Computing the Feng-Rao Distances for Codes from Order Domains

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Abstract

We compute the Feng-Rao distance of a code coming from an order domain with a simplicial value semigroup. The main tool is the Apéry set of a semigroup that can be computed using a Gröbner basis.

1 Introduction

An important family of error correcting codes is the Algebraic-Geometry Codes introduced by Goppa. In 1982 Tsfasman, Vlăduţ and Zink constructed a sequence of error correcting codes whose parameters exceed the Gilbert-Varshamov bound. In 1993 Feng and Rao presented the majority voting test that corrects up to the Feng-Rao distance for a class of AG codes, the one-point AG-codes [8]. Campillo and Farrán used the Apéry set of a semigroup in $\mathbb{N}_0$ to compute the Feng-Rao distance of the one point codes [2], i.e. for semigroups in $\mathbb{N}_0$.

The one-point AG-codes can be extended to a new family, the evaluation codes and their duals using order or weight functions that take values in $\mathbb{N}_0$ [8]. This new construction has an easier description and it seemed more general, but in fact the two families are equal [10].

An important improvement of the evaluation codes consists of considering weight functions with values in a finitely generated semigroup Γ of $\mathbb{N}_0^r$. This new class of codes contains also the one-point codes and the Feng-Rao algorithm extend in a natural way to a suitable class of codes [12, 9] in this more general setting where one can consider for instance toric rings. Weight functions can be constructed from order domains and give rise to the so called codes from order domains [5, 6].

For a code coming from an order domain $C_{\boldsymbol{\lambda}}$ one has the Feng-Rao distances, that are lower bounds for the minimum distance $d(C_{\boldsymbol{\lambda}}) \geq d_{\varphi}(\boldsymbol{\lambda}) \geq d(\boldsymbol{\lambda})$. The bound $d(\boldsymbol{\lambda})$ depends only on the semigroup $\Gamma \subset \mathbb{N}_0^r$.

The computation of the Feng-Rao distances involves the computation of the numbers $\mu_{\boldsymbol{\lambda}}$ for a infinite set of $\boldsymbol{\lambda}$'s, $\boldsymbol{\lambda} \in \Gamma$. In this paper we compute $d(\boldsymbol{\lambda})$ for

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simplicial semigroups in $\mathbb{N}_0^r$, for every r. Geil reduces the computation of $d_\varphi(\boldsymbol{\lambda})$ to the computation of a finite set of $\mu_{\boldsymbol{\lambda}}$'s [5], where $\mu_{\boldsymbol{\lambda}}$ is computed directly from the definition. In this paper we compute $\mu_{\boldsymbol{\lambda}}$ in a more effective way for simplicial semigroups using the Apéry set of the semigroup.

The Apéry set is obtained in [13] by computing a Gröbner basis with respect to the negative lexicographical ordering (`ls` in SINGULAR). If we fix a monomial order in a semigroup Γ, the Apéry set of Γ relative to the generators of the extremal rays allows us to write its elements in a unique way as a linear combination of the elements of the semigroup and the generators of the extremal rays of the cone. In this work we use this way of representing the elements of the semigroup to compute $\mu_{\boldsymbol{\lambda}}$ and $d(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \Gamma$, for simplicial semigroups. The algorithms have been implemented in the computer algebra system SINGULAR [7].

2 Codes coming from order domains

In this section we introduce the weight functions with values in a semigroup of $\mathbb{N}_0^r$, the order domains and the codes coming from order domains. The results of this section are from [5, 6]. We use definitions and results of Gröbner basis theory, all taken from [4].

Definition 2.1. Let $\Gamma \subseteq \mathbb{N}_0^r$ be a finitely generated semigroup and consider a monomial ordering $<_{\mathbb{N}_0^r}$ in $\mathbb{N}_0^r$. We extend the semigroup Γ, to $\Gamma_{-\infty} = \Gamma \cup \{-\infty\}$. Let $\mathbb{F}$ be a finite field and $I \subset \mathbb{F}[X_1, \ldots, X_m]$ be a prime ideal. A map $\rho : \mathbb{F}[X_1, \ldots, X_m]/I \to \Gamma_{-\infty}$ is said to be a (finitely generated) **weight function**, if it is surjective and it satisfies the following conditions:

$$\begin{cases} (0) & \rho(f) = -\infty \text{ if and only if } f = 0 \\ (1) & \rho(cf) = \rho(f) \text{ for all nonzero } c \in \mathbb{F} \\ (2) & \rho(f+g) \leq_{\mathbb{N}_0^r} \max_{<_{\mathbb{N}_0^r}} \{\rho(f), \rho(g)\} \\ (3) & \text{If } \rho(f) = \rho(g) \neq -\infty\text{, then there exists a nonzero } c \in \mathbb{F} \text{ such that} \\ & \rho(f - cg) <_{\mathbb{N}_0^r} \rho(g) \\ (4) & \rho(fg) = \rho(f) + \rho(g) \end{cases}$$

for all $f, g \in \mathbb{F}[X_1, \ldots, X_m]/I$.

In the case that a weight function ρ exists, then $\mathbb{F}[X_1, \ldots, X_m]/I$ is called an **order domain over** $\mathbb{F}$ and Γ is called the **value semigroup of** ρ.

In [6] a weight function is a surjective map from an $\mathbb{F}$-algebra R to a well ordered semigroup Γ such that conditions (0) to (4) are satisfied. That seems to be more general but whenever the semigroup Γ is finitely generated, then up to isomorphism R is of the previous form. Therefore the definition 2.1 describes all possible weight functions with finitely generated value semigroup. By [6, Corollary 5.7] we can assume that r in the definition 2.1 is equal to the transcendence degree of $\mathrm{Quot}(\mathbb{F}[X_1, \ldots, X_m]/I)$.

In the following we use the notation $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r) \in \mathbb{N}_0^r$. In order to characterize the order domains we need some terminology. Let us consider weights $\boldsymbol{w}(X_i) \in \mathbb{N}_0^r \setminus \{0\}$ for $i = 1, \ldots, m$. The weight of a monomial in $\mathbb{F}[X_1, \ldots, X_m]$ is then $\boldsymbol{w}(\mathbf{X}^{\boldsymbol{\alpha}}) = \sum_{i=1}^m \alpha_i \boldsymbol{w}(X_i)$.

