Some slides for 22nd Lecture, Algebra 1

Diego Ruano

Department of Mathematical Sciences
Aalborg University
Denmark

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Simple groups

A group G is called simple if $\{e\}$ and G are the only normal subgroups of H. Otherwise G is called solvable.

Examples:

- $\mathbb{Z}/p\mathbb{Z}$, with p prime.
- A_n, for $n \geq 5$ (Theorem 2.9.19 using lemma 2.9.18).

Simple finite groups form the building blocks for all finite groups.

Feit and Thomson's theorem: the order of a non-abelian finite simple group must be even.

In 2004: classification of simple groups, 18 families and 26 exceptions. See wikipedia.

Lemma 2.9.18

Every permutation in A_n is a product of 3-cycles if $n \geq 3$.

Proof:

- A permutation in A_n is product of an even number of transpositions
- $(a\ b)(c\ d) = (a\ d\ c)(a\ b\ c)$
- $(a\ b)(b\ c) = (a\ b\ c)$

For self-study (lecture 24)

Theorem 2.9.19

The alternating group A_n is simple for $n \geq 5$.

Actions of groups

Let G be a group and S a set. We will say that G acts (from the left) on S if there is a map

$$\begin{array}{rcl} \alpha : G \times S & \to & S \\ (g,s) & \mapsto & \alpha(g,s) = g \cdot s = gs \end{array}$$

such that

- $e \cdot s = s$ for every $s \in S$
- $(g \cdot h) \cdot s = g \cdot (h \cdot s)$, $\forall g, h \in G$ and $\forall s \in S$.

Let $\alpha : G \times S \to S$ be an action of G on S, $X \subset S$ subset of S an element of S.
$G \cdot s = Gs = \{gs : g \in G\}$ is called the orbit of s (under the action of G)

The set of orbits $\{Gs : s \in S\}$ is denoted S/G.

Actions of groups

Action of G acts (from the left) on S, $\alpha : G \times S \to S$,
$\alpha(g, s) = g \cdot s = gs$

Let $\alpha : G \times S \to S$ be an action of G on S, $X \subset S$ subset of S an element of S.
$G \cdot s = Gs = \{gs : g \in G\}$, orbit of s (under the action of G)

Let $g \cdot X = gX = \{gx : x \in X\}$, where $g \in G$. Then

$$G_X = \{g \in G : gX = X\}$$

is called the **stabilizer of X**
If $X = \{x\}$, we denote G_X by G_x (instead of by $G_{\{x\}}$)

A **fixed point for the action** is an element $s \in S$ s.t. $gs = s$ for every $g \in G$. The set of fixed points is denoted by S^G.

Examples

Proposition 2.10.5

Let $\alpha : G \times S \to S$ be an action

- Let $X \subset S$ be a subset of S. Then G_X is a subgroup of G.
- The set S is the union of G-orbits

$$S = \bigcup_{s \in S} Gs$$

  where $Gs \neq Gt$ implies that $Gs \cap Gt = \emptyset$, if $s, t \in S$.
- Let $x \in S$. Then

$$\begin{array}{rcl} \tilde{f} : G/G_x & \to & Gx \\ gG_x & \mapsto & gx \end{array}$$

  is a well defined and bijective map between the left cosets of G_x and the orbit Gx.

Corollary 2.10.7

Let $G \times S \to S$ be an action, where S is a finite set. Then

$$|S| = |S^G| + \sum_x |G/G_x|,$$

where the sumation is done by picking out an element x from each orbit with more than one element.

Conjugacy classes

This map is an action of G on G. It is called conjugation:

$$\begin{aligned} \alpha : G \times G &\to G \\ (g, h) &\mapsto ghg^{-1} \end{aligned}$$

The orbit

$$G \cdot h = C(h) = \{ghg^{-1} : g \in G\}$$

is denoted $C(h)$ and called the conjugacy class containing h.

We denote by $Z(h)$ (centralizer of h) the stabilized G_h.

The set of fixed points

$$G^G = Z(G) = \{g \in G : gx = xg \ \forall x \in G\}$$

is denoted $Z(G)$ and called the center of G.

Conjugacy classes

- There is at least one fixed point for the conjugation action, namely $e \in Z(G)$.
- $Z(G)$ is an abelian normal subgroup of G (exercise 2.50)

The stabilizer of a subgroup $H \subset G$

$$G_H = N_G(H) = \{g \in G : gHg^{-1} = H\}$$

is called the normalizer of H in G.

H is an normal subgroup if and only if $N_G(H) = G$ (ex 2.51).

If G is a finite group, corollary 2.10.7:

$$|G| = |Z(G)| + \sum_{h \in G} |G/Z(h)|,$$

where the sumation is done by picking out one element h from each conjugacy class with more than one element.