by NASA for the distribution of space-acquired data

I. Introduction

Recently, Diffie and Hellman (Ref. 3) introduced the notion of a *public-key cryptosystem* in which communication security is achieved without the need of periodic distribution of a secret key to the sender and receiver. This property makes

Corresponding to each irreducible polynomial of degree t over $GF(2^m)$, there exists a binary irreducible Goppa code of length $n = 2^m$, dimension $k \geqslant n - tm$, capable of correcting any pattern of t or fewer errors. Moreover, there exists a fast algorithm for decoding these codes. [Algorithm due to

This is what McEliece said:

A Public-Key Cryptosystem Based On Algebraic Coding Theory

II. Description of the System

We base our system on the existence of *Goppa codes*. For the full theory of such codes the reader is referred to (Ref. 5, Chapter 8), but here we summarize the needed facts.

the code, which could be in canonical, for example row-reduced echelon, form.

Having generated G, the system designer now "scrambles" G by selecting a random dense $k \times k$ nonsingular matrix S, and a random $n \times n$ permutation matrix P. He then computes

---

$G' = SGP$, which generates a linear code with the same rate and minimum distance as the code generated by G. We call G' the public generator matrix, since it will be made known to the outside world.

and an astronomical number of choices for S and P. The dimension of the code will be about $k = 1024 \text{-} 50 \cdot 10 = 524$. Hence, a brute-force approach to decoding based on comparing $\mathbf{x}$ to each codeword has a work factor of about $2^{524} = 10^{158}$; and a brute-force approach based on coset leaders has a

We could present it differently:

A. Couvreur's IDEA

- $\mathcal{F}$ denotes a family of codes of length n and dimension k.
- $\mathcal{S}$ denotes a set of "*secrets*" with a surjective map $\mathcal{C}: \mathcal{S} \longrightarrow \mathcal{F}$ sending a secret $s \in \mathcal{S}$ into a code $\mathcal{C}(s)$.
- To any $s \in \mathcal{S}$ is associated a decoding algorithm $\mathcal{D}(s)$ for $\mathcal{C}(s)$ correcting up to t errors.

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- To any $s \in \mathcal{S}$ is associated a decoding algorithm $\mathcal{D}(s)$ for $\mathcal{C}(s)$ correcting up to t errors.

Secret Key: $s \in \mathcal{S}$

Public Key: (G, t) where G is a $k \times n$ generator matrix for $\mathcal{C}(s)$.

Encryption: $\mathrm{Enc}(\mathbf{m}) = \mathbf{m}\mathrm{G} + \mathbf{e}$ where $\mathbf{e} \in \mathbb{F}_q^n$ is a uniformly random vector of weight t.

Decryption: Apply $\mathcal{D}(s)$ to $\mathbf{m}G + \mathbf{e}$ to recover $\mathbf{m}$.

Example - GRS codes

- Let $n,\ k$ be positive integers with $1 \leq k \leq n \leq q$.
- $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{F}_q^n$ with $a_i \neq a_j$ for all $i \neq j$.
- $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{F}_q^n$ with $b_i \neq 0$ for all i.
- **Polynomial space:**

$$L_k = \mathbb{F}_q[X]_{<k} = \{f(X) \in \mathbb{F}_q[X] \mid \deg(f) < k\}$$

  L_k is a vector space of dimension k, with canonical basis: $\mathcal{B} = \{1, \ldots, x^{k-1}\}$

- **Evaluation map:**

$$\begin{array}{rccc} \mathrm{ev}_{\mathbf{a},\mathbf{b}}: & L_k & \longrightarrow & \mathbb{F}_q^n \\ & f(X) & \longmapsto & \mathbf{b} * f(\mathbf{a}) = (b_1 f(a_1), \ldots, b_n f(a_n)) \end{array}$$

Definition: GRS codes

$$\mathrm{GRS}_k(\mathbf{a}, \mathbf{b}) = \{\mathrm{ev}_{\mathbf{a},\mathbf{b}}(f) \mid f \in L_k\}$$

Example - GRS codes

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$$\mathrm{GRS}_k(\mathbf{a},\mathbf{b}) = \{\mathrm{ev}_{\mathbf{a},\mathbf{b}}(f) \mid f \in L_k\}$$

- $\mathcal{F}$ the set of $[n,k]$ GRS codes,
- $\mathcal{S} = \left\{(\mathbf{a},\mathbf{b}) \in \mathbb{F}_q^n \times \mathbb{F}_q^n \mid a_i \neq a_j \text{ and } b_i \neq 0, \ \ \forall i,j \in \{1,\dots,n\}, \ \ i \neq j\right\}$
- $\mathcal{D}(s)$ is our favorite decoder for GRS, e.g. Berlekamp Welch algorithm with $t = \lfloor \frac{n-k}{2} \rfloor$

Example - Alternant codes

Definition: Alternant codes

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{F}_{q^m}^n$ be a vector with distinct entries and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{F}_{q^m}^n$ with $b_i \neq 0$. An alternant code of degree r is the code:

$$\mathcal{A}_r(\mathbf{a}, \mathbf{b}) = \mathrm{GRS}_r(\mathbf{a}, \mathbf{b})^{\perp} \cap \mathbb{F}_q^n$$

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$$\mathcal{A}_r(\mathbf{a}, \mathbf{b}) = \mathrm{GRS}_r(\mathbf{a}, \mathbf{b})^\perp \cap \mathbb{F}_q^n$$

- $\mathcal{F}$ the set of alternant codes of length n and degree r.
- $\mathcal{S} = \left\{(\mathbf{a}, \mathbf{b}) \in \mathbb{F}_{q^m}^n \times \mathbb{F}_{q^m}^n \mid a_i \neq a_j \text{ and } b_i \neq 0, \ \ \forall i, j \in \{1, \dots, n\}, \ \ i \neq j\right\}$
- $\mathcal{D}(s)$ is our favorite decoder for alternant codes, e.g. Berlekamp Welch algorithm.