Given the inner ordering $<_{\mathbb{N}_0^r}$ on Γ, a weight $\boldsymbol{w}$ and another monomial ordering $<_{\mathcal{M}}$ one define the **generalized weighted degree ordering** $<_{\boldsymbol{w}}$ on the monomials of $\mathbb{F}[X_1, \ldots, X_m]$ as $\mathbf{X}^{\boldsymbol{\alpha}} <_{\boldsymbol{w}} \mathbf{X}^{\boldsymbol{\beta}}$ if and only if

$$\begin{array}{ll} (1) & \boldsymbol{w}(\mathbf{X}^{\boldsymbol{\alpha}}) <_{\mathbb{N}_0^r} \boldsymbol{w}(\mathbf{X}^{\boldsymbol{\beta}}) \quad \text{or} \\ (2) & \boldsymbol{w}(\mathbf{X}^{\boldsymbol{\alpha}}) = \boldsymbol{w}(\mathbf{X}^{\boldsymbol{\beta}}) \text{ and } \mathbf{X}^{\boldsymbol{\alpha}} <_{\mathcal{M}} \mathbf{X}^{\boldsymbol{\beta}} \end{array}$$

Let $\Delta(I)$ be the footprint of I, i.e. the monomials that are not the leading monomial of any polynomial in I with respect to $<_{\boldsymbol{w}}$. The following result [5, Theorem 1] characterizes the order domains and gives a construction of its weight function.

Theorem 2.2. *Let $R = \mathbb{F}[X_1, \ldots, X_m]/I$, let $\mathcal{G}$ be a Gröbner basis of the ideal I with respect to $<_{\boldsymbol{w}}$. Suppose that the elements of its footprint $\Delta(I)$ have mutually distinct weights, and that every element of $\mathcal{G}$ has exactly two monomials of highest weight in its support. We consider the following finitely generated semigroup $\Gamma = \langle \boldsymbol{w}(\boldsymbol{X}^{\boldsymbol{\alpha}}) \mid \boldsymbol{X}^{\boldsymbol{\alpha}} \in \Delta(I) \rangle$.*

Let F be the remainder of $f \in R$ after division by $\mathcal{G}$. Then R is an order domain with weight function ρ defined by:

$$\begin{array}{rccl} \rho: & R & \rightarrow & \Gamma_{-\infty} \\ & f & \mapsto & \max_{<_{\mathbb{N}_0^r}} \{\boldsymbol{w}(\boldsymbol{X}^{\boldsymbol{\alpha}}) \mid \boldsymbol{X}^{\boldsymbol{\alpha}} \in \mathrm{Supp}(F)\} \text{ for } f \neq 0 \\ & 0 & \mapsto & -\infty \end{array}$$

Moreover: if R is an order domain with a weight function with finitely generated value semigroup, then R can be described by the above construction.

A subset $\mathcal{B} \subset R$ of an order domain with weight function ρ is called an **order basis** if ρ restricted to $\mathcal{B}$ is a bijection. The set $\mathcal{B}_\rho = \{f_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Gamma\}$ totally ordered by $<_{\mathbb{N}_0^r}$ is called a **well-behaving basis for** R. The order basis is a basis for R as a vector space over $\mathbb{F}$.

We define the codes from order domains, they are nothing other than evaluation codes as in [8] except that the weight function ρ now takes values at $\mathbb{N}_0^r$.

Definition 2.3. We consider an order domain $R = \mathbb{F}[X_1, \ldots, X_m]/I$ with weight function $\rho : R \to \Gamma_{-\infty}$, where Γ is ordered by $<_{\mathbb{N}_0^r}$. Assume that a surjective morphism of $\mathbb{F}$-algebras $\varphi : R \to \mathbb{F}^n$ is given. Let $\{f_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Gamma\}$ be a well-behaving basis for R. For $\boldsymbol{\lambda} \in \Gamma$ the **evaluation code** $E_{\boldsymbol{\lambda}}$ is defined

$$E_{\boldsymbol{\lambda}} = \langle \varphi(f_{\boldsymbol{\lambda}'}) \mid \boldsymbol{\lambda}' \leq_{\mathbb{N}_0^r} \boldsymbol{\lambda} \rangle = \langle \varphi(f) \mid f \in R,\ \rho(f) \leq_{\mathbb{N}_0^r} \boldsymbol{\lambda} \rangle$$

and **its dual** denoted by $C_{\boldsymbol{\lambda}}$ is

$$C_{\boldsymbol{\lambda}} = \{\boldsymbol{c} \in \mathbb{F}^n \mid \boldsymbol{c} \cdot \varphi(f_{\boldsymbol{\lambda}'}) = 0,\ \forall\ \boldsymbol{\lambda}' \leq_{\mathbb{N}_0^r} \boldsymbol{\lambda}\}$$

Where $\cdot$ is the inner product. The definition does not depend on the choice of the basis.

Also, one can extend the definition of the Feng-Rao distance for evaluation codes, also called order bounds. We redefine them for semigroups in $\mathbb{N}_0^r$.

Definition 2.4. Given $\boldsymbol{\lambda} \in \Gamma$ we define:

$$N_{\boldsymbol{\lambda}} = \{(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \in \Gamma^2 \mid \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}\}, \quad \mu_{\boldsymbol{\lambda}} = \# N_{\boldsymbol{\lambda}}$$

and the **Feng-Rao distances**

$$d(\boldsymbol{\lambda}) = \min\{\mu_{\boldsymbol{\eta}} \mid \boldsymbol{\lambda} <_{\mathbb{N}_0^r} \boldsymbol{\eta}\}$$

$$d_\varphi(\boldsymbol{\lambda}) = \min\{\mu_{\boldsymbol{\eta}} \mid \boldsymbol{\lambda} <_{\mathbb{N}_0^r} \boldsymbol{\eta},\ C_{\boldsymbol{\eta}'} \neq C_{\boldsymbol{\eta}}\ \forall\ \boldsymbol{\eta}' <_{\mathbb{N}_0^r} \boldsymbol{\eta}\}$$

The Feng-Rao distances are lower bounds for the minimal distance, namely

$$d(C_{\boldsymbol{\lambda}}) \geq d_\varphi(\boldsymbol{\lambda}) \geq d(\boldsymbol{\lambda})$$

The bound $d(\boldsymbol{\lambda})$ depends only on Γ. However $d_\varphi(\boldsymbol{\lambda})$ also depends on the morphism φ. A strategy to obtain codes with good parameters may be to look for semigroups Γ with $d(\boldsymbol{\lambda})$ as large as possible and then look for an order domain that has Γ as its semigroups of values or modify the morphism φ in order to obtain families of codes with $d_\varphi(\boldsymbol{\lambda})$ as large as possible. That will probable allow us to correct more errors.

3 Feng-Rao distances

We first introduce the Apéry set of a semigroup which is our main tool in this work. Then we compute $\mu_{\boldsymbol{\lambda}}$ and $d(\boldsymbol{\lambda})$ for a simplicial semigroup.

