# Exam for Mathematics for Computer Graphics at Aalborg University

## For MED5 (Medialogiuddannelsen) and SP7 (Spiluddannelsen)

## Thursday, January 14th 2010. From 9:00 to 13:00.

You are allowed to use books, notes and a calculator during the exam. However computers, pdas or phones are not allowed.
A reasoned explanation should follow the solution of the exercises. Moreover, the intermediate steps leading to the solution should also be written down. You can write the exam in Danish or English.
The percentage following each exercise number stands for the exercise's value in the final mark.

**Exercise 1** *(7 %)* Let $\overrightarrow{u_0} = (1,1)^T$ and $\overrightarrow{u_1} = (0,-1)^T$ in $\mathbb{R}^2$. Apply the Gram-Schmidt orthogonalization to $\{\overrightarrow{u_0}, \overrightarrow{u_1}\}$.

**Exercise 2** *(15 %)* Let $P_0 = (1,1,1)^T$, $P_1 = (2,3,4)^T$, $P_2 = (1,0,0)^T$, $P_3 = (0,1,1)^T$ be points in the affine space $\mathbb{R}^3$.

1. Check that $P_0, P_1, P_2, P_3$ are a simplex (i.e. they are affinely independent).
2. Compute the centroid of the previous simplex.
3. Compute the barycentric coordinates of $P = (0,0,0)^T$ with respect to the previous simplex.

**Exercise 3** *(7 %)*

1. Compute the spherical coordinates of the point with cartesian coordinates $(0,-1,-1)$.
2. Compute the cartesian coordinates of the point with polar coordinates $r = 2$, $\theta = 30$ degrees.

**Exercise 4** *(20 %)* Let $\tau : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\tau((1,0,0)^T) = (1,1,0)^T,\ \tau((0,1,0)^T) = (0,2,1)^T,\ \tau((0,0,1)^T) = (1,1,1)^T,$$

1. Compute the matrix $A$ such that $\tau(\overrightarrow{x}) = A\overrightarrow{x}$.
2. Compute $\tau((1,2,3)^T)$.
3. Compute the determinant of $A$. What is the null space and the range of $\tau$?
4. Argue that there is an inverse for $A$. Compute $A^{-1}$.
5. What is the inverse of $A^{-1}$?

**Exercise 5** *(12 %)* Consider the plane defined by the points $P_0 = (1,2,3)^T$, $P_1 = (2,1,3)^T$, $P_2 = (1,3,2)^T$ in the affine space $\mathbb{R}^3$.

1. Compute the distance from the point $(1,1,1)^T$ to the previous plane.
2. Compute the distance from the point $(-1,4,3)^T$ to the previous plane. What does this mean?

**Exercise 6** *(16 %)*

1. Compute the quaternion that corresponds to a 60 degrees rotation around the axis given by the vector $(1,1,0)^T$ (hint: $(1,1,0)^T$ is not normalized).
2. Compute the quaternion that corresponds to a 120 degrees rotation around the axis given by the vector $(0,1,0)^T$.
3. Compute the quaternion that corresponds to apply first the rotation of 1. and then the rotation of 2.

**Exercise 7** *(8 %)* Let $P_0 = (1,0,0)^T$, $P_1 = (0,1,0)^T$, $P_2 = (0,2,2)^T$, $P_3 = (0,1,1)^T$.

1. Compute the Bézier curve of $P_0, P_1, P_2, P_3$ (of order 3).
2. Compute $P_0'$ and $P_3'$ in $\mathbb{R}^3$ such that the Hermite curve with sample positions $\{P_0, P_3\}$ and tangent vectors $\{P_0', P_3'\}$ is equal to the previous Bézier curve.

**Exercise 8** *(15 %)* Let $p = 0+(1,0,0)$ and $q = 0+(0,1,0)$ be two quaternions representing two orientations.

1. Compute $lerp(p,q,2/3)$ (linear interpolation)
2. Compute $slerp(p,q,t)$ (spherical linear interpolation)
3. Compute $slerp(p,q,2/3)$
4. Let $p'$ and $q'$ be two quaternions representing two orientations. Assume that $p' \cdot q' = 0.999$. If you had to interpolate the orientation between $p'$ and $q'$, would you consider lerp or slerp? (hint: think in finite floating-point precision)

*Husk at skrive jeres fulde navn på hver side af besvarelsen. Nummerer siderne, og skriv antallet af afleverede ark på 1. side af besvarelsen. God arbejdslyst.*

## Ex 1



$$\vec{w}_0 = (1,1)$$

$$\vec{w}_1 = (0,-1) - \frac{(0,-1)(1,1)}{2}(1,1) = \left(\frac{1}{2}, -\frac{1}{2}\right)$$

$$\|\vec{w}_0\| = \sqrt{2} \quad , \quad \|\vec{w}_1\| = \sqrt{1/2}$$

$$\hat{\vec{w}}_0 = \left(\sqrt{2}/2 , \sqrt{2}/2\right) \quad , \quad \hat{\vec{w}}_1 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

## Ex 2

1) $P_1 - P_0 = (1,2,3)$ , $P_2 - P_0 = (0,-1,-1)$ ,

$P_3 - P_0 = (-1,0,0)$

$a_1(P_1 - P_0) + a_2(P_2 - P_0) + a_3(P_3 - P_0) = \vec{0} \Rightarrow$

$$\Rightarrow \begin{matrix} a_1 - a_3 = 0 \\ 2a_1 - a_2 = 0 \\ 3a_1 - a_2 = 0 \end{matrix} > a_1 = 0 > a_3 = 0 \Rightarrow a_2 = 0$$

$\Rightarrow (P_1 - P_0)$ , $(P_2 - P_0)$ , $(P_3 - P_0)$ are l.i.