Example - Classical Goppa codes - McEliece 1978

Definition: Classical Goppa codes

Let $\mathbf{a} \in \mathbb{F}_{q^m}^n$ be a vector with distinct entries and $g \in \mathbb{F}_{q^m}[x]_{\leq t}$ be a polynomial sucha that $g(a_i) \neq 0 \forall i$. The Goppa code associated to $(\mathbf{a}, g)$ is

$$\mathcal{G}(\mathbf{a}, g) = \mathcal{A}_{\deg(g)}(\mathbf{a}, g(\mathbf{a})^{-1}) \cap \mathbb{F}_q^n$$

where $g(\mathbf{a})^{-1} = (g(a_1)^{-}1, \ldots, g(a_n)^{-1})$

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where $g(\mathbf{a})^{-1} = (g(a_1)^-1, \ldots, g(a_n)^{-1})$

- $\mathcal{F}$ the set of classical Goppa codes of length n and degree r.
- $\mathcal{S} = \left\{(\mathbf{a}, g) \in \mathbb{F}_{q^m}^n \times \mathbb{F}_{q^m}[x]_{\leq t} \mid \ldots\right\}$
- $\mathcal{D}(s)$ is our favorite decoder for Goppa codes.

Example - MDPC codes

Definition: QC-MDPC codes

Let n be a positive even integer and $f, g \in \mathbb{F}_2[x]_{\leq n}$ be two polynomials of weight $\mathcal{O}(\sqrt{n})$. An $[2n, n]$ QC-MPDC code is the kernel of the sparse matrix:

$$\left(\begin{array}{cccc|cccc} f_0 & f_1 & \cdots & f_{n-1} & g_0 & g_1 & \cdots & g_{n-1} \\ f_{n-1} & f_0 & \cdots & f_{n-2} & g_{n-1} & g_0 & \cdots & g_{n-2} \\ \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & \end{array}\right)$$

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- $\mathcal{F}$ the set of $[2n, n]$ MDPC codes
- $\mathcal{S} = \{(f, g) \in \mathbb{F}_q[x]_{\leq n}$ of weight $\mathcal{O}(\sqrt{n})\}$
- $\mathcal{D}(s)$ is our favorite decoder for MDPC codes, e.g. Bit Flipping algorithm.

Example - Algebraic geometry codes

Definition: Algebraic geometry codes

Let $\mathcal{X}$ be a smooth projective geometrically connected curve over $\mathbb{F}_q$, G be a divisor on $\mathcal{X}$ and $\mathcal{P} = (P_1, \ldots, P_n)$ be a set of $\mathbb{F}_q$-points of $\mathcal{X}$. We define

$$\mathcal{C}_L(\mathcal{X}, \mathcal{P}, G) = \{(f(P_1), \ldots, f(P_n) \mid f \in \mathcal{L}(G)\}$$

Example - Algebraic geometry codes

Definition: Algebraic geometry codes
Let $\mathcal{X}$ be a smooth projective geometrically connected curve over $\mathbb{F}_q$, G be a divisor on $\mathcal{X}$ and $\mathcal{P} = (P_1, \ldots, P_n)$ be a set of $\mathbb{F}_q$-points of $\mathcal{X}$. We define

$$\mathcal{C}_L(\mathcal{X}, \mathcal{P}, G) = \{(f(P_1), \ldots, f(P_n) \mid f \in \mathcal{L}(G)\}$$

- $\mathcal{F}$ the set of AG codes of length n from the curve $\mathcal{X}$
- $\mathcal{S} = \left\{(\mathcal{P}, G) \in \mathcal{X}(\mathbb{F}_q)^n \times \mathrm{Div}_{\mathbb{F}_q}(\mathcal{X}) \mid P_i \neq P_j \forall i \neq j\right\}$
- $\mathcal{D}(s)$ is our favorite decoder for AG codes, e.g. Error Correcting Pairs algorithm.

Efficient Decoding Algorithms

The following classes of codes:

- CGeneralized Reed-Solomon codes (GRS codes).
- Cyclic codes
- Alternant codes
- Goppa codes
- Algebraic geometry codes (AG codes)

... have efficient decoding algorithms:

- Arimoto, Peterson, Gorenstein, Zierler
- Berlekamp, Massey, Sakata
- Justensen et al. Vladut-Skorobatov
- Error-correcting pairs (ECP)

Attacks on the McEliece PKC

We have mainly 2 different ways of cryptanalyzing the McEliece cryptosystem:

1. Generic Decoding Attacks - Message recovery attacks The best known techniques needs **exponential** time in the code length.

2. Structural Attacks - Key recovery attacks Retrieve the code structure rather than use an unspecific decoding algorithm, i.e. recover $s \in \mathcal{S}$ such that the public key $\mathcal{C}_{\mathrm{pub}} = \mathcal{C}(s)$.

   Requirement: Distinguishing a prescribed structure code from a random one.

We focus on **Key Recovery Attacks** on this talk.

Security Proofs of McEliece

We reduce the problem of **Bounded decoding problem** to the security of McEliece under the assumption:

The generator matrix of the public $[n, k]$ code looks random.

That is:

The uniform distribution on the public $[n, k]$ code in family $\mathcal{F}$ is computationally indistinguishable from the uniform distribution on the whole family of $[n, k]$ codes.

