Computer Algebra (2012)-Aalborg University
Third set of exercises

The deadline for this set of exercises is Thursday 6/12. I would like to get (by email) an electronic file with your solutions. Furthermore, a (brief) **reasoned explanation should follow the solution** of the exercises, you can write your explanation in the electronic file or print the computer solution and hand-write your explanation. For instance: if you use a command, explain the input and output. If you write a program, comment it. If you are using an algorithm or theorem in the book, mention it.

Solve the following exercises using a Computer Algebra System. You are welcome to use Maple/Sage commands unless the exercise asks us to trace an algorithm or to implement a command:

Exercise 1 Trace the division algorithm for y^2x divided by $\{yx - y, y^2 - x\} \subset \mathbb{Q}[x, y]$ with respect grevlex with $y > x$. Check your result using a command.

Exercise 2

1. Consider two different polynomials f, g in $\mathbb{F}_4[x, y, z]$ such that their leading term is different for lex, deglex and grevlex. Show their leading terms with respect to the 3 monomial oders.

2. Compute the S-polynomial of f, g using a command in Maple or Sage with respect to the 3 monomial orders defined in the course. Show the multidegree of the S-polynomial and the expected degree of the combination before cancellations.

Exercise 3

1. Let $I = \langle\{xy - x, -y + x^2\}\rangle \subset \mathbb{Q}[x, y]$ and consider the lex order with $x < y$. Show that $\{xy - x, -y + x^2\}$ is not a Gröbner basis with respect to the previous order.

2. Compute a Gröbner basis of I with respect to the previous order using a command in Maple or singular.

3. Trace the Buchberger algorithm for computing a Gröbner basis for I with respect to the previous order.

4. Compute a minimal Gröbner basis of I (using the Lemma 21.36 in [GG]).

5. Compute the reduced Gröbner basis of I with respect to the previous order.

Exercise 4 Let $I = \langle x - y^2, xy - x\rangle \subset \mathbb{F}_5[x, y]$. Compute $G \subset \mathbb{F}_5[x, y]$ such that it is a Gröbner basis for I with respect to the 3 monomial orders defined in the lecture with $x > y$ and $y > x$ (to the 6 of them at the same time).

Exercise 5 Solve the following system of equations over $\mathbb{C}$ (computing a Gröbner base, not by using a command in Maple/Sage):

$$\begin{cases} x^2+y+z=1 \\ x+y^2+z=1 \\ x+y+z^2=1 \end{cases}$$

(Hint: there are 5 solutions).

Exercise 6 Let $I = \langle x^4y - z^6, x^2 - y^3z, x^3z^2 - y^3 \rangle \subset \mathbb{F}[x, y, z]$.

1. Using lex order find a Gröbner basis G for I and a collection of monomials that spans (over $\mathbb{F}$) the space of remainders modulo G.
2. Consider now the grlex order. How do your sets of monomials compare?, why?

Best regards,

Diego