Some slides for 11th Lecture, Algebra 1

Diego Ruano

Department of Mathematical Sciences
Aalborg University
Denmark

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Group homomorphisms

Let G and K be groups. A map $f : G \to K$ is called a group homomorphism if

$$f(xy) = f(x)f(y)$$

for every $x, y \in G$.

- Example: exponential function
- Example: determinant
- Example: $\pi : G \to G/N$ for a normal subgroup N of G.

The kernel of a group homomorphism $f : G \to K$ is

$$\mathrm{Ker}(f) = \{g \in G : f(g) = e\}$$

The image of f is

$$f(G) = \{f(g) : g \in G\}$$

A bijective group homomorphism is called a group isomorphism.
We write $G \cong K$ and say G and K are isomorphic.

Proposition 2.4.9

Let $f : G \to K$ be a group homomorphism.

1. The image $f(G) \subset K$ is a subgroup of K
2. The kernel $\mathrm{Ker}(f) \subset G$ is a normal subgroup of G.
3. f is injective if and only if $\mathrm{Ker}(f) = \{e\}$

Proof: (1)

- $e \in f(G)$?: $f(e) = f(ee) = f(e)f(e) \Rightarrow f(e) = e$
- $f(x)^{-1} \in f(G)$?: Yes, $f(x)^{-1} = f(x^{-1})$. For $x \in G$,

$$e = f(e) = f(xx^{-1}) = f(x)f(x^{-1})$$

$$e = f(e) = f(x^{-1}x) = f(x^{-1})f(x)$$

- $f(x)f(y) \in f(G)$?: For $x, y \in G$, $f(x)f(y) = f(xy)$

Proof: (2), $\ker(f)$ is a subgroup

- $e \in \ker(f)$?: $f(e) = e$
- $x^{-1} \in \ker(f)$?: For $x \in \ker(f)$, $e = f(x) = f(x)^{-1} = f(x^{-1})$
- $xy \in \ker(f)$?: For $x, y \in \ker(f)$, $f(xy) = f(x)f(y) = ee = e$

Proof: (2), the subgroup $N = \ker(f)$ is a normal subgroup.
$N = gNg^{-1}, \forall g \in G$.

- $gNg^{-1} \subset N$: For $x \in N$,
  $f((gx)g^{-1}) = (f(g)f(x))f(g^{-1}) = f(g)f(g)^{-1} = e$.
- $gNg^{-1} \supset N$: Consider the previous statement for g^{-1}:
  $g^{-1}Ng \subset N$. Then $Ng \subset gN$ and $N \subset gNg^{-1}$.

Proof: (3) f is injective $\Leftrightarrow \mathrm{Ker}(f) = \{e\}$

- $\Rightarrow$): For f injective, $Ker(f) = e$ since $f(e) = e$.
- $\Leftarrow$): For $\mathrm{Ker}(f) = \{e\}$ and $f(x) = f(y)$,

$$e = f(y)^{-1}f(x) = f(y^{-1})f(x) = f(y^{-1}x)$$

Then, $y^{-1}x \in \ker(f)$, and therefore $y^{-1}x = e$ and $x = y$.

To think: The previous result tells us that the kernel of any homomorphism is a normal subgroup. Is the converse true?

Something useful:

Tricks

Let $f : G \to K$ be a group homomorphism.

- $f(e) = e$
- $f(x^{-1}) = (f(x))^{-1}$

Theorem 2.5.1-The isomorphism theorem

Let G and K be groups and $f : G \to K$ a group homomorphism and $N = \ker(f)$. Then

$$\begin{array}{rcl} \tilde{f} : G/N & \to & f(G) \\ gN & \mapsto & f(g) \end{array}$$

is a well defined map and a group isomorphism

How do we understand G/N? Finding a group K, a surjective morphism $f : G \to K$ such that $N = \ker(f)$

Theorem 2.5.1

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Proof: well defined and injective. For $x, y \in G$:

- $f(x) = f(y) \Leftrightarrow$
- $f(y)^{-1} f(x) = e \Leftrightarrow$
- $f(y^{-1}) f(x) = e \Leftrightarrow$
- $f(y^{-1} x) = e \Leftrightarrow$
- $y^{-1} x \in N \Leftrightarrow$
- $xN = yN$

Theorem 2.5.1

Let G and K be groups and $f : G \to K$ a group homomorphism and $N = \ker(f)$. Then

$$\begin{aligned} \tilde{f} : G/N &\to f(G) \\ gN &\mapsto f(g) \end{aligned}$$

is a well defined map and a group isomorphism

Proof: $\tilde{f}$ is a group homorphism

$$\tilde{f}((g_1N)(g_2N)) = \tilde{f}((g_1g_2)N) = f(g_1g_2) = f(g_1)f(g_2) = \tilde{f}(g_1N)\tilde{f}(g_2N$$

Proof: $\tilde{f}$ is surjective
f is surjective onto $f(G)$

Examples