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2014: More attacks on QC and QD codes

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2017: NIST's call for post-quantum crypto

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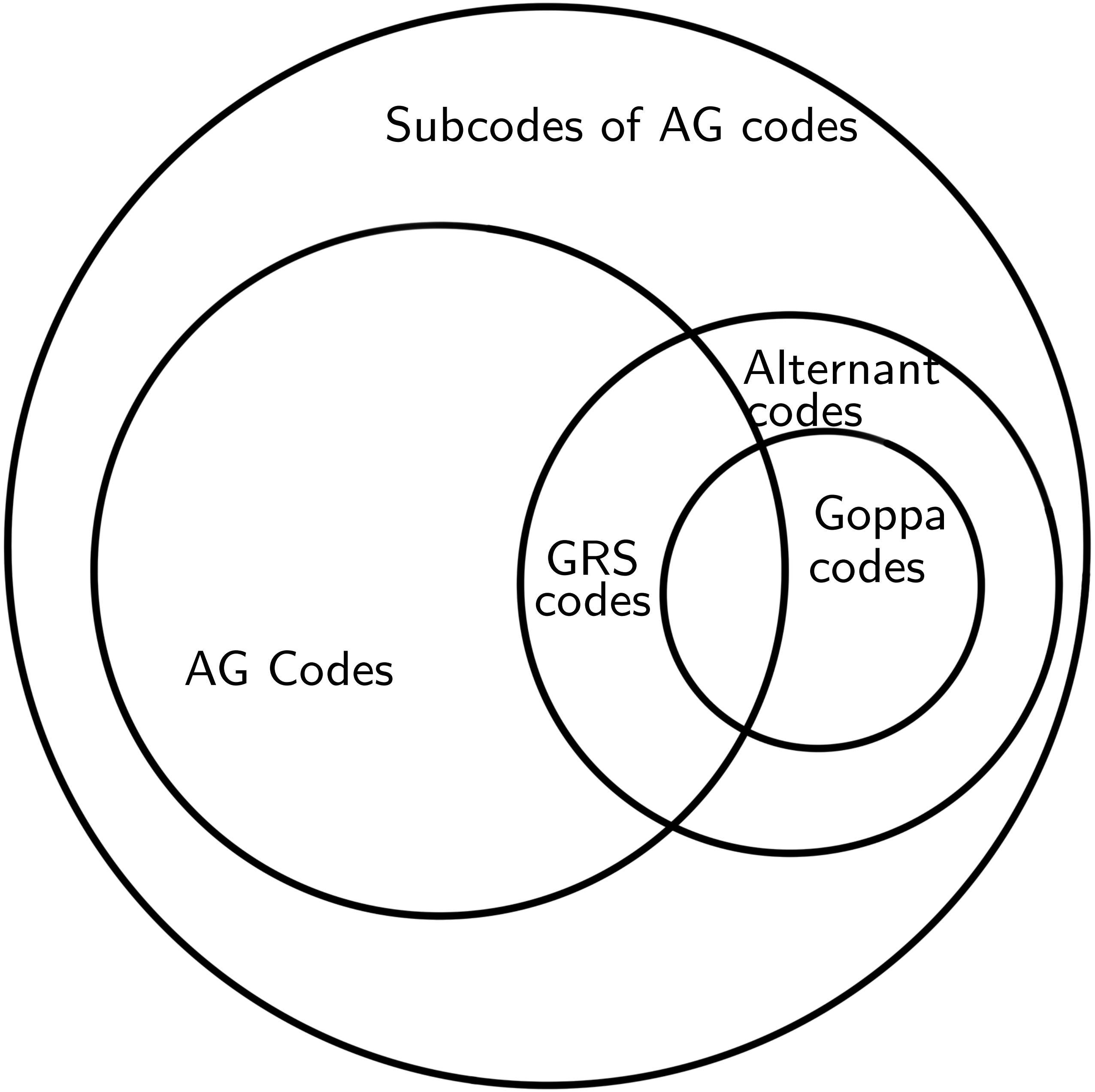
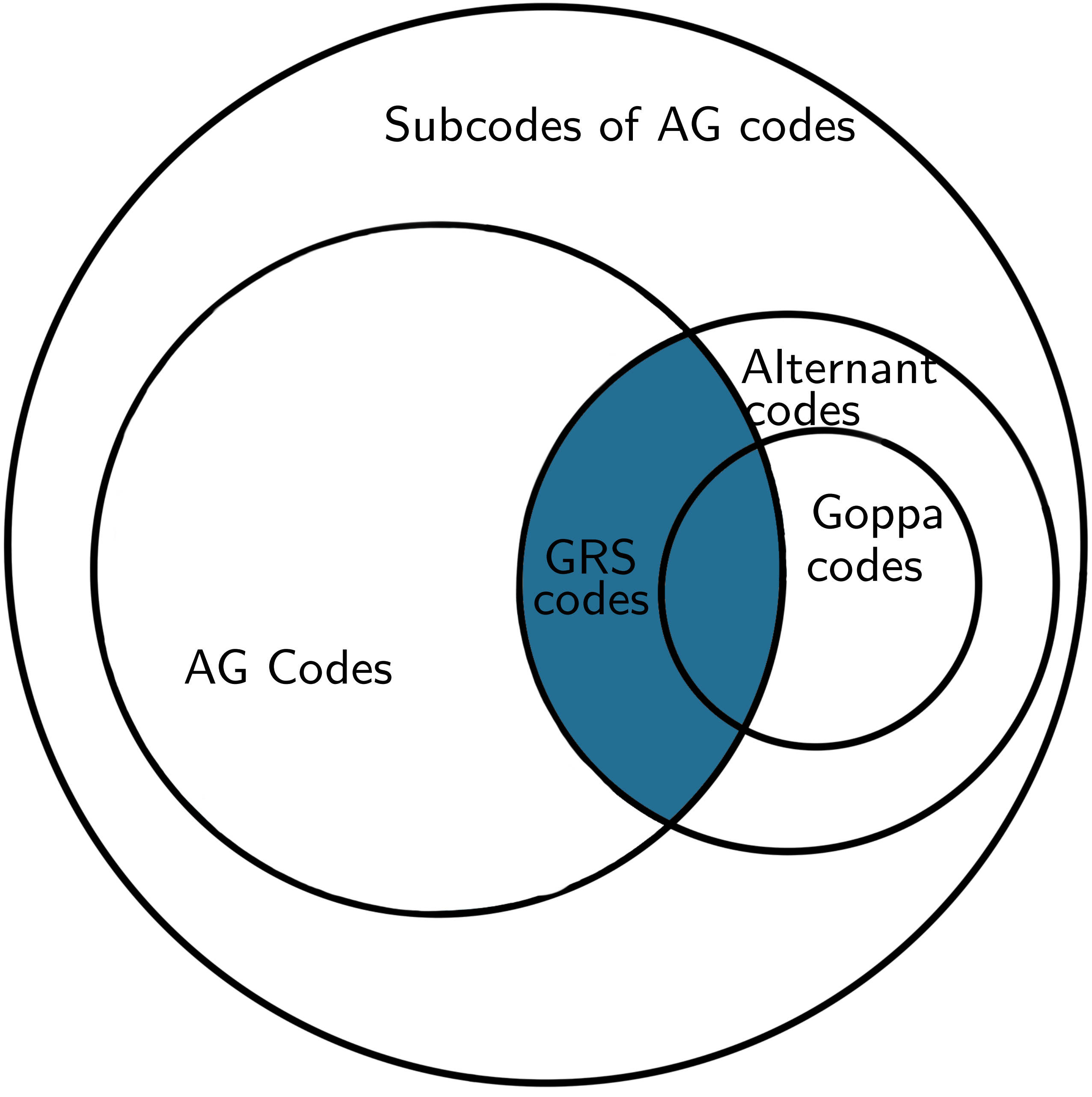
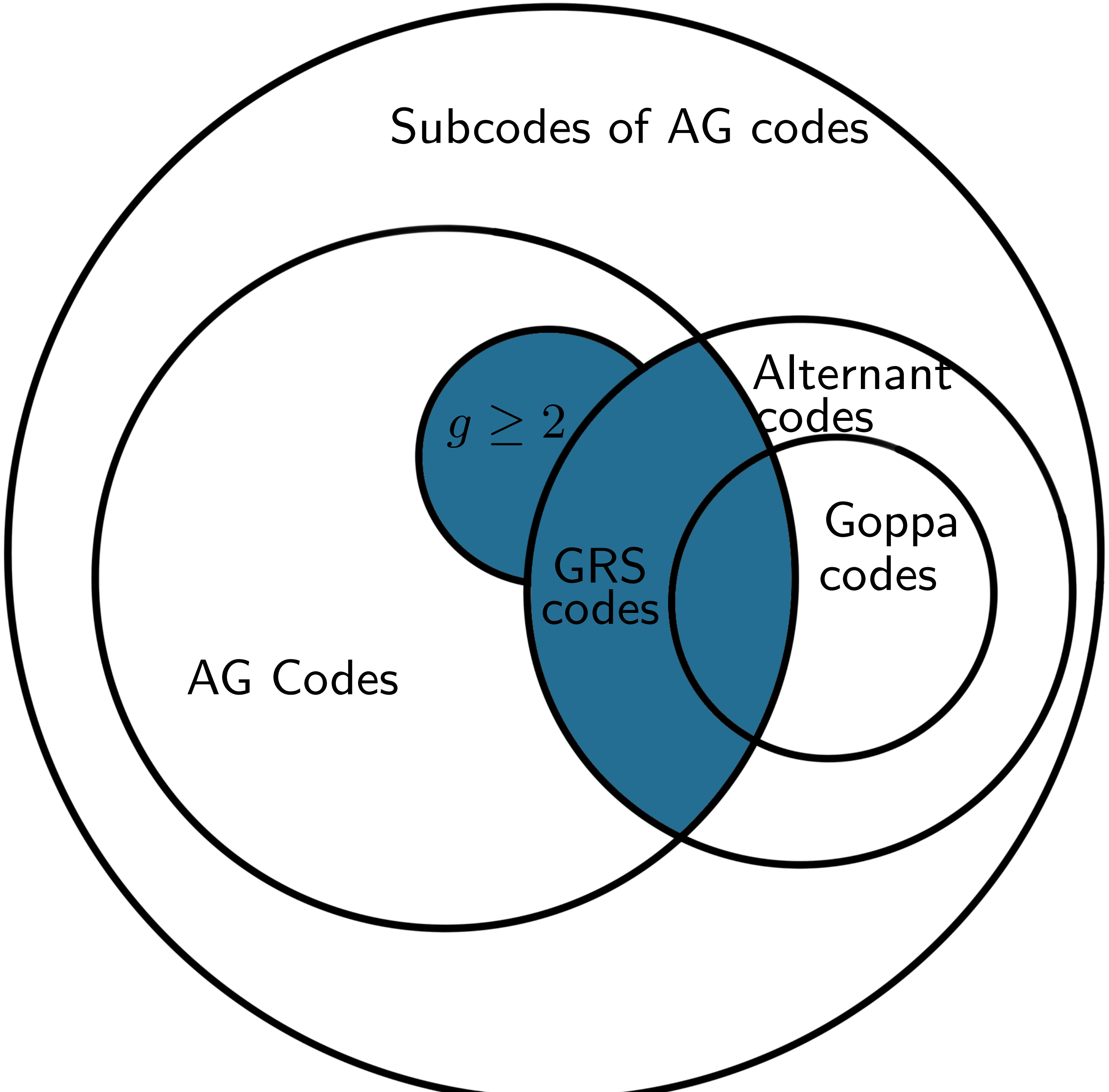


