Computer Algebra (2012)-Aalborg University
Lecture 20, November 12th

20th Lecture: Monday November 12th, 8:15-12:00 at room G5-109.

- 8:15-10:00 Lecture: Repetition + Buchberger algorithm, Geometric applications and complexity (pages 598–607).

- 10:00-12:00 Work in groups. Exercises from [GG]: A, B, C, D, E, F, G, H, I, J, 21.21, 21.23, 21.17, 21.9 (only i).

  Exercise A: Let $R = \mathbb{F}_3[X, Y]$. Let $f = X^2Y + 2XY^2 + XY + X$, $f_1 = X + 2Y^2 + 1$, $f_2 = Y^2 + Y$. Divide f by $\{f_1, f_2\}$ considering the monomial order $<_{\text{lex}}$. Divide f by $\{f_1, f_2\}$ considering now the monomial order $<_{\text{grlex}}$.

  Exercise B: Investigate how to define monomial orders in Maple and Sage.

  Exercise C: Compute, in Maple and Sage, the S-polynomial of two polynomials in $\mathbb{F}_q[x_1, x_2, x_3, x_4]$.

  Exercise D: Compute, in Maple and Sage, a Gröbner basis of an ideal in $\mathbb{F}_q[x_1, x_2, x_3, x_4]$ with respect to the 3 monomial orders considered in the lecture.

  Exercise E: Compute a Gröbner basis (with Maple or Sage) of the following ideal with respect to the lexicographical ordering with $z > y > x$:

  $$I = \langle x^5 + y^3 + z^2 - 1, x^2 + y^2 + z - 1, x^6 + y^5 + z^3 - 1\rangle \subset \mathbb{Q}[x, y, z]$$

  Exercise F: Compute a Gröbner basis (with Maple or Sage) of

  $$I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1\rangle \subset \mathbb{Q}[x, y, z]$$

  using lex and grevlex with $x > y > z$. Is there any difference?

  Exercise G: Now we change one exponent in exercise F: Compute a Gröbner basis (with Maple or Sage) of

  $$I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1\rangle \subset \mathbb{Q}[x, y, z]$$

  using lex and grevlex with $x > y > z$. Is there any difference?

  Exercise H: Compute a Gröbner basis (with Maple or Sage) of

  $$I = \langle x^4 - yz^2w, xy^2 - z^3, x^3z - y^3w\rangle \subset \mathbb{Q}[x, y, z, w]$$

  using lex and grevlex with $x > y > z > w$. Is there any difference (hint: grevlex is not always better than lex). Actually one can prove that $z^{n^2+1} - y^{n^2}w$ is in the reduced Gröbner basis of the ideal

  $$\langle x^{n+1} - yz^{n-1}w, xy^{n-1} - z^n, x^nz - y^nw\rangle$$

  w.r.t. grevlex with $x > y > z > w$.

  Exercise I: Let $I = \langle f_1 = x^2y - 1, f_2 = xy^2 - x\rangle$ and consider the lex order.

  1. Show that $\{f_1, f_2\}$ is not a Gröbner basis for I.

2. Trace the Buchberger algorithm for computing a Gröbner basis for I. You can use Sage or Maple for computing the S-polynomials.

Exercise J: Consider the ideal in Example 21.21 and the Gröbner basis $\{f_1, \ldots, f_5\}$ computed in pages 598 and 599. Compute a minimal Gröbner basis this ideal using lemma 21.36. Use Maple or Sage to compute the reduced Gröbner basis of this ideal.

Best regards,

Diego