Robotics. A hierarchical approach

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Previous remark.- The current document is an introduction to a *stratified approach* for several relevant problems in *Robotics* concerning to modeling, planning, tracking, navigation, kinematic and dynamics issues. They provide the core models and techniques for applicaitons to Humanoid Robots and Animats which are developed in the last two modules. These notes are not a general introduction to Robotics which would include a lot of additional issues regarding electronic devices, sensors, actuators or distributed architectures to integrate all of them on complex devices.

The main goal of these notes consists of providing a unified language for different aspects appearing in the literature, including some AI recent configurations. Unification is performed in terms of a stratified approach, the intensive use of different fields and their description in terms of different kinds of (discrete, continuous, infinitesimal) symmetries. I don't claim originality about fundamental results, because most of them are well known in the literature; the main contribution concerns to unification of several related approaches under a common mathematical umbrella.

The book [Mur94] [1] provides the nearest approach to the developed here, because it incorporates main tools arising from Differential Geometry for kinematics and dynamic analysis. Some aspects which are not included in this book concern to

- The use of Lie actions (from algebraic and infinitesimal viewpoints) not only for motion planning and execution, but for the interplay between Kinematic and dynamic aspects, including Optimization and Control issues.
- The use of stratified approaches on Semi-analytic spaces denoted as $\mathcal{X} = (X, \mathcal{O}_X)$ where X is the base space and $\mathcal{O}_X$ the set of regular functions. They provide the support for the Perception-Action Cycle (PAC) in Robotics linking Perception $\mathcal{P}$ and Action $\mathcal{A}$ spaces, and their structural relations (in terms of topological fibrations with Configurations (or Joints) space $\mathcal{C}$ and the Working space $\mathcal{W}$. The resulting global scheme will be labeled as the PACW (Perception-Action-Configurations-Working) cycle.
- The incorporation of AI based methods along all the stages of the PACW cycle, by reformulating some basic principles of Machine Learning in terms of vector and covector fields, and their joint management

[1] R.M. Murray, Z. Li and S. S.Sastry: *A Mathematical Introduction to Robotic Manipulation*, CRC Press, 1994.

in terms of their formal products given by Tensor fields. In particular, TensorFlow provide the support for computational implementation, whose estimation is performed by using Tensor Voting procedures.

The above aspects are "transversal" to the six modules of this matter. They use a simplification of well known notions in Mathematics and Physics, but with a scarce use in Robotics. Interactions between mathematicians and engineers in the Robotics field is a hazardous history with summits and valleys. At last decades of the 20th century, most mathematicians considered that Robotics was not a mathematician field, and most engineers consider that mathematicians are too theoretical ones to be useful for solving practical problems.

If one looks to usual practices of both communities, both of them are partially right .... but the interpenetration is a reality which has given to an increasing acknowledgement from the late nineties. A symptom of this mutual recognition is the acknowledgement of Robotics and Computer Vision as areas of Mathematics performed by the Mathematical Society of France along the early years of the 21st century. Luckily, things have changed from the last years of the 20th century.

Some aspects of the *interplay between Mathematics and Robotics* concern to effective numerical solutions of equations, local and global topological methods for solving motion planning problems, geometric models for design, dynamical aspects for stability, attraction and interpolation issues, analytical aspects for prolongation of solutions (including singular cases), and so on. Lie groups and associated structures (principal bundles, mainly) were already introduced in Robotics at the early 1990s. In ther same way as for Physics, they provide a unification for modeling and interacting at different levels.

Currently, there are computational tools (models and algorithms) for all of them. In practice, it is necessary to solve interoperability issues between different computational tools (models and software), but this is a hard problem which in beyond the reaching of these notes. Our more modest proposal consists of developing the *Geometry and its mechanical extensions to Kinematics and Dynamics* to provide the initial framework to unify all the above approaches. Hence, these notes are focused towards mathematicians with interest in Robotics and mechanical engineers with a reasonably good formation in maths, wishing to know how connect different approaches under a geometric umbrella.

There are several excellent books which are focused from a geometrical viewpoint, but some of them are too theoretical, others use a mathematical formalism which can be applied only to a very limited cases of use, and most of them include a very limited use of algebraic properties (symmetries (as unifying principle), analytical structures (to improve adaptive behaviors) or a lower use of Machine Learning approches (for self-learning and adaptation ot changing environments).

Nevertheless remarkable advances in Geometric approaches (arising from Bundles and Geometric Algebra, mainly), their extension to optimization and Controlissues in Robot Kinematics and Dynamics is still scarce. Usual approaches are restricted to the regular case, by ignpring possible changes of state or phase transitions; they can be considered as "singularities" on the base manifold M or their extension to Phase space $P = TM$ and TP. Their completion by using "limits of tangent data" in stratified spaces provides a support for managing critical situations and solve related problems appearing in Automatic Navigation in Kinematics or under-actuated Control in Dynamics.

