Exam for Mathematics for Computer Graphics at Aalborg University

For MED5 (Medialogiuddannelsen)

Tuesday, March 9th 2010. From 9:00 to 13:00.

You are allowed to use books, notes and a calculator during the exam. However computers, pdas or phones are not allowed.
A reasoned explanation should follow the solution of the exercises. Moreover, the intermediate steps leading to the solution should also be written down. You can write the exam in Danish or English.
The percentage following each exercise number stands for the exercise's value in the final mark.

Exercise 1 *(9 %)* Consider a game where there are two helicopters. Helicopter 1 is at position $(1, 2, 3)$ and Helicopter 2 at position $(4, 5, 6)$. If Helicopter 1 is looking in direction $(-1, 3, 0)$ and Helicopter 2 is looking in direction $(4, 0, 0)$,

1. Can Helicopter 1 see Helicopter 2?
2. Can Helicopter 2 see Helicopter 1?

Exercise 2 *(9 %)* Consider a game where there is a car moving over the xy plane and upwards is defined by $(0, 0, 1)$ in our reference frame. The car moves with velocity $(1, 2, 0)$ and it wants to move in direction $(-1, 1, 0)$ instead. Should the car turn left or right?

Exercise 3 *(20 %)* Let $\tau : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ such that

$$\tau((1, 0, 0, 0)^T) = (1, 2, 3, 4)^T,\ \tau((0, 1, 0, 0)^T) = (2, 5, 6, 8)^T,$$

$$\tau((0, 0, 1, 0)^T) = (3, 6, 9, 13)^T,\ \tau((0, 0, 0, 1)^T) = (4, 8, 13, 16)^T$$

1. Compute the matrix A such that $\tau(\overrightarrow{x}) = A\overrightarrow{x}$.
2. Compute $\tau((1, -1, 0, 2)^T)$.
3. Compute the determinant of A (hint: it is easy if you use elementary row operations).
4. Argue that there is an inverse for A (but you do not need to compute it).
5. Compute the determinant of A^{-1} (hint: you can compute this without computing the matrix A^{-1}).

Exercise 4 *(16 %)* Consider the plane through the point $(4, 5, 6)^T$ with normal $\vec{n} = (0, 1, 0)^T$. We now perform a shear with respect to this plane and the shear vector $\vec{s} = (1, 2, 3)^T$.

1. Determine the 4×4-matrix that describes this shear (hint: the center of transformation is not the origin).

2. Find the transformation of the point $(-1, 0, 0)^T$.

Exercise 5 *(16 %)*

1. Compute the quaternion that corresponds to a 90 degrees rotation around the axis given by the vector $(0, 1, 1)^T$ (hint: $(0, 1, 1)^T$ is not normalized).

2. Compute the quaternion that corresponds to a 180 degrees rotation around the axis given by the vector $(0, 1, 0)^T$.

3. Compute the quaternion that corresponds to apply first the rotation of 1. and then the rotation of 2.

4. Compute the quaternion that corresponds to apply first the rotation of 2. and then the rotation of 1.

Exercise 6 *(20 %)* Consider the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

1. Show that A is a rotation matrix.

2. What happens to the point $(1, 2, 3)^T$ under rotation?

3. Determine the rotation angle and the rotation axis of the rotation matrix A.

4. Determine the quaternion that corresponds to the rotation matrix A.

Exercise 7 *(10 %)* Let $Q(u)$, $u = 0, \ldots, 1$, be the Hermite curve with $Q(0) = (1, 0, 0)^T$, $Q'(0) = (0, 1, 0)^T$, $Q(1) = (0, 2, 2)^T$, $Q'(1) = (0, 1, 0)^T$. Compute $Q(\frac{1}{2})$.

Husk at skrive jeres fulde navn på hver side af besvarelsen. Nummerer siderne, og skriv antallet af afleverede ark på 1. side af besvarelsen. God arbejdslyst.

Ex 1

$\vec{h_1} = (1,2,3)$, $\vec{h_2} = (4,5,6)$

$\vec{v_1} = \vec{h_2} - \vec{h_1} = (3,3,3)$

$(3,3,3)\cdot(-1,3,0) = 6 > 0 \Rightarrow$ Yes

$\vec{v_2} = \vec{h_1} - \vec{h_2} = (-3,-3,-3)$

$(-3,-3,-3)\cdot(4,0,0) = -12 < 0 \Rightarrow$ No

Ex 2

$(1,2,0) \times (-1,1,0) = (0,0,1) - (0,0,-2) = (0,0,3)$

$3 > 0 \Rightarrow$ LEFT

Ex 3

1)

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 6 & 9 & 13 \\ 4 & 8 & 13 & 16 \end{pmatrix}$$

2)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 6 & 9 & 13 \\ 4 & 8 & 13 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 13 \\ 23 \\ 28 \end{pmatrix}$$

3)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 6 & 9 & 13 \\ 4 & 8 & 13 & 16 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1$$