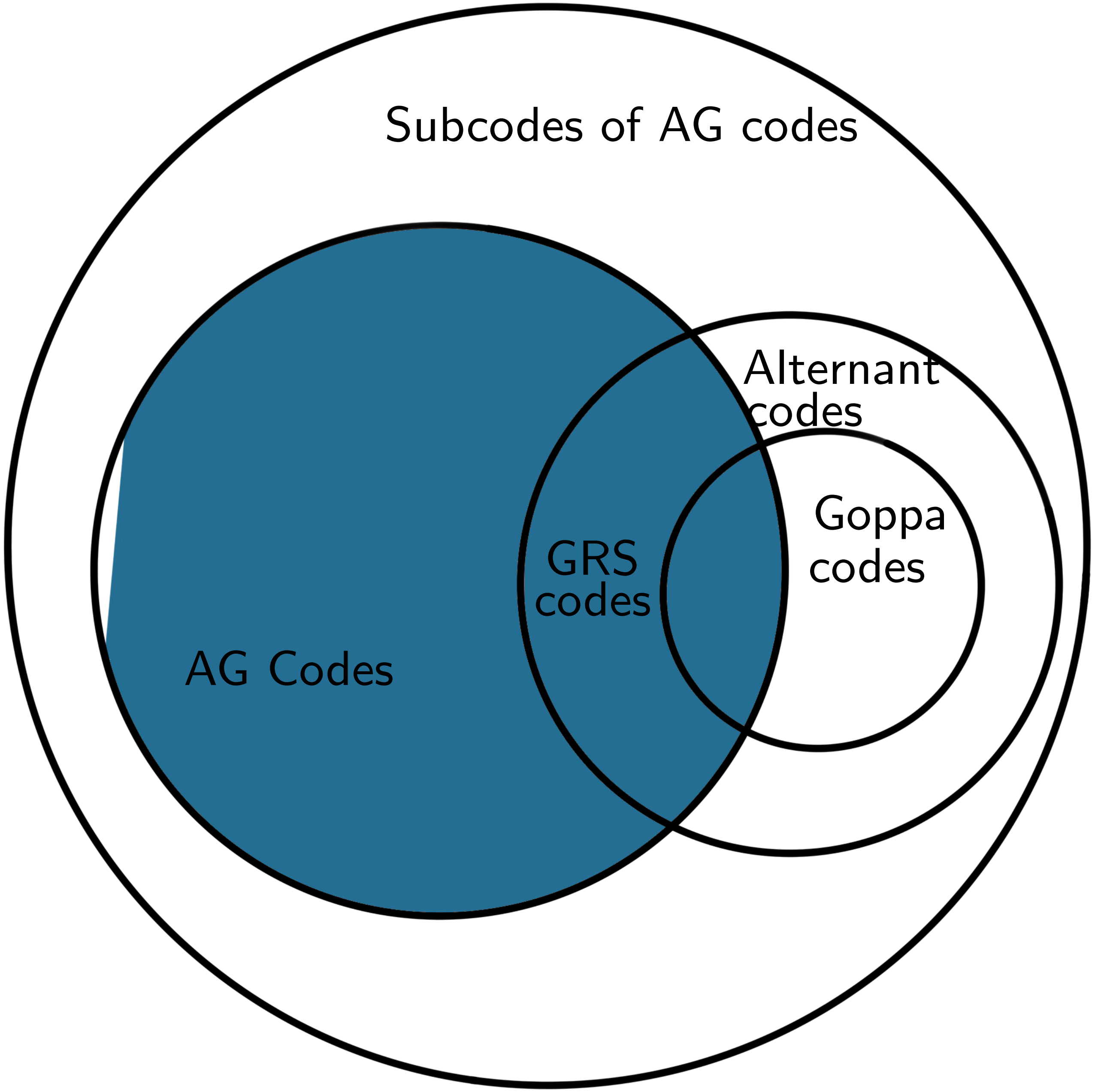
What is still surviving? - Sidelnikov-Shestakov 1992