In all the above issues, "stratification" play a fundamental role, since they involve the introdudction of "natural hierarchies" between Geometric, Kinematic and Dynamic levels of Mechanics.

- Stratifications can be adapted to Perception $\mathcal{P}$, Action $\mathcal{A}$, Configurations $\mathcal{C}$ and Working $\mathcal{W}$spaces appearing in the PACW cycle can be described as (semi)analytic spaces, i.e. as pairs $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ are the corresponding regular functions on X. This representations unifies morphological and functional aspects for each $\mathcal{X}$, and relate properties of spaces by means stratified maps $\Phi : \mathcal{X} \to \mathcal{Y}$ for each pair of spaces.

- Furthermore maps between the support for morphological properties, the formulation of functional properties is performed in terms Jets spaces (as extensions of maps linking the spaces appearing in the PACW). This formulation allows the recovery of structural connections between differential and integral approaches in terms of motion's equations and variational principles (by using Jets spaces for functional aspects).

- In addition, due to the existence of topological fibrations $\mathcal{X} \to \mathcal{Y}$ for each pair of spaces appearing in the PACW cycle, a simultaneous treatment of trajectories and constraints can be described in terms of distributions $\mathcal{D}$ of vector fields and systems $\mathcal{S}$ of differential forms, whose product gives tensor fields (r and covector fields. This reformulation can be easily adapted to TensorFlow appearing in the DNN framework under PyTorch.

Formally, a *stratification* is a decomposition of a topological space in a disjoint union of subsets ("cells" or topological varieties with boundary, usually) verifying "good incidence conditions" w.r.t. the adherence of adjacent subsets which are "topologically trivial". stratifications can be given for spaces having in account properties w.r..t elements of refernce involving static or evolving environments. Thus, they can involve not only the support $\mathcal{X}$ and maps $\mathcal{X} \to \mathcal{Y}$, but succesive extensions to be formulated in terms of jets spaces, e.g.

A typical example is given by the decomposition of $(n \times p)$-matrices A by the rank $rk(A)$, where the global space is a vector space which contains a singular locus given by non-regular matrices, i.e. matrices A with rank $rk(A) < min(n,p)$. Stratifications can be absolute or relative, i.e. w.r.t. a map. The most relevant decompositions for robot dynamics are relative stratifications which are defined w.r.t. to transfer maps between configurations and working spaces, and their "extensions".

The introduction of G-actions invovling the structure or the motion, induces G-stratification principles involve also to symmetries in a two-fold way, involving

- Structural *hierarchies between groups* induce a decomposition between G-orbits and their topological closures; typical examples appear in regard to rigid motions $SE(n) = SO(n) \ltimes \mathbb{R}^n$ (Euclidean Group), affine transformations $A(n) = GL(n) \ltimes \mathbb{R}^n$ or more general projective transformations (for scene representations), symplectic transformations (preserving motion's equations) or contact transformations (for interaction with the environments), and their corresponding infinitesimal versions for Kinematics and Dynamics.

- *Equivariant bifurcations* appearing in changes of state for geometric aspects in X or phase transitions for Kinematics on $P_X = TX$ can be explained in terms of breaking (local or infinitesimal) symmetries. In this case, breaking symmetries require an exxtension of traditional formulations (where the group is fixed). This extension is well know in Dynamical Systems and their applications in Engineering; from the 1980s it has been extended to much more general situations in Theoretical Physics (labeled as super-geometry)[2]

Both ideas are easily translated to optimization and control issues, and their reformulation in terms of multivectors which provide a more compact

[2]See the module A_{24} (Geometric Topology) for more details.

presentation of phenomena linked to the management of robot kinematics and dynamics.

The compatibility with algebraic or infinitesimal symmetries (arising from relation between Lie groups G and their Lie algebras $\mathfrak{g} = T_eG$), and the introduction of symmetries for conservation laws (linked to Classical Mechanics and Variational Principles) suggests to develop a more systematical approach for Robotics modeling which is based on *Equivariant Stratifications*. Roughly speaking, they can be understood as some "symmetric space" superimposed to configurations and working spaces which is locally given by a finite collection of homogeneous space with dense orbits by the algebraic or infinitesimal action.