For the definitions and results of semigroup theory we refer to [3, 13] (and their references). All the results of semigroup theory in this section are valid for a cancellative finitely generated commutative semigroup Γ that has a zero element and $\Gamma \cap (-\Gamma) = (0)$, in particular for a value semigroup $\Gamma \subset \mathbb{N}_0^r$ of a weight function. Now we introduce the Apéry set which was introduced in [1] in order to study numerical semigroups of curve singularities.

Let $\Gamma = \langle \Lambda \rangle = \langle \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n \rangle$ be a finitely generated semigroup, take a fixed partition $\Lambda = E \cup A$, $E \cap A = \emptyset$, of the set of generators Λ. The **Apéry set** Q **of** Γ **relative to** E is defined to be

$$Q = \{\boldsymbol{q} \in \Gamma \mid \boldsymbol{q} - \boldsymbol{e} \notin \Gamma,\ \forall\ \boldsymbol{e} \in E\}$$

One can write $\boldsymbol{\lambda} \in \Gamma$ as the sum of an element of Q and a linear combination of the elements of E. Therefore $\Gamma = \langle E \cup Q \rangle$.

Let $\mathbb{F}[\Gamma]$ be the semigroup $\mathbb{F}$-algebra, $\mathbb{F}[\Gamma] = \bigoplus_{\boldsymbol{\lambda} \in \Gamma} \mathbb{F}\chi^{\boldsymbol{\lambda}}$ (where $\chi^{\boldsymbol{\lambda}} \chi^{\boldsymbol{\lambda}'} = \chi^{\boldsymbol{\lambda} + \boldsymbol{\lambda}'}$). The **ideal of** Γ **relative to** Λ, denoted by J, is $\ker(\varphi_0)$, where φ_0 is the $\mathbb{F}$-algebra morphism

$$\varphi_0 : \mathbb{F}[\mathbf{X}] \to \mathbb{F}[\Gamma]$$

defined by $\varphi_0(X_i) = \chi^{\boldsymbol{\lambda}_i}$. φ_0 is surjective, and hence $\mathbb{F}[\Gamma] \simeq \mathbb{F}[\mathbf{X}]/\ker(\varphi_0)$. Consider $\mathbb{F}[\Gamma]$ with the natural Γ-grading and $\mathbb{F}[\mathbf{X}]$ as a Γ-graded ring, assigning degree $\boldsymbol{\lambda}_i$ to X_i. J is graded because φ_0 is a Γ-graded morphism of degree zero.

Let C_Γ be the cone generated by Γ, then C_Γ is strongly convex. The semigroup Γ is called **simplicial** if the the number of extremal rays of C_Γ is equals to the dimension of $\mathbb{F}[\Gamma]$. In particular any semigroup of dimension 1 and 2 is simplicial.

Let $\Gamma = \langle E, A\rangle \subset \mathbb{N}_0^r$, where $E = \{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_f\}$ is the set of the generators of the extremal rays and $A = \{\boldsymbol{\lambda}_{f+1}, \dots, \boldsymbol{\lambda}_n\}$ (if $f = r$ then Γ is simplicial). Let $\mathbb{F}[\mathbf{X}_E]$ be the polynomial ring in the r indeterminates associated with E and let $\mathbb{F}[\mathbf{X}_A]$ be the polynomial ring in the $n - f$ indeterminates associated with A. Consider the local ordering *lex-inf* (`ls` in SINGULAR) in $\mathbb{F}[\mathbf{X}] = \mathbb{F}[\mathbf{X}_E, \mathbf{X}_A]$

$$\mathbf{X}^{\boldsymbol{\alpha}} >_{lex\text{ - }inf} \mathbf{X}^{\boldsymbol{\beta}} \Longleftrightarrow \mathbf{X}^{\boldsymbol{\alpha}} <_{lex} \mathbf{X}^{\boldsymbol{\beta}}$$

where *lex* is the lexicographic order with $X_1 > \cdots > X_n$.

This ordering is not a monomial ordering. However, since there exists only a finite number of monomials of Γ-degree $\boldsymbol{\lambda} \in \Gamma$, a Gröbner basis of J can be computed from any Γ-graded generating set of J. The time complexity of computing a Gröbner basis is exponential in terms of the number of variables and this makes it impossible to achieve some computations involving a Gröbner basis. But J is a toric ideal [11], and one has that any reduced Gröbner basis of a toric ideal can be computed in polynomial time [11, Theorem 12.24].

Assume that $\mathcal{G}$ is the reduced Gröbner basis of J for *lex-inf*. Let Δ_A be the set of monomials $\mathbf{X}_A^{\boldsymbol{\alpha}}$ which are not divisible by any leading monomial of $\mathcal{G}$, i.e., the footprint restricted to $\mathbb{F}[\mathbf{X}_A]$. The Apéry set of a semigroup $\Gamma = \langle E, A\rangle$ with respect to E is computed in [13] as

$$Q = \{\boldsymbol{\lambda} \in \Gamma \mid \boldsymbol{\lambda} = \sum_{i=f+1}^{n} \alpha_i \boldsymbol{\lambda}_i, \text{ where } \mathbf{X}_A^{\boldsymbol{\alpha}} \in \Delta_A\}$$

and in particular Q is finite.

For a simplicial semigroup $\Gamma = \langle E, A\rangle$, where E is the set of the generators of the extremal rays, one has that the elements of E must be $\mathbb{Q}$-linearly independent. Therefore if $E = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_r\}$ we assume in the following without loss of generality that $\boldsymbol{e}_i$ is in the i-th axis, for $i \in \{1, \dots, r\}$, and we say in this case that E is aligned with the coordinate axes. That is, if pr_i is the i-th projection then $\mathrm{pr}_i(\boldsymbol{e}_i) \neq 0$ and $\mathrm{pr}_j(\boldsymbol{e}_i) = 0,\ \forall\ j \neq i$.

For a semigroup where the generators of the extremal rays are not aligned with the coordinate axes one can consider a $\mathbb{Q}$-linear transformation, $\phi : \mathbb{Q}^r \to \mathbb{Q}^r$, such that $\phi(\boldsymbol{\lambda}_i)$ is in the i-th axis, and the associated matrix has nonzero determinant (because they are linearly independent). Finally we multiply this transformation by the least common multiple of the denominators of $\{\phi(\boldsymbol{\lambda}_i) \mid i \in \{1, \dots, r\}\}$ (in order to have all the coordinates in $\mathbb{Z}$).