What is still surviving? - Faurer-Minder 2008



What is still surviving? Couvreur-M.-Pellikaan 2014



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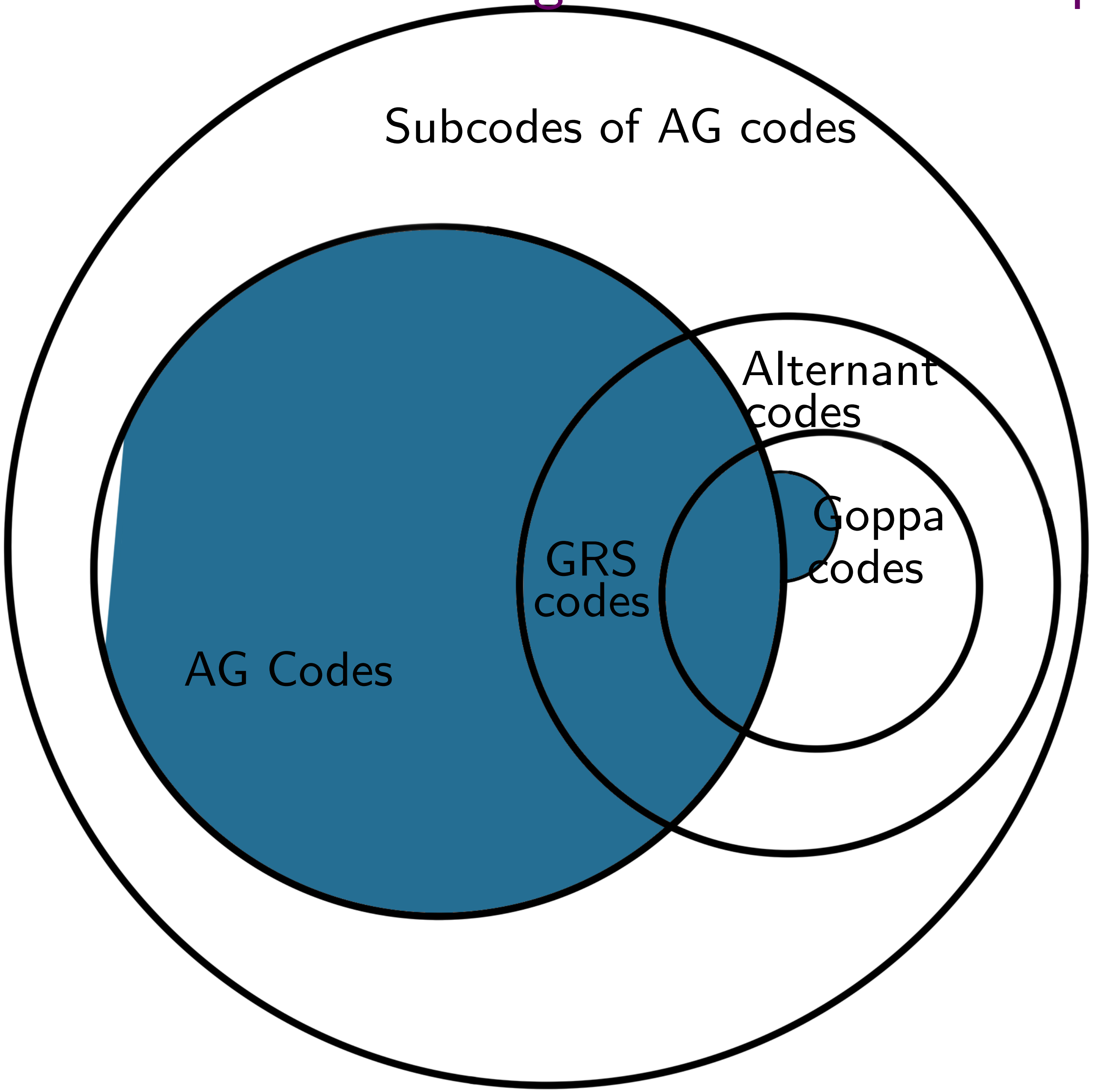
Couvreur-Otmani- Tillich & Faugère-Perret-Portzamparc 2014



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Algebraic world:

- Binary Goppa code (NIST's classic McEliece and NTS KEM).
- Goppa codes for $m \geq 2$.
- Goppa codes with "*small*" automorphism group.
- Subfield subcodes of AG codes.

Advantages: Short ciphertexts, no decoding failure.

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Promising alternatives:

– HQC, RQC.

◦ **Advantages:** do not rely on indistinguishability. Promising application of algebraic codes.

Original Proposal - Binary Goppa codes

1978: McEliece's - Binary Goppa codes

Public Key Size: 32kB for 65 bits of security

(with respect to Prange algorithm).

2018: NIST proposals with Binary Goppa codes:

– **Classic McEliece**
  Public Key Size: 1-1.3MByte for > 256 bits of security.
– **NTS KEM** 319 KBytes for > 128 bits security.

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During these 40 years many attempts to get shorter keys... but **HOW?**

IDEA 1 : Reducing the extension degree

Definition: Alternant codes

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{F}_{q^m}^n$ be a vector with distinct entries and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{F}_{q^m}^n$ with $b_i \neq 0$. An alternant code of degree r is the code:

$$\mathcal{A}_r(\mathbf{a}, \mathbf{b}) = \mathrm{GRS}_r(\mathbf{a}, \mathbf{b})^{\perp} \cap \mathbb{F}_q^n$$

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Fact. The larger the m the worse the parameters. But:

- The case $m = 1$ is broken (Sidelnikov-Shestakov 1992).
- Some specific cases of m=2 and 3 called wild Goppa codes are broken too:
  - Couvreur, Otmani, Tillich, 2014.
  - Faugère, Perret, de Portzamparc, 2014.

IDEA 1 : Reducing the extension degree

Further construction from GRS codes

- **2001:** Berger-Loidreau

  Subcodes of GRS codes.
- **2006:** Wieschebrink

  Adds random columns in GRS code's generator matrix.
- **2013:** Baldi, Bianchi, Chiaraluce, Rosenthal, Schipani

  Multiply the GRS code by a sparse matrix.
- **2016:** Wang's RLCE system

  Replaces some columns of a GRS's generator matrix by linear combinations of GRS and random columns.

IDEA 2 : Using codes with a non trivial automorphism group

Definition: Given a code $\mathcal{C} \subseteq \mathbb{F}_q^n$ with a group action $\mathcal{G}$, one can define the invariant code:

$$\mathcal{C}^{\mathcal{G}} = \{\mathbf{x} \in \mathcal{C} \mid \forall \sigma \in \mathcal{G} \sigma(\mathbf{x}) = \mathbf{x}\}$$

If the action of $\mathcal{G}$ is public, then $\mathcal{C}^{\mathcal{G}}$ is computable in polynomial time.

IDEA 2 : Using codes with a non trivial automorphism group

Definition: Given a code $\mathcal{C} \subseteq \mathbb{F}_q^n$ with a group action $\mathcal{G}$, one can define the invariant code:

$$\mathcal{C}^{\mathcal{G}} = \{\mathbf{x} \in \mathcal{C} \mid \forall \sigma \in \mathcal{G} \sigma(\mathbf{x}) = \mathbf{x}\}$$

If the action of $\mathcal{G}$ is public, then $\mathcal{C}^{\mathcal{G}}$ is computable in polynomial time.

