Some slides for 4th Lecture, Algebra 2

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Construction of the rational numbers

Field of fractions

Proposition 3.4.1

Let R be a domain with field of fractions Q, let L be a field and let $\varphi : R \to L$ be an injective ring homomorphism. Then there exists a unique injective ring homomorphism $\overline{\varphi} : Q \to L$ such that $\overline{\varphi} \circ i = \varphi$.

Corollary 3.4.2

Let R be a domain contained in the field L. The smallest subfield in L containing R is

$$K = \{as^{-1} : a \in R, s \in R \setminus \{0\}\}$$

The field of fractions of R is isomorphic to K.

Divisibility and greatest common divisor in a domain

We assume from now on that R is a domain.

Suppose that $x, y \in R$. If $x = ry$ for some $r \in R$, we say that y is a divisor (divisor) of x and we denote it by $y|x$

- $y|x$ if and only if $\langle x \rangle \subset \langle y \rangle$.
- If $x = uy$, where $u \in R^*$, then $\langle x \rangle = \langle y \rangle$.
- If $\langle x \rangle = \langle y \rangle$, then $x = ry$ and $y = sx$ for some s, r. Therefore $x = (rs)x$ and $rs = 1$. This implies that $r, s \in R^*$ and there exists $u \in R^*$ s.t. $x = uy$ and we say that x and y are associated elements (associerede elementer) of R.

An element $d \in R$ is a greatest common divisor (største fælles divisor) of $a, b \in R$ if d is a common divisor of a and b and every common divisor of a and b divides d.

Let R be a principal ideal domain. We know that for every $a, b \in R$ there is $d \in R$ s.t.

$$\langle a, b \rangle = \langle d \rangle$$

What is d?, d is the greatest common divisor of a and b.

Proof:

- d is a common divisor of a and b since $\langle a \rangle \subset \langle d \rangle$ and $\langle b \rangle \subset \langle d \rangle$
- If e is a common divisor of a and b, then $\langle e \rangle \supset \langle a, b \rangle = \langle d \rangle$. That is e divides d.