3.1 Computing $\mu_{\boldsymbol{\lambda}}$

Let $\Gamma \subset \mathbb{N}_0^r$ be a finitely generated semigroup of values of a weight function. Let Q be the Apéry set of Γ relative to the generators of the extremal rays. Then $\Gamma = \langle E, Q \rangle$ where $E = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r\}$ is the set of generators of the extremal rays. We assume that E is aligned with the coordinate axes.

Let $\boldsymbol{\eta} \in \Gamma$, we denote by $\overline{\boldsymbol{\eta}}$ the class of $\boldsymbol{\eta}$ in

$$M = \frac{\mathbb{Z}}{(\mathrm{pr}_1(\boldsymbol{e}_1))} \bigoplus \cdots \bigoplus \frac{\mathbb{Z}}{(\mathrm{pr}_r(\boldsymbol{e}_r))}$$

Let Q be the Apéry set of Γ relative to E. If $\boldsymbol{\lambda} \in \Gamma$, there exists $\boldsymbol{q} \in Q$ (not unique) such that $\overline{\boldsymbol{\lambda}} = \overline{\boldsymbol{q}}$, and therefore one can write $\boldsymbol{\lambda}$ as the sum of $\boldsymbol{q}$ and a linear combination of the elements of E. One has Γ as a (in general) non-disjoint union

$$\Gamma = \bigcup_{\boldsymbol{q} \in Q} \{\boldsymbol{q} + \mathbb{N}_0 \boldsymbol{e}_1 + \cdots + \mathbb{N}_0 \boldsymbol{e}_r\}$$

We order the elements of the Apéry set with the same class by the lexicographical ordering. We write

$$Q = \bigcup_{\boldsymbol{i} \in I} \{\boldsymbol{q}_{\boldsymbol{i}}^1, \ldots, \boldsymbol{q}_{\boldsymbol{i}}^{t_i}\}$$

where $I = \{\overline{\boldsymbol{q}} \mid \boldsymbol{q} \in Q\}$ and $\boldsymbol{q}_{\boldsymbol{i}}^1 <_{lex} \cdots <_{lex} \boldsymbol{q}_{\boldsymbol{i}}^{t_i}$.

Let $\boldsymbol{\lambda} \in \Gamma$ then there is a unique $\boldsymbol{q}_{\boldsymbol{d}}^c \in Q$ called the **Apéry element of $\boldsymbol{\lambda}$** and a unique $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r) \in \mathbb{N}_0^r$ such that

$$\boldsymbol{\lambda} = \boldsymbol{q}_{\boldsymbol{d}}^c + \sum_{k=1}^r \beta_k \boldsymbol{e}_k$$

with $\overline{\boldsymbol{\lambda}} = \overline{\boldsymbol{q}_{\boldsymbol{d}}^c}$ and $\nexists \boldsymbol{\beta}' \in \mathbb{N}_0^r$ such that $\boldsymbol{\lambda} = \boldsymbol{q}_{\boldsymbol{d}}^{c'} + \sum \beta_k' \boldsymbol{e}_k \;\; \forall\, c' < c$.

Remark 3.1. If the simplicial semigroup Γ is Cohen-Macaulay then each $\boldsymbol{q} \in Q$ has a different class in M and it is not necessary to order the elements of Q. Therefore the computations for Cohen-Macaulay semigroups are much simpler.

One has that $\mu_{\boldsymbol{\lambda}} = \# N_{\boldsymbol{\lambda}}$ where $N_{\boldsymbol{\lambda}} = \{(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \in \Gamma^2 \mid \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}\}$. Let $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \in N_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} = \boldsymbol{q}_{\boldsymbol{d}}^c + \sum \beta_k \boldsymbol{e}_k$, $\boldsymbol{\lambda}_1 = \boldsymbol{q}_{\boldsymbol{i}}^p + \sum \gamma_k \boldsymbol{e}_k$ then $\overline{\boldsymbol{\lambda}_2} = \overline{\boldsymbol{d} - \boldsymbol{i}}$. Set $\boldsymbol{j} = \overline{\boldsymbol{d} - \boldsymbol{i}}$ and let $\boldsymbol{\lambda}_2 = \boldsymbol{q}_{\boldsymbol{j}}^l + \sum \delta_k \boldsymbol{e}_k$. We consider

$$N_{\boldsymbol{\lambda}}^{\boldsymbol{i},p,l} = \{(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \in \Gamma^2 \mid \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}, \boldsymbol{\lambda}_1 = \boldsymbol{q}_{\boldsymbol{i}}^p + \sum \gamma_k \boldsymbol{e}_k, \boldsymbol{\lambda}_2 = \boldsymbol{q}_{\boldsymbol{j}}^l + \sum \delta_k \boldsymbol{e}_k\}$$

for $\boldsymbol{i} \in I$, $p \in \{1, \ldots, t_i\}$, $l \in \{1, \ldots, t_j\}$. Therefore

$$\mu_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{i} \in I, p=1,\ldots,t_i, l=1\ldots,t_j} \# N_{\boldsymbol{\lambda}}^{\boldsymbol{i},p,l}$$

One has that $\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 = \boldsymbol{\lambda} \Leftrightarrow \sum(\gamma_k + \delta_k)\boldsymbol{e}_k = \boldsymbol{\lambda} - \boldsymbol{q}_{\boldsymbol{i}}^p - \boldsymbol{q}_{\boldsymbol{j}}^l$ so

$$\gamma_k + \delta_k = \mathrm{pr}_k(\boldsymbol{\lambda} - \boldsymbol{q}_{\boldsymbol{i}}^p - \boldsymbol{q}_{\boldsymbol{j}}^l)/\mathrm{pr}_k(\boldsymbol{e}_k) \quad \text{for all } k$$

let $B_k = \mathrm{pr}_k(\boldsymbol{\lambda} - \boldsymbol{q}_{\boldsymbol{i}}^p - \boldsymbol{q}_{\boldsymbol{j}}^l)/\mathrm{pr}_k(\boldsymbol{e}_k)$

We are interested in the case that $B_k \geq 0\ \forall k$ (i.e.$\boldsymbol{B} = (B_1, \ldots, B_r) \geq_{NAT} 0$) because in the other case the equality $\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}$ is not possible. If $p = l = 1$ there are $\prod_{k=1}^r (B_k + 1)$ possible values for $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ and therefore $\# N_{\boldsymbol{\lambda}}^{i,1,1} = \prod_{k=1}^r (B_k + 1)$. But if $k \neq 1$ or $l \neq 1$ there are fewer pairs because some of them should be written using another Apéry element and if we do not discard them they would be counted several times.