In 2005 Gaborit proposes to use codes with a non-trivial automorphism group $\mathcal{G}$:

- Quasi-cyclic codes (QC-codes) : $\mathcal{G} = \mathbb{Z}/l\mathbb{Z}$
- Quasi-dyadic codes (QC-codes) : $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{\gamma}$

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- Advantage: Permits to reduce the public key size.
- Advantage: No incidence on the security w.r.t. generic decoding.
- Disadvantage: Affect the security w.r.t. key recovery attacks.

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Some key recovery attacks:

- QC-BCH codes
  Otmani-Tillich-Dallot (2008)

- QC-Alternant codes
  Faugère - Otmani - Perret - Tillich (2010)

- QC and QD - Alternant codes
  Faugère - Otmani - Perret - Tillich - Portzamparc (2016)

- QD - Alternant codes (DAGS)
  Barelli - Couvreur (2018)

Outline

1. History of code-based cryptography

2. Algebraic cryptanalysis in code-based cryptography

   2.1. Sidelnikov-Shestakov like attack.
   2.2. Algebraic attacks by solving a polynomial system.
   2.3. $\star$-product.

3. How to design secure schemes with codes?

Sidelnikov-Shestakov - 1992

Theorem: Sidelnikov-Shestakov
Given as input any matrix G generating the code $\mathrm{GRS}_k(\mathbf{a}, \mathbf{b})$, there exists an algorithm running in time $\mathcal{O}(n^4)$ that outputs $\mathbf{a}', \mathbf{b}'$ such that:

$$\mathrm{GRS}_k(\mathbf{x}, \mathbf{y}) = \mathrm{GRS}_k(\mathbf{x}', \mathbf{y}')$$

Moreover,

$$\mathbf{x} = \tfrac{a\mathbf{x}+b\mathbf{1}}{c\mathbf{x}+d\mathbf{1}} \quad \text{and} \quad \mathbf{y}' = \lambda \mathbf{y}$$

Sidelnikov-Shestakov - 1992

Public Key: $\mathcal{C} \subseteq \mathrm{GRS}_k(\mathbf{x}, \mathbf{y})$

Secret Key: $s = (\mathbf{x}, \mathbf{y})$

Sidelnikov-Shestakov Attack:

Step 1. In our search for $\mathbf{x}$, one can arbitrarily fix 3 points, say:

$$x_{n-2} = 1, \quad x_{n-1} = 0 \quad \text{and} \quad x_n = \infty$$

Step 2. From a generator matrix G of a code $\mathrm{GRS}_k(\mathbf{x}, \mathbf{y})$ compute two minimum weight codewords whose supports are close.

From G, by Gaussian elimination we can find:

$$\begin{array}{lllllllllll} \mathbf{u} = (& 0 & \dots & 0 & 0 & 1 & u_{k+1} & \dots & u_n &) & \longrightarrow & f(x) \\ \mathbf{v} = (& 0 & \dots & 0 & 1 & 0 & v_{k+1} & \dots & v_n &) & \longrightarrow & g(x) \end{array}$$

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Lemma If two elements $f, g \in \mathbb{F}_q[x]_{\leq k}$ share $k-2$ zeroes, then

$$\phi(x) = \frac{f(x)}{g(x)} = \frac{\alpha x + \beta}{\gamma x + \delta}$$

$$\mathbf{u} \star \mathbf{v}^{-1} = (\quad 0 \quad \dots \quad 0 \quad \perp \quad \frac{u_{k+1}}{v_{k+1}} \quad \dots \quad \frac{u_n}{v_n} \quad) \quad \longrightarrow \quad f(x)$$

1. **Solve** in $(\alpha, \beta, \gamma, \delta)$ the system $\phi(x_i) = \frac{u_i}{v_i}$ with $i = n-2, n-1, n$ we find ϕ
2. **Solve** the equation $\phi(x_i) = \frac{u_i}{v_i}$ for each $i \in [k+1, n-3]$ we find $x_{k+1}, \dots, x_{n-3}$

Step 3. Once $\mathbf{x}$ is known, one can easily find a valid $\mathbf{y}$ by solving a linear system.

Sidelnikov-Shestakov - 1992

Remark: Computing minimum weight codewords is hard but...

is only **Gaussian elimination** for **GRS codes**!!!

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Some attacks deriving form Sidelnikov-Shestakov:

- Minder-Shokrollahi (2007) - Broke Sidelnikov's proposal based on binary Reed-Muller codes.

  Subexponential time attack

- Faure-Minder (2008) - Broke AG codes from hyperelliptic curves.

  The attack has exponential cost in the curve's genus

  In orange due to the cost of computing minimum weight codewords.

Algebraic attacks by polynomial system solving

Idea: A code $\mathcal{A}_r(\mathbf{x}, \mathbf{y})$ is contained in the kernel of a matrix of the form:

$$H = \begin{pmatrix} y_1 & \cdots & y_n \\ x_1 y_1 & \cdots & x_n y_n \\ \vdots & \ddots & \vdots \\ x_1^{r-1} y_1 & \cdots & x_n^{r-1} y_n \end{pmatrix}$$

Put x_i, y_i as formal variables X_i, Y_i and solve the polynomial system:

$$H(X_i, Y_i)^T \cdot G = 0$$

For usual McEliece parameters, the resolution of such a polynomial system is out of reach. But...

