Refined analysis of RGHWs of code pairs coming from Garcia-Stichtenoth's second tower

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1 Introduction

Relative generalized Hamming weights (RGHW) of two linear codes are fundamental for evaluating the security of ramp secret sharing schemes and wire-tap channels of type II [3, 4]. Until few years ago only for MDS codes and a few other examples of codes the hierarchy of the RGHWs was known [6], but recently new results were discovered for one-point algebraic geometric codes [3], q-ary Reed-Muller codes [7] and cyclic codes [8]. In [2] it was discussed how to obtain asymptotically good sequences of ramp secret sharing schemes by using one-point algebraic geometric codes defined from good towers of function fields. The tools used here were the Goppa bound and the Feng-Rao bounds. In the present paper we focus on secret sharing schemes coming from the Garcia-Stichtenoth second tower [1]. We demonstrate how to obtain refined information on the RGHW's when the codimension is small. For general co-dimension we give an improved estimate on the highest RGHW. The new results are obtained by studying in detail the sequence of Weierstrass semigroups related to a sequence of rational places [5].

We recall the definition of RGHWs and briefly mention their use in connection with secret sharing schemes.

Definition 1 *Let $C_2 \subsetneq C_1$ be two linear codes. For $m = 1, \ldots, \dim C_1 - \dim C_2$ the m-th relative generalized Hamming weight (RGHW) of C_1 with respect to C_2 is*

$$M_m(C_1,C_2) = \min\{\#\mathrm{Supp} D \mid D \subseteq C_1 \text{ is a linear space,}$$
$$\dim D = m, D \cap C_2 = \{\vec{0}\}\}. \tag{1}$$

Here $\mathrm{Supp} D = \#\{i \in \mathbb{N} \mid$ *exists* $(c_1, \ldots, c_n) \in D$ *with* $c_i \neq 0\}$.
For $m = 1, \ldots, \dim C_1$ the m-th generalized Hamming weight (GHW) $d_m(C_1)$ is equal to $M_m(C_1, \{\vec{0}\})$.

It was proved in [3, 4] that a secret sharing secret scheme obtained from two linear codes $C_2 \subsetneq C_1$ has $r_m = n - M_{\ell-m+1}(C_1,C_2) + 1$ reconstruction and $t_m = M_m(C_2^\perp, C_1^\perp) - 1$ privacy for $m = 1, \ldots, \ell$. Here, r_m and t_m are the unique numbers such that the following holds: It is not possible to recover m q-bits of information about the secret with only t_m shares, but it is possible with some t_m+1 shares. With any r_m shares it is possible to recover m q-bits of information about the secret, but it is not possible to recover m q-bits of information with some r_m-1 shares.

We shall focus on one-point algebraic geometric codes $C_{\mathscr{L}}(D,G)$ where $D=P_1+\cdots+P_n$, $G=\mu Q$, and $P_1,\ldots,P_n,Q$ are pairwise different rational places over a function field. By writing v_Q for the valuation at Q, the Weierstrass semigroup corresponding to Q is

$$H(Q)=-v_Q\left(\bigcup_{\mu=0}^{\infty}\mathscr{L}(\mu Q)\right)=\{\mu\in\mathbb{N}_0\mid\mathscr{L}(\mu Q)\neq\mathscr{L}((\mu-1)Q)\}. \quad (2)$$

We denote by g the genus of the function field and by c the conductor of the Weierstrass semigroup.

We consider $C_1=C_{\mathscr{L}}(D,\mu_1 Q)$ and $C_2=C_{\mathscr{L}}(D,\mu_2 Q)$, with $-1\leq\mu_2<\mu_1$. Observe that for $\ell=\dim(C_1)-\dim(C_2)$ and $\mu=\mu_1-\mu_2$ we have that $\ell\leq\mu$, with equality if $2g\leq\mu_2<\mu_1\leq n-1$ holds.

From [2, Proposition 23 and its proof] we have the following result:

Proposition 2 *If $1\leq m\leq\min\{\ell,c\}$, then*

$$M_m(C_1,C_2)\geq n-\mu_1+(m-1)+(m-c+g+h_{c-m}) \quad (3)$$

where $h_{c-m}=\#(H(Q)\cap(0,c-m])$. If $2g\leq\mu_1\leq n-1$, then

$$M_m(C_1,C_2)\geq n-\dim C_1+2m-c+h_{c-m} \quad (4)$$

Applying Proposition 2 to code pairs coming from Garcia-Sticthenoth's second tower [1] the following asymptotically result was obtained in [2, Corollary 40]:

Corollary 3 *Let q be an even power of a prime and $0\leq\rho\leq\frac{1}{\sqrt{q}-1}$. There exists a sequence of one-point algebraic geometric codes $C_i=C_{\mathscr{L}}(D,\mu_i Q)$ and a sequence of positive integers m_i such that for i going to infinity: $n_i=n(C_i)\to\infty$, $\dim C_i/n_i\to R$, $\mu_i/n_i\to\tilde{R}$, $m_i/n_i\to\rho$. Let $\delta=\liminf d_{m_i}(C_i)/n_i$, we have that:*

$$\delta\geq 1-\tilde{R}+2\rho. \quad (5)$$

If $\frac{1}{\sqrt{q}-1}\leq R\leq 1$, we have that:

$$\delta\geq 1-R+2\rho-\frac{1}{\sqrt{q}-1}. \quad (6)$$

From Garcia-Stichtenoth's second tower [1] one obtains codes over any field $\mathbb{F}_q$ where q is an even power of a prime. Garcia and Stichtenoth analyzed the asymptotic behavior of the number of rational places and the genus, from which it is clear that the codes beat the Gilbert-Varshamov bound for $q \geq 49$. Remarkably, a complete description of the Weierstrass semigroups corresponding to a sequence of rational places was given in [5]. This description is what allows us to refine in the present paper the analysis of the RGHWs.

2 Small codimension

In this section we give a sharper bound on the RGHWs of two one-point algebraic geometric codes coming from Stichtenoth-Garcia's towers when the codimension is small.

Proposition 4 *Let ν be an even positive integer and q an even power of a prime. Consider two one-point algebraic geometric codes $C_2 \subsetneq C_1$ defined from the ν-th Garcia-Stichtenoth function field over $\mathbb{F}_q$. For $\mu < q^{\frac{\nu+1}{2}}$ and $m = 1, \dots, \mu$, we have that:*

$$M_m(C_1, C_2) \geq n - \mu_1 + \min \left\{ (m-1) q^{\frac{\nu}{4} - \frac{1}{2}u} + \left\lfloor q^{u - \frac{1}{2}} \left(1 - q^{-\frac{1}{2}}\right) \right\rfloor : u \in \left\{ \left\lceil \log_q(m-1) + \frac{1}{2} \right\rceil, \left\lfloor \log_q(\mu - 1) + \frac{1}{2} \right\rfloor \right\} \right\}. \quad (7)$$

Note that there are some cases where the minimum is reached for $u = \lceil \log_q(m-1) + \frac{1}{2} \rceil$ and other cases where it is reached for $u = \left\lfloor \log_q(\mu - 1) + \frac{1}{2} \right\rfloor$. For this reason in Proposition 4 the value u is not univocal.

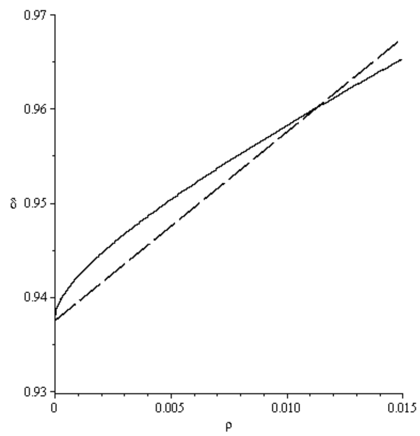
As Proposition 2, this result has an asymptotic implication:

Corollary 5 *Let q be an even power of a prime, $0 \leq \tilde{R}_2 \leq \tilde{R}_1 < 1$, and $\tilde{R} = \tilde{R}_1 - \tilde{R}_2 < \frac{1}{\sqrt{q}-1}$. There exists a sequence of pairs of one-point AG codes $C_{2,i} = C_{\mathscr{L}}(D_i, \mu_{2,i} Q) \subsetneq C_{1,i} = C_{\mathscr{L}}(D_i, \mu_{1,i} Q)$, such that: $n_i = n(C_{2,i}) = n(C_{1,i}) \to \infty$, $\mu_{j,i}/n_i \to \tilde{R}_j$ for $j = 1, 2$ for $i \to \infty$. For a given ρ let m_i be such that $m_i/n_i \to \rho$ for $i \to \infty$ and let $M = \liminf M_{m_i}(C_{1,i}, C_{2,i})/n_i$. The sequence of code pairs satisfies:*

$$M \geq 1 - \tilde{R}_1 + \min_{u \in \{\rho, \tilde{R}\}} \left\{ \rho \left(u(q - \sqrt{q})\right)^{-\frac{1}{2}} + \frac{u}{q}(q - \sqrt{q}) \right\}. \quad (8)$$

Note that if we assume that $C_{2,i}$ are zero codes for all i, then $\lim M_{m_i}(C_{1,i},\{\vec{0}\})$ is the asymptotically value of the m_i-th general Hamming weight of $C_{i,1}$. For $\tilde{R} < \frac{1}{4(q-\sqrt{q})}$, the bound in Corollary 5 is sharper than the one obtained in Corollary 3.