2) $\frac{1}{4}(1,1,1) + \frac{1}{4}(2,3,4) + \frac{1}{4}(1,0,0) + \frac{1}{4}(0,1,1) =$

$= (1, 5/4, 6/4)$

3) $P = a_0 P_0 + a_1 P_1 + a_2 P_2 + a_3 P_3$ , $\Sigma a_i = 1$

$P - P_0 = a_1(P_1 - P_0) + a_2(P_2 - P_0) + a_3(P_2 - P_0)$

$(-1,-1,-1) = a_1(1,2,3) + a_2(0,-1,-1) + a_3(-1,0,0)$

$\Rightarrow a_1 = 0$ , $a_2 = 1$ , $a_3 = 1$ $\Rightarrow a_0 = 1 - 1 - 1 = -1$

$P = (-1, 0, 1, 1) \leftarrow$ BARYCENTRIC COORDINATES

## Ex3)

1) $\rho = \sqrt{2}$

$\phi = \text{arctan2}(1, -1) = 135° = \frac{3\pi}{4}$

$\theta = \text{arctan2}(-1, 0) = 270° = \frac{3\pi}{2}$

2) $x = 2\cos 30 = \sqrt{3}$

$y = 2\sin 30 = 1$

## Ex4

1) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

2) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$

3) $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(2-1) + 1(1) = 2.$

$\text{Det}(A) \neq 0 \Rightarrow$
$\text{Ker}(T) = \{(0)\}$
$\text{Range}(T) = \mathbb{R}^3$

4) $\det(A) \neq 0$ and it is 3x3 matrix $\Rightarrow$ A has inverse.

$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1 \end{array}\right)$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1 \end{array}\right)$

$\Rightarrow A^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1 \\ -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 1 \end{pmatrix}$

5) $(A^{-1})^{-1} = A$

## Ex 5

$P_1 - P_0 = (1,-1,0)$ , $P_2 - P_0 = (0,1,-1)$

$\vec{n} = (1,-1,0) \times (0,1,-1) = (1,1,1)$

$\Pi: x + y + z + d = 0$

$1 + 2 + 3 + d = 0 \Rightarrow d = -6$

Normalize equation plane: $\frac{1}{\sqrt{3}} x + \frac{1}{\sqrt{3}} y + \frac{1}{\sqrt{3}} z - \frac{6}{\sqrt{3}} = 0$

$\text{Dist}((1,1,1),\pi) = \left| -\frac{3}{\sqrt{3}} \right| = \frac{3}{\sqrt{3}}$

$\text{Dis}((-1,4,3),\pi) = |0| = 0 \Rightarrow (-1,4,3) \in \Pi$

## Ex 6

1) $\hat{\vec{r}} = (\sqrt{2}/2, \sqrt{2}/2, 0)$

$w = \cos(60/2) = \sqrt{3}/2$ , $\vec{v} = \sin(60/2)\hat{\vec{r}} =$

$= \frac{1}{2}(\sqrt{2}/2, \sqrt{2}/2, 0)$

$q = \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, 0\right)$

2) $\vec{r} = \hat{r}$ , $w = \cos(60) = 1/2$

$\vec{v} = \sin(60)\vec{r} = \sqrt{3}/2\,(0,1,0)$

$q = \frac{1}{2} + \left(0, \frac{\sqrt{3}}{2}, 0\right)$

3) $q_2 \cdot q_1 = \frac{\sqrt{3}}{4} - \frac{\sqrt{6}}{8}, (0, \frac{3}{4}, 0) + (\frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{8}, 0) + (0, 0, -\frac{\sqrt{6}}{8})$

$$= \frac{2\sqrt{3} - \sqrt{6}}{8} + \left(\frac{\sqrt{2}}{8}, \frac{6+\sqrt{2}}{8}, -\frac{\sqrt{6}}{8}\right)$$

## Ex 7:

1) $Q(u) = (1-u)^3 P_0 + 3u(1-u)^2 P_1 + 3u^2(1-u) P_2 + u^3 P_3$

$$= \begin{pmatrix} (1-u)^3 \\ 3u(1-u)^2 + 6u^2(1-u) + u^3 \\ 6u^2(1-u) + u^3 \end{pmatrix}$$

2) $P_0' = 3(P_1 - P_0) = \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}$

$P_3' = 3(P_3 - P_2) = \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}$

## Ex 8

1) $\text{lerp}(p, q, t) = (1-t)p + tq = 0 + (1-t, t, 0) = r$

Normalize $r \rightsquigarrow 0 + \left(\frac{1-t}{\sqrt{1-2t+2t^2}}, \frac{t}{1-2t+2t^2}, 0\right)$

$\text{lerp}(p, q, 2/3) = 0 + (1/\sqrt{5}, 2/\sqrt{5}, 0)$

2) $\text{slerp}(p, q, t) = \frac{\sin((1-t)90)[0+(1,0,0)] + \sin(t\,90)[0+(0,1,0)]}{1}$

$p \cdot q = 0 \Rightarrow \theta = 90°$

3) $\text{slerp}(p, q, 2/3) = [0 + (1/2, 0, 0)] + [0 + (0, \frac{\sqrt{3}}{2}, 0)]$

$= 0 + (1/2, \sqrt{3}/2, 0)$

4) lerp → see page 468 from book