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- Attacks on QC and QD Alternant codes
  Faugère- Otmani- Perret- Portzamparc, Tillich (2010)

$\star$ product

For all $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n$ we define

$\star$ **product:**

$$\mathbf{a} * \mathbf{b} = (a_1 b_1, \ldots, a_n b_n) \in \mathbb{F}_q^n$$

$\star$ **product of two codes:** Let $A,\ B \subseteq \mathbb{F}_q^n$ we define

$$A * B = \langle \{\mathbf{a} * \mathbf{b} \mid \mathbf{a} \in A \text{ and } \mathbf{b} \in B\} \rangle$$

For $B = A$ then we denote by $A^2 = A * A$

$\star$ product - Attack

Theorem: Cascudo-Cramer-Mirandola-Zémor 2013
Let $\mathcal{C}$ be a random $[n, k]$-code then

$$\mathrm{Prob}\left(\dim(\mathcal{C}^2) < \min\left(n, \binom{k+1}{2}\right)\right) \xrightarrow{n\to\infty} 0$$

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Proposition:

- If $k \leq \frac{n+1}{2}$. Then, $\mathrm{GRS}_k(\mathbf{a},\mathbf{b})^2 = \mathrm{GRS}_{2k-1}(\mathbf{a},\mathbf{b}*\mathbf{b})$
- If $k > \frac{n+1}{2}$ then,
  $\left(\mathrm{GRS}_k(\mathbf{a},\mathbf{b})^{\perp}\right)^2 = \mathrm{GRS}_{2(n-k)-1}(\mathbf{a},\mathbf{b}^{\perp}*\mathbf{b}^{\perp})$

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Proposition: Similar result for AG codes

$$\mathcal{C}_L(\mathcal{X},\mathcal{P},G)^2 = \mathcal{C}_L(\mathcal{X},\mathcal{P},2G)$$

under some condition on $\deg(G)$.

First use of $\star$-product - Wieschebrink 2010

- Berger-Loidreau propose in 2005 to use **subcodes of GRS codes**
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Fact: With high probability:

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Wieschebrink's attack:

Step 1. Compute $\mathcal{C}^2$

Step 2. Perform Sidelnikov-Shestakov attack on $\mathcal{C}^2$ to recover $(\mathbf{a}, \mathbf{b} * \mathbf{b})$

Step 3. Deduce $(\mathbf{a}, \mathbf{b})$.

Filtration Attack (Application of $\star$-product)

Illustrative example on GRS codes

Suppose we know the codes

$$\mathcal{C}_k = \mathrm{GRS}_k(\mathbf{a}, \mathbf{b}) \quad \text{and} \quad \mathcal{C}_{k-1} = \mathrm{GRS}_{k-1}(\mathbf{a}, \mathbf{b})$$

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Proposition: If $2k-1 \leq n-2$, then:

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can be computed as the set

$$\mathbf{c} \in \mathcal{C}_{k-1} \quad \text{y} \quad \mathbf{c} * \mathcal{C}_k \subseteq \left(\mathcal{C}_{k-1}\right)^2$$

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Proof: [Sketch]

$$\begin{array}{ccccccc} & \mathcal{C}_{k-1} & * & \mathcal{C}_k & = & (\mathcal{C}_{k-1})^2 \\ & (\mathbf{b} * f(\mathbf{a})) & * & (\mathbf{b} * g(\mathbf{a})) & = & (\mathbf{b} * \mathbf{b})\,(fg)(\mathbf{a}) \\ \text{with} & \deg(f) < k-1 & , & \deg(g) < k & \Rightarrow & \deg(fg) < 2k-2 \end{array}$$

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Then, reiterate the process we deduce the filtration:

$$\mathrm{GRS}_k(\mathbf{a}, \mathbf{b}) \supseteq \mathrm{GRS}_{k-1}(\mathbf{a}, \mathbf{b}) \supseteq \mathrm{GRS}_{k-2}(\mathbf{a}, \mathbf{b}) \supseteq \cdots \supseteq \mathrm{GRS}_1(\mathbf{a}, \mathbf{b})$$

Thus we get: $\mathrm{GRS}_1(\mathbf{a}, \mathbf{b}) = \left\{\alpha \mathbf{b} \mid \alpha \in \mathbb{F}_q^*\right\}$

Where we can deduce $\mathbf{b}$ and $\mathbf{a}$ solving a linear system.

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Remark: We do not need to know both $\mathrm{GRS}_k(\mathbf{a}, \mathbf{b})$ and $\mathrm{GRS}_{k-1}(\mathbf{a}, \mathbf{b})$ but $\mathrm{GRS}_{k-1}(\mathbf{a}, \mathbf{b})$ can be replaced by a shortening of $\mathrm{GRS}_k(\mathbf{a}, \mathbf{b})$ at one position.

Filtration Attack (Application of $\star$-product)

Same idea is behind:

- Alternative attack on GRS codes

  Couvreur-Gautier-Gaborit-Otmani-Tillich (2015)
- AG codes and their subcodes

  Couvreur-M.-Pellikaan (2014-17)
- Wild Goppa codes for $m = 2$

  Couvreur-Otmani-Tillich (2014-17)

Remark: No more need to compute minimum weight codewords!!

Outline

1. History of code-based cryptography

2. Algebraic cryptanalysis in code-based cryptography

3. How to design secure schemes with codes?

How to evaluate the security of algebraic codes?

- **Sufficiently many codes in the family**
  Support Splitting Algorithm (N. Sendrier).

- **Low weight codewords should be hard to compute.**
  Avoid Sidelnikov-Shestakov like attacks.

- **No square code distinguiser**.
  – $\mathcal{C}^2$, $(\mathcal{C}^{\perp})^2$ should behave like random codes.
  – Also their shortenings.

- And if you use some automorphism group, check the above properties for both $\mathcal{C}$ and $\mathcal{C}^{\mathcal{G}}$.

- It should resist attacks by algebraic systems solving – This is difficult to analyze.

Thanks