$\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ have Apéry element $\boldsymbol{q}_{\boldsymbol{i}}^p$ and $\boldsymbol{q}_{\boldsymbol{j}}^l$ if and only if

$$\begin{cases} \boldsymbol{q}_{\boldsymbol{i}}^p + \sum \gamma_k \boldsymbol{e}_k \not\geq_{NAT} \boldsymbol{q}_{\boldsymbol{i}}^g, & g = 1, \ldots p-1 \\ \boldsymbol{q}_{\boldsymbol{j}}^l + \sum \delta_k \boldsymbol{e}_k \not\geq_{NAT} \boldsymbol{q}_{\boldsymbol{j}}^h, & h = 1, \ldots l-1 \end{cases}$$

Since $\overline{\boldsymbol{q}_{\boldsymbol{i}}^g - \boldsymbol{q}_{\boldsymbol{i}}^p} = 0$ and $\overline{\boldsymbol{q}_{\boldsymbol{j}}^h - \boldsymbol{q}_{\boldsymbol{j}}^l} = 0$ one has that there exist $\boldsymbol{r}^g, \boldsymbol{s}^h \in \mathbb{Z}^r$ such that $\boldsymbol{q}_{\boldsymbol{i}}^g - \boldsymbol{q}_{\boldsymbol{i}}^p = \sum r_k^g \boldsymbol{e}_k$ for $g = 1, \ldots, p-1$ and $\boldsymbol{q}_{\boldsymbol{j}}^h - \boldsymbol{q}_{\boldsymbol{j}}^l = \sum s_k^h \boldsymbol{e}_k$ for $h = 1, \ldots, l-1$. We are only interested in the case that $\boldsymbol{r}^g, \boldsymbol{s}^h \subset \mathbb{N}_0^r$ because in the other case the equality $\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 = \boldsymbol{\lambda}$ is not possible.

Therefore one has

$$\begin{cases} \boldsymbol{\gamma} \not\geq_{NAT} \boldsymbol{r}^g, & g = 1, \ldots, p-1 \\ \boldsymbol{\delta} \not\geq_{NAT} \boldsymbol{s}^h, & h = 1, \ldots, l-1 \end{cases}$$

And since $\gamma_k + \delta_k = B_k\ \forall\ k$ one has equivalently

$$\begin{cases} \boldsymbol{\gamma} \not\geq_{NAT} \boldsymbol{r}^g, & g = 1, \ldots, p-1 \\ \boldsymbol{\gamma} \not\leq_{NAT} \boldsymbol{B} - \boldsymbol{s}^h, & h = 1, \ldots, l-1 \end{cases}$$

For all possible pairs, $X = \{\boldsymbol{\gamma} \in \mathbb{N}_0^r \mid \boldsymbol{\gamma} \leq_{NAT} \boldsymbol{B}\}$, one has to discard the pairs with Apéry element $\boldsymbol{q}_{\boldsymbol{i}}^g$, $Y_g = \{\boldsymbol{\gamma} \in X \mid \boldsymbol{\gamma} \geq_{NAT} \boldsymbol{r}^g\}$ $g = 1, \ldots, p-1$, and with Apéry element $\boldsymbol{q}_{\boldsymbol{j}}^h$, $Y'_h = \{\boldsymbol{\gamma} \in X \mid \boldsymbol{\gamma} \leq_{NAT} \boldsymbol{B} - \boldsymbol{s}^h\}$ $h = 1, \ldots, l-1$. Therefore

$$N_{\boldsymbol{\lambda}}^{\boldsymbol{i},p,l} = X \setminus \left((\bigcup_{g=1}^{l-1} Y_g) \bigcup (\bigcup_{h=1}^{l-1} Y'_h) \right)$$

We have proved the following result

Theorem 3.2. *Let $\Gamma \subset \mathbb{N}_0^r$ be a simplicial value semigroup with Apéry set $Q = \cup_{\boldsymbol{i} \in I}\{\boldsymbol{q}_{\boldsymbol{i}}^1, \ldots, \boldsymbol{q}_{\boldsymbol{i}}^{t_i}\}$ relative to the generators of the extremal rays, with notations as above. Let $\boldsymbol{\lambda} = \boldsymbol{q}_{\boldsymbol{d}}^c + \sum \beta_k \boldsymbol{e}_k$. Then*

$$\mu_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{i} \in I, p=1,\ldots,t_i, l=1,\ldots,t_j} \# N_{\boldsymbol{\lambda}}^{\boldsymbol{i},p,l} = \# \left(X \setminus \left((\bigcup_{g=1}^{l-1} Y_g) \bigcup (\bigcup_{h=1}^{l-1} Y'_h) \right) \right)$$

where $\overline{\boldsymbol{j}} = \overline{\boldsymbol{d} - \boldsymbol{i}}$.

One can compute $\mu_{\boldsymbol{\lambda}}$ using the previous result and the inclusion-exclusion principle. To achieve this computation one only need to know the value of the following cardinals

$$\begin{cases} \#(X) = \prod_{k=1}^{r}(B_k+1) \\ \#(Y_g) = \prod_{k=1}^{r}(B_k+1-\max\{0, r_k^g\}) \\ \#(Y'_h) = \prod_{k=1}^{r}(B_k+1-\max\{0, s_k^l\}) \\ \#(Y_{g_1}\cap\cdots\cap Y_{g_e}\cap Y'_{h_1}\cap\cdots\cap Y'_{h_f}) = \\ \prod_{k=1}^{r}(B_k+1-\max\{0, r_k^{g_1},\ldots,r_k^{g_e}\}-\max\{0, s_k^{h_1},\ldots,s_k^{h_f}\}) \end{cases}$$

Remark 3.3. We have computed $\mu_{\boldsymbol{\lambda}}$ using the Apéry set. This computation will be used to compute $d(\boldsymbol{\lambda})$ but it can be used also to improve the computation of $d_\varphi(\boldsymbol{\lambda})$.

Let $R = \mathbb{F}[X_1,\ldots,X_m]/I$ be an order domain with weight function ρ. The surjective morphism φ considered in [5] is what is known as the **evaluation map** and it is just the evaluation at the points of the variety defined by the ideal I (over $\mathbb{F}$, where $\mathbb{F}$ is the finite field with q elements).

Let I' be the radical ideal $I' = I + \langle X_1^q - X_1,\ldots,X_m^q - X_m\rangle$. One has that

$$d_\varphi(\boldsymbol{\lambda}) = \min\{\mu_{\boldsymbol{\lambda}'} \mid \boldsymbol{\lambda} <_{\mathbb{N}_0^r} \boldsymbol{\lambda}',\ \boldsymbol{\lambda}' \in \{\rho(M+I) \mid M \in \Delta(I')\}\}$$

In [5] $\mu_{\boldsymbol{\lambda}}$ is computed directly from the definition, but one can improve that computation considering the computation of $\mu_{\boldsymbol{\lambda}}$ using the Apéry set as in this section if the value semigroup is simplicial.