4) A squared $\wedge \det(A) \neq 0 \Rightarrow \exists A^{-1}$

5) $\det(A^{-1}) = \dfrac{1}{\det(A)} = -1$

Ex 4

1) $M = \begin{bmatrix} A & (I-A)\vec{x} \\ 0^T & 1 \end{bmatrix}$, $A = \begin{bmatrix} I + s \otimes \hat{n} & 0 \\ 0^T & 1 \end{bmatrix}$ $\quad \vec{x} = (4,5,6)$, $\vec{n} = (0,1,0)$, $\vec{s} = (1,2,3)$

$$I + \vec{s} \otimes \hat{n} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (0\ 1\ 0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

$$(I - A)\vec{x} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -5 \\ -10 \\ -15 \end{pmatrix}$$

$$M = \begin{bmatrix} 1 & 1 & 0 & -5 \\ 0 & 3 & 0 & -10 \\ 0 & 3 & 1 & -15 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) $\begin{bmatrix} 1 & 1 & 0 & -5 \\ 0 & 3 & 0 & -10 \\ 0 & 3 & 1 & -15 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -10 \\ -15 \\ 1 \end{bmatrix}$ $\quad (-6, -10, -15)$

Ex 5

1) $w = \cos(45) = \frac{\sqrt{2}}{2}$, $\hat{r} = (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

$\vec{v} = \sin(45) \cdot \vec{r} = (0, \frac{1}{2}, \frac{1}{2}) \Rightarrow q_1 = \frac{\sqrt{2}}{2} + (0, \frac{1}{2}, \frac{1}{2})$

2) $w = \cos(90) = 0$, $\hat{r} = (0, 1, 0)$

$\vec{v} = \sin(90) \cdot \vec{r} = (0, 1, 0) \Rightarrow q_2 = 0 + (0, 1, 0)$

3) $q_2 \cdot q_1 = w_1 w_2 - \vec{v_1} \cdot \vec{v_2}$, $w_1 \vec{v_2} + w_2 \vec{v_1} + \vec{v_2} \times \vec{v_1}$

$\vec{v_2} \times \vec{v_1} = (0, 1, 0) \times (0, \frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, 0, 0) - (0, 0, 0)$

$q_2 \cdot q_1 = 0 - \frac{1}{2}$, $(0, \frac{\sqrt{2}}{2}, 0) + (\frac{1}{2}, 0, 0) = -\frac{1}{2} (\frac{1}{2}, \frac{\sqrt{2}}{2}, 0)$

4) $q_1 \cdot q_2 = 0 - \frac{1}{2}$, $(0, \frac{\sqrt{2}}{2}, 0) + (-\frac{1}{2}, 0, 0) = -\frac{1}{2} (-1/2, \frac{\sqrt{2}}{2}, 0)$

$\vec{v_2} \times \vec{v_1} = -\vec{v_1} \times \vec{v_2}$

Ex 6

1) $\det\begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix}=1 \quad \wedge \quad \begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix}\begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix}=Id \Rightarrow$ Rotation Matrix

2) $\begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-1\\2\\-3\end{pmatrix}$

3) One can use formula at page 145 (rotation around R_y)
or:

$\theta = \arccos\left(\frac{1}{2}(\text{trace}(A)-1)\right) = \arccos(-1) = 180° = \pi$ radians

$y = \frac{1}{2}\sqrt{A_{11}-A_{00}-A_{22}+1} = \frac{1}{2}\sqrt{4} = 1$

$x = \frac{A_{01}}{2y} = 0 \qquad z = \frac{A_{12}}{2y} = 0 \quad \Rightarrow \vec{r} = (0,1,0)$

4) $w = \cos(90) = 0 \quad , \quad \vec{v} = \sin(90)\cdot(0,1,0) = (0,1,0)$

$q = 0 + (0,1,0)$

Ex 7:

$Q(u) = [2(P_0-P_1)+P_0'+P_1']u^3 + [3(P_1-P_0)-2P_0'-P_1']u^2 + P_0'u + P_0$

$2(P_0-P_1)+P_0'+P_1' = 2\begin{pmatrix}1\\-2\\-2\end{pmatrix}+\begin{pmatrix}0\\2\\0\end{pmatrix}=\begin{pmatrix}2\\-2\\-4\end{pmatrix}$

$3(P_1-P_0)-2P_0'-P_1' = 3\begin{pmatrix}-1\\2\\2\end{pmatrix}+\begin{pmatrix}0\\-3\\0\end{pmatrix}=\begin{pmatrix}-3\\3\\6\end{pmatrix}$

$Q(1/2) = \frac{1}{8}\begin{pmatrix}2\\-2\\4\end{pmatrix}+\frac{1}{4}\begin{pmatrix}-3\\3\\6\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\end{pmatrix}=\frac{1}{8}\begin{pmatrix}4\\8\\8\end{pmatrix}=\begin{pmatrix}1/2\\1\\1\end{pmatrix}$

or $\quad Q(u) = (2u^3-3u^2+1)P_0 + (-2u^3+3u^2)P_1 + (u^3-2u^2+u)P_0' + (u^3-u^2)P_1'$

$2u^3-3u^2+1 = 1/2 \quad ; \quad -2u^3+3u^2 = 1/2 \quad ; \quad u^3-2u^2+u = 1/8$

$u^3-u^2 = -1/8$

$Q(1/2) = \frac{1}{2}\begin{pmatrix}1\\0\\0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0\\2\\2\end{pmatrix}+\frac{1}{8}\begin{pmatrix}0\\1\\0\end{pmatrix}-\frac{1}{8}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}1/2\\1\\1\end{pmatrix}$