# Computer Algebra (2014)-Aalborg University
# Second set of exercises

**The deadline for this set of exercises is Friday January 23rd**. A (brief) reasoned explanation should follow the solution of the exercises. I would like to get (by email) an electronic file with your solutions and a printed copy in my mail box.

Solve the following exercises using a Computer Algebra System. You are welcome to use Sage commands unless the exercise asks us to trace an algorithm or to implement a command:

**Exercise 1** Trace the division algorithm for $y^2x$ divided by $\{yx - y, y^2 - x\} \subset \mathbb{Q}[x, y]$ with respect grevlex with $y > x$. Check your result using a command.

**Exercise 2**

1. Consider two different polynomials $f, g$ in $\mathbb{F}_4[x, y, z]$ such that their leading term is different for lex, deglex and grevlex. Show their leading terms with respect to the 3 monomial orders.
2. Compute the S-polynomial of $f, g$ using a command in Maple or Sage with respect to the 3 monomial orders defined in the course. Show the multidegree of the S-polynomial and the expected degree of the combination before cancellations.

**Exercise 3**

1. Compute a Gröbner basis (with Sage) of the following ideal with respect to the lexicographical ordering with $z > y > x$:

$$I = \langle x^5 + y^3 + z^2 - 1, x^2 + y^2 + z - 1, x^6 + y^5 + z^3 - 1\rangle \subset \mathbb{Q}[x, y, z]$$

2. Compute a Gröbner basis (with Sage) of

$$I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1\rangle \subset \mathbb{Q}[x, y, z]$$

   using lex and grevlex with $x > y > z$. Is there any difference?

3. Now we modify one single exponent: Compute a Gröbner basis (with Sage) of

$$I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1\rangle \subset \mathbb{Q}[x, y, z]$$

   using lex and grevlex with $x > y > z$. Is there any difference?

4. Compute a Gröbner basis (with Sage) of

$$I = \langle x^4 - yz^2w, xy^2 - z^3, x^3z - y^3w\rangle \subset \mathbb{Q}[x, y, z, w]$$

   using lex and grevlex with $x > y > z > w$. Is there any difference? (hint: grevlex is not always better than lex). Actually one can prove that $z^{n^2+1} - y^{n^2}w$ is in the reduced Gröbner basis of the ideal

$$\langle x^{n+1} - yz^{n-1}w, xy^{n-1} - z^n, x^nz - y^nw\rangle$$

   w.r.t. grevlex with $x > y > z > w$.)

**Exercise 4**

1. Let $I = \langle \{xy - x, -y + x^2\} \rangle \subset \mathbb{Q}[x, y]$ and consider the lex order with $x < y$. Show that $\{xy - x, -y + x^2\}$ is not a Gröbner basis with respect to the previous order.

2. Compute a Gröbner basis of $I$ with respect to the previous order using a command in Sage.

3. Trace the Buchberger algorithm for computing a Gröbner basis for $I$ with respect to the previous order.

4. Compute a minimal Gröbner basis of $I$ (using the Lemma 21.36 in [GG]).

5. Compute the reduced Gröbner basis of $I$ with respect to the previous order.

**Exercise 5** Let $I = \langle x - y^2, xy - x \rangle \subset \mathbb{F}_5[x, y]$. Compute $G \subset \mathbb{F}_5[x, y]$ such that it is a Gröbner basis for $I$ with respect to the 3 monomial orders defined in the lecture with $x > y$ and $y > x$ (with respect to the 6 of them at the same time!).

**Exercise 6** Solve the following system of equations over $\mathbb{C}$ (computing a Gröbner base, not by using a command in Sage):

$$\begin{cases} x^2 + y + z = 1 \\ x + y^2 + z = 1 \\ x + y + z^2 = 1 \end{cases}$$

(Hint: there are 5 solutions).

**Exercise 7** Let $I = \langle x^4 y - z^6, x^2 - y^3 z, x^3 z^2 - y^3 \rangle \subset \mathbb{F}[x, y, z]$.

1. Using lex order find a Gröbner basis $G$ for $I$ and a collection of monomials that spans (over $\mathbb{F}$) the space of remainders modulo $G$.

2. Consider now the grlex order. How do your sets of monomials compare?, why?

Best regards,

Diego