3.2 Computing $d(\boldsymbol{\lambda})$

For a simplicial semigroup Γ we compute $d(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \Gamma$. The following theorem allows us to do that in a finite number of steps.

Theorem 3.4. *Let $\Gamma \subset \mathbb{N}_0^r$ be a finitely generated simplicial semigroup, let Q be its Apéry set with respect to the extremal rays. Let $\overline{\boldsymbol{\lambda}}$ be the class of $\boldsymbol{\lambda} \in \Gamma$ in M as before. Let $\boldsymbol{\lambda}, \boldsymbol{\eta} \in \Gamma$ such that $\boldsymbol{\lambda} <_{NAT} \boldsymbol{\eta}$ and $\overline{\boldsymbol{\lambda}} = \overline{\boldsymbol{\eta}}$ then we have $\mu_{\boldsymbol{\lambda}} < \mu_{\boldsymbol{\eta}}$.*

Proof.

Let $\boldsymbol{\lambda} = \boldsymbol{q} + \sum \beta_k \boldsymbol{e}_k$ and $\boldsymbol{\eta} = \boldsymbol{q} + \sum \gamma_k \boldsymbol{e}_k$. Since $\boldsymbol{\lambda} <_{NAT} \boldsymbol{\eta}$ one has that $\beta_k \leq \gamma_k$ for all k and there exists $j \in \{1,\ldots,r\}$ such that $\beta_j < \gamma_j$. Let $\boldsymbol{\lambda}' = \boldsymbol{\lambda} + \boldsymbol{e}_j$.

We consider $N_{\boldsymbol{\lambda}} = \{(\boldsymbol{a}_1, \boldsymbol{b}_1),\ldots,(\boldsymbol{a}_{\mu_{\boldsymbol{\lambda}}}, \boldsymbol{b}_{\mu_{\boldsymbol{\lambda}}})\}$ then $(\boldsymbol{a}_i + \boldsymbol{e}_j, \boldsymbol{b}_i) \in N_{\boldsymbol{\lambda}'}$ for $i = 1,\ldots,\mu_{\boldsymbol{\lambda}}$ and they are different two by two. Moreover $(0, \boldsymbol{\lambda}') \in N_{\boldsymbol{\lambda}'}$ and it is different from the previous pairs. Therefore $\#N_{\boldsymbol{\lambda}'} \geq \#N_{\boldsymbol{\lambda}} + 1$ and consequently $\mu_{\boldsymbol{\lambda}} < \mu_{\boldsymbol{\lambda}'}$.

Proceeding by induction we obtain $\mu_{\boldsymbol{\lambda}} < \mu_{\boldsymbol{\eta}}$. □

Let Γ be a value semigroup with Apéry set Q. Since

$$d(\boldsymbol{\lambda}) = \min\{\mu_{\boldsymbol{\eta}} \mid \boldsymbol{\lambda} <_{\mathbb{N}_0^r} \boldsymbol{\eta}\} = \min_{\boldsymbol{q}\in Q}\{\mu_{\boldsymbol{\eta}} \mid \boldsymbol{\lambda} <_{\mathbb{N}_0^r} \boldsymbol{\eta}, \boldsymbol{\eta} = \boldsymbol{q} + \sum \beta_i \boldsymbol{e}_i\}$$

and Q is a finite set, the computation of $d(\boldsymbol{\lambda})$ can be achieved if one computes the minimum in each subsemigroup of the form $\Gamma_{\boldsymbol{q}} = \boldsymbol{q} + \mathbb{N}_0\boldsymbol{e}_1 + \cdots + \mathbb{N}_0\boldsymbol{e}_r$, for $\boldsymbol{q} \in Q$.

If $<_{NAT}$ was a total order one would have to check only the lowest element (with respect to $<_{NAT}$) that is greater than $\boldsymbol{\lambda}$ (with respect to $<_{\mathbb{N}_0^r}$) in each $\Gamma_{\boldsymbol{q}}$ with $\boldsymbol{q} \in Q$. But $<_{NAT}$ is a total order only for $\Gamma \subset \mathbb{N}_0$.

Since $\leq_{NAT}$ is a partial order for $\Gamma \in \mathbb{N}_0^r$ we have to consider more values, but a finite number of values. We define $T_{\boldsymbol{q}} \subset \Gamma_{\boldsymbol{q}}$ finite such that for all $\boldsymbol{\eta} >_{\mathbb{N}_0^r} \boldsymbol{\lambda}$ there exists $\boldsymbol{\eta}' \in T_{\boldsymbol{q}}$ such that $\mu_{\boldsymbol{\eta}} \geq \mu_{\boldsymbol{\eta}'}$. Therefore

$$d(\boldsymbol{\lambda}) = \min_{\boldsymbol{q}\in Q}\{\mu_{\boldsymbol{\eta}} \mid \boldsymbol{\eta} \in T_{\boldsymbol{q}}\}$$

Now we define $T_{\boldsymbol{q}}$ for the different monomial orderings ($1 < X_k \ \forall k$). We give one definition for a degree or weighted degree ordering and one for the lexicographical ordering.

$T_{\boldsymbol{q}}$ for a degree or weighted degree ordering: That is `dp`, `Dp`, `wp` or `Wp` in Singular. Let $\beta_k \in \mathbb{N}_0$ such that $\boldsymbol{\eta}_k = \boldsymbol{q} + \beta_k\boldsymbol{e}_k >_{\mathbb{N}_0^r} \boldsymbol{\lambda}$ and $\boldsymbol{q} + (\beta_k - 1)\boldsymbol{e}_k \leq_{\mathbb{N}_0^r} \boldsymbol{\lambda}$ (if $\beta_k > 0$), for $k = 1, \ldots, r$. One has that β_i exists because $<_{\mathbb{N}_0^r}$ is a degree ordering.

Let $T_{\boldsymbol{q}} = \{\boldsymbol{q} + \sum \gamma_k\boldsymbol{e}_k >_{\mathbb{N}_0^r} \boldsymbol{\lambda} \mid \boldsymbol{\gamma} \leq_{NAT} \boldsymbol{\beta}\}$. $T_{\boldsymbol{q}}$ is a finite set. Furthermore one can discard all the elements of $T_{\boldsymbol{q}}$ such that there exist other elements of $T_{\boldsymbol{q}}$ smaller with respect to $\leq_{NAT}$.