In the following graph we compare the bound from Corollary 3 (the dashed curve) with the bound from Corollary 5 (the solid curve). The first axis represents $\rho = \lim m_i/n_i$, and the second axis represents $\delta = \liminf M_{m_i}(C_{1,i},\{\vec{0}\})$.



3 The highest RGHW

In this section for $2g \leq \mu_2 < \mu_1 < n-1$, we obtain a new bound for the highest RGHW of two one-point algebraic geometric codes obtained from Stichtenoth-Garcia's second tower.

Proposition 6 *Let ν be an even positive integer and $2g \leq \mu_2 < \mu_1 < n-1$. Consider two one-point algebraic geometric codes $C_2 \subsetneq C_1$ built on the ν-th Garcia-Stichtenoth tower. We have that:*

$$M_\ell(C_1,C_2) = n - \dim C_2 \quad \text{if } \ell \geq q^{\frac{\nu-1}{2}} \tag{9}$$

$$M_\ell(C_1,C_2) \geq n - \dim C_2 - \left(q^{\frac{\nu-1}{2}} \sum_{i=1}^{\lfloor \frac{\nu+1}{2} - \log_q(\ell) \rfloor - 1} (q^{1-\frac{i}{2}} - q^{-\frac{i}{2}}) + \right.$$
$$\left. + (q^{\frac{\nu+1}{2} - \lfloor \frac{\nu+1}{2} - \log_q(\ell) \rfloor} - \ell) q^{\frac{\lfloor \frac{\nu+1}{2} - \log_q(\ell) \rfloor}{2}} \right) \quad \text{if } \ell < q^{\frac{\nu-1}{2}} \tag{10}$$

For $\ell \geq q^{\frac{\nu-1}{2}}$, the Singleton bound is reached. For $\ell < q^{\frac{\nu-1}{2}}$ it is still an interesting bound because we are able to estimate h_{c-m}. This bound has an asymptotically implication as well:

Corollary 7 *Let q be an even power of a prime, $\frac{2}{\sqrt{q}-1} \leq \tilde{R}_2 \leq \tilde{R}_1 < 1$, and $\tilde{R} = \tilde{R}_1 - \tilde{R}_2$. There exists a sequence of one-point algebraic geometric codes $C_{2,i} = C_{\mathscr{L}}(D_i, \mu_{2,i}Q) \subsetneq C_{1,i} = C_{\mathscr{L}}(D_i, \mu_{1,i}Q)$, $\mu_i = \mu_{1,i} - \mu_{2,i}$, such that: $n_i = n(C_{2,i}) = n(C_{1,i}) \to \infty$, $\mu_{j,i}/n_i \to \tilde{R}_j$ for $j = 1,2$ for $i \to \infty$. Let $\ell_i = \dim C_{1,i} - \dim C_{2,i}$, $M = \liminf M_{\ell_i}(C_{1,i}, C_{2,i})/n_i$, $R_j = \lim \frac{\dim C_{i,j}}{n_i}$ for $j = 1,2$, and $R = R_1 - R_2$, we have that:*

$$M = 1 - R_2 \quad \text{if} \quad R \geq \frac{1}{q - \sqrt{q}} \tag{11}$$

and

$$M \geq 1 - R_2 - \left(\frac{1}{q - \sqrt{q}} \left(\sum_{i=1}^{-\lfloor \log_q (R(1-\frac{1}{\sqrt{q}})) \rfloor - 1} (q^{1-\frac{i}{2}} - q^{-\frac{i}{2}}) + \right.\right.$$
$$\left.\left. + q^{1+\frac{1}{2}\lfloor \log_q (R(1-\frac{1}{\sqrt{q}})) \rfloor} \right) - R q^{-\frac{1}{2}\lfloor \log_q (R(1-\frac{1}{\sqrt{q}})) \rfloor} \right) \quad \text{if} \quad R < \frac{1}{q - \sqrt{q}}. \tag{12}$$

In Corollary 3, ρ is smaller than or equal to $\frac{1}{\sqrt{q}-1}$. If we assume $C_{2,i}$ to be the zero codes for all i, then the value M of Corollary 7 represents the asymptotically value of the highest generalized Hamming weight of $C_{i,1}$. By using Corollary 3 for $R = \frac{1}{\sqrt{q}-1}$ it is possible to obtain a similar value, but for the other values of R it is a new bound.

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