The "dense character" of generic orbits concerns to "regular behaviors", but some generic singularities are allowed in the adherence of orbits which provide a support for "phase transitions". In this framework, phase transitions are controlled by subgroups or subalgebras represented as symmetries breaking linked to equivariant bifurcation phenomena. This idea is developed at last two modules of the Course in regard to some phenomena involving humanoid robots and animats, respectively.

According to the above remarks, materials of the current introduction are organized as follows:

1. We start with a *general introduction* where we develop a triply articulated approach around algebraic transformations (Klein, Lie, Cartan), semi-analytical framework (as an extension of the smooth case) and a multivector approach (in the vein of Geometric Algebra as natural extension of the well-known quaternionic calculus commonly used in geometric reformulations of Robotics).

2. Second section is devoted to display how to perform an *integration of geometric, kinematic and dynamic aspects* in a common mechanical framework. This integration is performed in a double sense (which could be labeled as geometric-topological and analytical approaches), in terms of morphological and functional aspects, according to the classical distinction for live beings.

3. Next, we develop an *articulation between deterministic and probabilistic models*, in terms of differential (or more generally, semi-analytic) approach and statistical approach. Instead of posing the accent on stochastic approach, we adopt a probabilistic approach which allows to incorporate different kinds of (scalar, vector, tensor) fields. This

choice eases the presentation of a feedback between deterministic and uncertainty principles for this knowledge field. The approach based on variants of sample consensus allow to incorporate learning processes and provide a natural feedback with expert systems, which allow the design of more efficient tools for learning and provide structural models for estimation issues.

4. The fourth section is devoted to some *recent mathematical developments* which are considered as relevant ones for Robotics. This section has a more advanced character, but it opens the door for a mutual collaboration in regard to the above topics. Furthermore, they contribute to a better understanding of distributed robotics in all their aspects relative to architecture, shape and function.

5. Last section displays the *overall structure* of all these materials in an enlarged course which is organized in *six modules* involving anchored robots, mobile platforms, kinematics, dynamics, humanoid robots and animats. The two first modules have a descriptive character and allow to understand the role of sensors and commands to ease the interaction with itself and the environment. The hard kernel is composed by an extended geometric approach to kinematics and dynamics which are developed along the central modules. Finally, last modules illustrate some of the most outstanding applications linked to humanoid robots (very focused towards assistance to disabled persons) and animats (in strong relation with biologically inspired animation in Computer Graphics and simulation of embedded collective intelligence).

Most of these materials have been developed and presented in different meetings, courses and invited talks along late nineties and the early years of 21st century. These presentations have a fragmentary character. Thus and due to some external requirements, I have reunified related materials in a common draft, by connecting several approaches which were initially sketched, only

1. Introduction

Following the Lagrange's line of thought at the early years of the 19th century, Mechanics can be understood as an extension of Geometry. Along these notes, any Geometry is understood in the Klein's sense, i.e., is characterized by a group G as follows: A Geometry linked to a group G is given as the set of properties (relative to objets and functionals) which remain invariant by the action of the group G. This description was originally introduced by F.Klein (Erlangen's Program 1873) for classical linear groups, i.e., for linear subgroups of the general linear group preserving a quadratic or a bilinear form.

Successive extensions of this description for Geometry linked to classical groups are relative to

- Algebra of Classical Groups or, equivalently, to geometries linked to finite-dimensional groups following the Erlangen's Program.
- Differential Analysis on Manifodls in terms of G-invariant differentia forms (evolving constraints) a la Cartan, or their dual G-invariant vecotr fields (for trajectories performed by control points).
- Topology by replacing linear or geometric groups by ∞-dimensional groups of homeormophisms on a PS-manifold M as base space,
- Kinematics in terms of diffeomorphisms forthe Phase space $P = TM$ to include variation rates of geometric "quantities".
- Dynamics where symmetries are linked to forces and moments on TP to explain interactions, or reformulating control issues in geometric terms, e.g.

Each one of the above extensions is meaningful for a G-equivariant approach to Robotics. Foudnations for Their corresponding computational versions have bneen developed in the modules B_{1k} for $1 \leq k \leq 6$ of the matter B_1 (Computational Mechanics of Continuous Media). Some of the most relevant inputs for signals arise from digital image and video, whose basic elements have been developed in the module B_{21} (Image Processing and Analysis). Along the modules of B_3, we will adapt all of them (expanding when necessary) to some Robotic applications.

To achieve these goals, it is necessary to introduce hierarchies (involving morphological and functionals aspects), which cna be described in terms of cell decompositions (for the support) and their dual (evaluation of linear functionals). The incorporation of kinematic and dynamic aspects requires an additional PS-structure (PS: Piecewise Smooht) which generalizes the notion of tangent space (or their "join" in the tangent bundle τ_M of a manifold M) to incorporate "events" such as changes of state or phase changes, e.g. Hence, one needs extend tangent and cotangante bundles to "stratificiation" with "good incidence conditions" for "confluent strata" at singualrities.