One has that $d(\boldsymbol{\lambda}) = \min_{\boldsymbol{q}\in Q}\{\mu_{\boldsymbol{\eta}} \mid \boldsymbol{\eta} \in T_{\boldsymbol{q}}\}$ because if $\boldsymbol{\eta} = \boldsymbol{q} + \sum \gamma_k\boldsymbol{e}_k$, $\boldsymbol{\eta} >_{\mathbb{N}_0^r} \boldsymbol{\lambda}$, $\boldsymbol{\eta} \notin T_{\boldsymbol{q}}$, there exists $\gamma_k > \beta_k$ and therefore $\boldsymbol{\eta} >_{NAT} \boldsymbol{\eta}_i = \boldsymbol{q} + \beta_i \in T_{\boldsymbol{q}}$.

$T_{\boldsymbol{q}}$ for a lexicographical ordering: That is `lp` in Singular. Let $<_{\mathbb{N}_0^r}$ be the lexicographical ordering with $X_1 > \cdots > X_r$.

Let $\beta_1 \in \mathbb{N}_0$ such that $\boldsymbol{\eta}_0 = \boldsymbol{q} + \beta_1\boldsymbol{e}_1 >_{\mathbb{N}_0^r} \boldsymbol{\lambda}$ and $\boldsymbol{q} + (\beta_1 - 1)\boldsymbol{e}_1 \leq_{\mathbb{N}_0^r} \boldsymbol{\lambda}$ (if $\beta_1 > 0$). If $\mathrm{pr}_1(\boldsymbol{q} + (\beta_1 - 1)\boldsymbol{e}_1) = \mathrm{pr}_1(\boldsymbol{\lambda})$ then let $\beta_2 \in \mathbb{N}_0$ such that $\boldsymbol{\eta}_1 = q + (\beta_1 - 1)\boldsymbol{e}_1 + \beta_2\boldsymbol{e}_2 >_{\mathbb{N}_0^r} \boldsymbol{\lambda}$ and $\boldsymbol{q} + (\beta_1 - 1)\boldsymbol{e}_1 + (\beta_2 - 1)\boldsymbol{e}_2 \leq_{\mathbb{N}_0^r} \boldsymbol{\lambda}$ (if $\beta_2 > 0$).

We define $\boldsymbol{\eta}_k$ in the same way for $k = 1, \ldots, j$ until $\beta_j \neq \mathrm{pr}_j(\boldsymbol{\lambda})$. We define $T_{\boldsymbol{q}} = \{\eta_0, \eta_1, \ldots, \eta_j\}$. Furthermore one can discard all the elements of $T_{\boldsymbol{q}}$ such that there exists other element of $T_{\boldsymbol{q}}$ smaller with respect to $\leq_{NAT}$.

One has that $d(\boldsymbol{\lambda}) = \min_{\boldsymbol{q}\in Q}\{\mu_{\boldsymbol{\eta}} \mid \boldsymbol{\eta} \in T_{\boldsymbol{q}}\}$ because let $\boldsymbol{\eta} = \boldsymbol{q} + \sum \gamma_k\boldsymbol{e}_k$, $\boldsymbol{\eta} >_{\mathbb{N}_0^r} \boldsymbol{\lambda}$, $\boldsymbol{\eta} \notin T_{\boldsymbol{q}}$. If $\mathrm{pr}_1(\boldsymbol{\eta}) \neq \mathrm{pr}_1(\boldsymbol{\lambda})$ then $\gamma_1 \geq \beta_1$ and therefore $\boldsymbol{\eta} \geq_{NAT} \boldsymbol{\eta}_0 \in T_{\boldsymbol{q}}$. In other case ($\mathrm{pr}_1(\boldsymbol{\eta}) = \mathrm{pr}_1(\boldsymbol{\lambda})$) there exists j_0 such that $\mathrm{pr}_k(\boldsymbol{\eta}) = \mathrm{pr}_k(\boldsymbol{\lambda})$ for $k = 1, \ldots j_0 - 1$ and $\mathrm{pr}_{j_0}(\boldsymbol{\eta}) \neq \mathrm{pr}_{j_0}(\boldsymbol{\lambda})$, or equivalently $\gamma_k = \beta_k - 1$, $k = 1, \ldots j_0 - 1$ and $\gamma_{j_0} \geq \beta_{j_0}$ and therefore $\boldsymbol{\eta} \geq_{NAT} \boldsymbol{\eta}_{j_0} \in T_{\boldsymbol{q}}$.

We have implemented the algorithms of this section in Singular: the computation of the Apéry set of a finitely generated semigroup; $\mu_{\boldsymbol{\lambda}}$ for a simplicial semigroup; and the Feng-Rao bound for a simplicial semigroup.

3.3 Example

Let $\Gamma = \langle E, A\rangle \subset \mathbb{N}_0^2$ generated by $E = \langle \boldsymbol{e}_1 = (2,0), \boldsymbol{e}_2 = (0,1)\rangle$ and $A = \langle (1,2),(3,1)\rangle$. The generators of the extremal rays are aligned with the coordinate axes. We consider Γ ordered by the lexicographical order $X_1 > X_2$.

The Apéry set of Γ relative to E is $Q = \{(0,0),(1,2),(3,1)\}$. Since $\overline{(1,2)} = \overline{(3,1)} = (1,0)$ we write $Q = \{\boldsymbol{q}^1_{(0,0)} = (0,0)\} \cup \{\boldsymbol{q}^1_{(1,0)} = (1,2), \boldsymbol{q}^2_{(1,0)} = (3,1)\}$.

We consider $\boldsymbol{\lambda} = (12,14) = (0,0) + 6\boldsymbol{e}_1 + 14\boldsymbol{e}_2$. We compute $\mu_{\boldsymbol{\lambda}}$. $\overline{\boldsymbol{\lambda}} = (0,0)$ therefore for $i = (0,0)$, $j = (0,0)$ and for $i = (1,0)$, $j = (1,0)$.

$$\mu_{\boldsymbol{\lambda}} = N^{(0,0),1,1} + N^{(1,0),1,1} + N^{(1,0),2,1} + N^{(1,0),1,2} + N^{(1,0),2,2}$$

$$\begin{cases} B_1 = \mathrm{pr}_1((12,14) - (0,0) - (0,0))/\mathrm{pr}_1(\boldsymbol{e}_1) = 6 \\ B_2 = \mathrm{pr}_2((12,14) - (0,0) - (0,0))/\mathrm{pr}_2(\boldsymbol{e}_2) = 14 \\ N^{(0,0),1,1} = (6+1)(14+1) = 105 \end{cases}$$

$$\begin{cases} B_1 = \mathrm{pr}_1((12,14) - (1,2) - (1,2))/\mathrm{pr}_1(\boldsymbol{e}_1) = 5 \\ B_2 = \mathrm{pr}_2((12,14) - (1,2) - (1,2))/\mathrm{pr}_2(\boldsymbol{e}_2) = 10 \\ N^{(1,0),1,1} = (5+1)(10+1) = 66 \end{cases}$$

$$\begin{cases} B_1 = \mathrm{pr}_1((12,14) - (3,1) - (1,2))/\mathrm{pr}_1(\boldsymbol{e}_1) = 4 \\ B_2 = \mathrm{pr}_2((12,14) - (3,1) - (1,2))/\mathrm{pr}_2(\boldsymbol{e}_2) = 11 \\ \#X = (4+1)(11+1) = 60 \\ \#Y_1 = (4+1)(11+1-1) = 55 \ \text{ because } \boldsymbol{r}^1 = -\boldsymbol{e}_1 + \boldsymbol{e}_2 \\ N^{(1,0),1,2} = N^{(1,0),2,1} = 60 - 55 = 5 \end{cases}$$

$$\begin{cases} B_1 = \mathrm{pr}_1((12,14) - (3,1) - (3,1))/\mathrm{pr}_1(\boldsymbol{e}_1) = 3 \\ B_2 = \mathrm{pr}_2((12,14) - (3,1) - (3,1))/\mathrm{pr}_2(\boldsymbol{e}_2) = 12 \\ \#X = (3+1)(12+1) = 52 \\ \#Y_1 = (3+1)(12+1-1) = 48 \ \text{ because } \boldsymbol{r}^1 = -\boldsymbol{e}_1 + \boldsymbol{e}_2 \\ \#Y_1' = (3+1)(12+1-1) = 48 \ \text{ because } \boldsymbol{s}^1 = -\boldsymbol{e}_1 + \boldsymbol{e}_2 \\ \#Y_1 \cap Y_1' = (3+1)(12+1-1-1) = 44 \\ N^{(1,0),2,2} = 52 - 48 - 48 + 44 = 0 \end{cases}$$

Therefore $\mu_{(12,14)} = 105 + 66 + 5 + 5 + 0 = 181$.

Now we compute $d(\boldsymbol{\lambda})$. For each $\boldsymbol{q} \in Q$ we compute $T_{\boldsymbol{q}}$.

$T_{(0,0)}$: $(0,0) + 7\boldsymbol{e}_1 > (12,14)$ and $(0,0) + 6\boldsymbol{e}_1 \leq (12,14)$, therefore $\boldsymbol{\eta}_0 = (14,0)$. Since $\mathrm{pr}_1((0,0) + 6\boldsymbol{e}_1) = \mathrm{pr}_1(\boldsymbol{\lambda}) = 12$ we consider $\boldsymbol{\eta}_1 = (0,0) + 6\boldsymbol{e}_1 + 15\boldsymbol{e}_2 = (12,15)$ because $(0,0) + 6\boldsymbol{e}_1 + 14\boldsymbol{e}_2 \leq (12,14)$. Then $T_{(0,0)} = \{(14,0),(12,15)\}$.

$T_{(1,2)}$: $(1,2) + 6\boldsymbol{e}_1 > (12,14)$ and $(1,2) + 5\boldsymbol{e}_1 \leq (12,14)$, therefore $\boldsymbol{\eta}_0 = (13,2)$. Since $\mathrm{pr}_1((1,2) + 5\boldsymbol{e}_1) \neq \mathrm{pr}_1(\boldsymbol{\lambda}) = 12$ one has $T_{(1,2)} = \{(13,2)\}$.

$T_{(3,1)}$: $(3,1) + 5\boldsymbol{e}_1 > (12,14)$ and $(3,1) + 4\boldsymbol{e}_1 \leq (12,14)$, therefore $\boldsymbol{\eta}_0 = (13,1)$. Since $\mathrm{pr}_1((3,1) + 4\boldsymbol{e}_1) \neq \mathrm{pr}_1(\boldsymbol{\lambda}) = 12$ one has $T_{(3,1)} = \{(13,1)\}$.

One has that $\mu_{(14,0)} = 8$, $\mu_{(12,15)} = 194$, $\mu_{(13,2)} = 26$ (that can be discarded), $\mu_{(13,1)} = 12$. Therefore

$$d((12,14)) = \min_{\boldsymbol{q} \in Q}\{\mu_{\boldsymbol{\eta}} \mid \boldsymbol{\eta} \in T_{\boldsymbol{q}}\} = 8$$

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References

[1] R. Apéry, Sur les branches superlinéaires des courbes algébriques, C.R. Acad. Sci. Paris. 222 (1946) 1198–1200.

[2] A. Campillo, J. I. Farrán, Computing Weierstrass semigroups and the Feng-Rao distance from singular plane models, Finite Fields Appl. 6 (2000) 71–92.

[3] A. Campillo, P. Gimenez, Syzygies of affine toric varieties, J. Algebra 225 (2000) 142–161.

[4] D. Cox, J. Little, D. O'Shea, Ideals, varieties and algorithms: an introduction to Computational Algebraic Geometry and Commutative Algebra, UTM, Springer-Verlag, 1992.

[5] O. Geil, On the construction of codes from order domains, Tech. Rep. R-00-2013, Aalborg University (2000).

[6] O. Geil, R. Pellikaan, On the structure of order domains, Finite Fields Appl. 8 (2002) 369–396.

[7] G.-M. Greuel, G. Pfister, H. Schönemann, Singular 3.0, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, `http://www.singular.uni-kl.de` (2005).

[8] T. Høholdt, J. H. van Lint, R. Pellikaan, Algebraic Geometry Codes in "Handbook of Coding Theory", Vol. 1, Elsevier, 1998, Ch. 10.

[9] J. B. Little, A key equation and the computation of error values for codes from order domains, ArXiv:math.AC/0303299.

[10] R. Matsumoto, Miura's generalization of one-point ag codes is equivalent to Høholdt, van Lint and Pellikaan's generalization, IEICE Trans. Fundamentals E82-A, no. 10 (1999) 2007–2010.

[11] E. Miller, B. Sturmfels Combinatorial commutative algebra, Graduate Texts in Mathematics 227, Springer-Verlag (2005).

[12] M. E. O'Sullivan, A generalization of the Berlekamp-Massey-Sakata algorithm, Preprint.

[13] P. Pisón, The short resolution of a lattice ideal, Proc. Amer. Math. Soc. 131 (4) (2002) 1081–1091.