The *notion of stratification* is the organizer principle which appears in all contexts in a recurrent way. Roughly speaking, a *stratification* of a variety V is a decomposition of V given as a disjoint union of k-dimensional "cells" c_i^k (subsets topologically equivalent to open sets of a cartesian space $\mathbb{R}^k$) which are called *strata* verifying "good" incidence conditions for their "frontiers" or adherences. Similarly, a *stratified map* $f : V \to W$ is a map between stratified spaces such that each stratum of W is the union of the images of strata of V. [3]

Different notions of stratification appear in multiple contexts (some of them with a wrong meaning in matehamtical terms). The notion used here is ubiquitous and crosses transversally the three main innovations of the approach developed along these notes:

- *Algebraic stratification*: It involves to natural hierarchies between Groups G (or their infinitesimal version in terms of algebras) with their corresponding Geometries. It induces a decomposition of spaces or superimposed structures (fiber bundles or principal bundles, e.g.) as a union of G-orbits; a typical example is linked to the orbital structure for the moment map which provides the core for the invariant reformulation of analytical mechanics in terms of symplectic or contact geometries.

- *Analytic stratification*: It involves to decomposition of eventually singular spaces of the complementary of regular locus, the relative decomposition w.r.t. a scalar function $f : M \to \mathbb{R}$ (or more generally a functional linked to optimization issues, e.g.), or w.r.t a map such as the transference map $\tau : \mathcal{C} \to \mathcal{W}$ given locally by a vector map. In all cases, the key is given by the differential map or its formal version given by the first order jet and their extensions. In all cases, stratifications are given by rank deficiency loci of involved maps such as the Jacobian map for robot kinematics, e.g.

[3] More refined versions of this notion will be introduced along the different modules

- *Cliffordian stratification*: An intrinsic representation of the rank deficiency loci can be locally described as the dependency locus of subsets of lines (columns or rows) of matrices representing multivectors linked to the Robot Dynamics; they are linked to the differential of transfer map (between robotic configurations and end-effectors) or their successive extensions.

A stratified approach allows not only to integrate precedent approaches, but incorporate the behavior at singularities, including optimization and control issues. In particular, reachability and controllability issues can be formulated in terms of rank conditions Additionally, the underlying analytic structure allows to extend control strategies from regular case to generic singularities. Thus, the stratified approach takes advantage of locally symmetric structures (associated to groups, symmetries breaking and equivariant bifurcations), linked to extrinsic properties (related to action functionals, e.g.) and/or intrinsic formulations (linked to the dependency loci in terms of contraction operators applied to multivectors).

To overcome the limitations of classical approaches (with a lot of switching procedures for different phases of tasks due to the lack of differentiability, mainly) and motivate the stratified models from the beginning, the draft follows an increasing complexity relative to robots and tasks to be performed. So, we start with the simplest models linked to planar or spatial anchored robots, evolving next towards mobile platforms and ending with multibody systems, including legged robots. From a mathematical viewpoint, there appears a natural hierarchy between

- *geometric aspects* related to constraints linked to robotic architecture (reaching, e.g.) and task-adapted design constraints, between others;
- *kinematic aspects* including velocities and accelerations at joints and end-effectors, generation of impulses at joints,m and transmission phenomena along the whole body (in terms of propagation models, e.g.); and
- *dynamic aspects* involving voluntary movements, inertial effects, anticipation and compensation strategies, correction of errors, attenuation of vibrations, e.g.) and all issues related to control and optimization procedures.

From Classical Mechanics it is well known the existence of a natural hierarchy between Geometry, Kinematics and Dynamics which is translated in a natural way to the Robotics. Firt fomrulations are dfue to Lagrange and Legendre in terms of relations between generalized coordinates $(\underline{q}, \underline{p}, \underline{r})$, where $\underline{p} = \dot{\underline{q}}$ and $\underline{r} = \dot{\underline{p}} = \ddot{\underline{q}}$- A problem to be solved is how to develop a unified framework able of integrating the space-time evolution of morphologicla and functional aspects in Robotics. To accomplish this goal, we have developed several complementary approaches which are based on:

- *Algebraic Lie formalism* which is formulated in terms of some Classical Groups linked initially given by the Euclidean group for rigid motions (rotations and translations) in Geometry, and Symplectic Groups preserfing ideal motion's equations, and extended to more general groups to include visualization of scenes (affine, projective, conformal groups), and structural models for interactions arising from motions. Its infinitesimal version provides some tools to represent small motions at joints, and to obtain motion integrals.
- *Analytic formalism* which is formulated in terms of jets-formalism to display how functionals involving kinematics (resp. dynamics) are an extension of functionals involving geometric (kinematic) quantities. This approach allows to incorporate singularities appearing at geometric, kinematic and dynamic levels, including control and optimization issues in a contact or symplectic .
- *Cliffordian formalism* (also called, Geometric Algebra) to manage "mixed quantities" (involving scalars and multivectors) and functionals defined on them. The reformulation of Analytical Mechanics (including phenomena linked to wave fronts) in terms of Clifford analysis is more sophisticated. To fix ideas, we shall illustrate it with some examples arising from dexterous manipulation and locomotion tasks.

The precedent formalism is transversal to the six modules of these notes (see section 5 of this introduction for details). Furthermore, each formalism sheds light on different aspects of Robotics, extending the homogeneous to a locally symmetric framework (compatible with analytic stratifications), and a global treatment in terms of the geometric analysis of multivectors.

The above triply articulated approach is illustrated with increasingly complex robots starting with planar or anchored robots, continuing with mobile platforms, till arriving to multilegged robots (humanoids and animats). Thus, instead of developing a general mathematical formalism from the beginning (till arriving to sometimes disappointing simple robots), one starts from several typical robotic architectures and one applies different mathematical approaches (linked to the above three items), by displaying their adaptability or their power to solve kinematic or dynamic problems linked to tasks to be performed. In a so large scientific domain it is not possible to claim originality about the materials contained along these notes; the originality arises mainly from the re-organization of materials which throws new insights on always renewing topics.

The introduction of a geometric language has several advantages which are related with robust modeling, the reuse of a geometric formulation of mechanics, the capability of providing an objective and parametric representation, of describing the mechanical architecture and articular configurations, motions representations, study of trajectories, analysis of dynamical effects associated to motions, capabilities for interpreting solutions linked to optimization procedures and/or the robustness of algorithms, e.g.. Next, we display some of these advantages in a more detailed and structured way.

1.1. Some basic notions

1.1.1. Notion of a robot

Following the old definition given by the Robot Institute of America (1979), a *robot* is a multi-functional reprogramable manipulator which is designed to move materials, pieces or specialized devices, through variable programmed movements for the realization of different tasks.

In particular, a robot includes mechanical components (acting as a skeleton), sensors and actuators (for capturing information and generating responses), and mechatronic devices (for optimization and control). All of them are managed by expert systems with reflex and supervised components for reactive or supervised tasks execution. Tasks supervision is performed by a central unit and distributed systems along the architecture, which are in charge of coordinating and generating semi-automatic responses (including reflex and voluntary movements).

The interaction with environment is a difficult task which is organized by following different kinds of constraints. Usual industrial robots operate in very controlled closed environments with a security perimeter for a more safe operation mode. The development of robots for operating in open environments is a hard challenge to solve; nevertheless spectacular advances from the nineties, we are still some far of achieving satisfactory solutions for complex open environments in presence of intelligent agents. Hence, we shall restrict ourselves to robots operating in (partially) structured environments.

1.1.2. Architecture of a robot

A *kinematic chain* is a finite connected collection of segments which are sequentially connected by joints. The *support for the mechanical architecture* of a robot is given by a finite collection of articulated kinematic chains which are connected to a fixed or mobile planar components (labeled as plats) by means of joints. A robotic arm has once a kinematic chain, which can be (hyper)redundant to provide several ways of executing a task (in regard to assisted surgery, e.g.). When several kinematic chains are connected to one or more "bodies", we shall say that one has a *multibody parallel robot*.

The introduction of sensors [4] and actuators [5] enlarge the interaction capabilities of robots in semi-structured environments. Jointly with the above mechanical architecture, they provide the *mechatronic architecture* of the robot.

- *Sensors* are designed to capture the information of an eventually changing environment; very often they go farther from the limitations of human perception. From the static viewpoint, they involve to acoustic (including ultrasounds, e.g.), tactile (including proximity and contact devices, e.g.) and visual information (including non-visible spectrum, e.g.). From a kinematic viewpoint they involve to different kinds of dead reckoning (including classical odometry, e.g.) or energy (acoustic, electromagnetic). From a dynamic viewpoint, they involve to force sensors, evaluation of phase-shift, frequency modulation and specific devices (magnetic compasses, gyroscopes, e.g.)

- *Actuators* are designed to transform signals in commands according to a system of rules managed by expert systems. Commands are specific of tasks to be developed and involve to an integration of the available information according to the planned tasks. Tasks can include self-localization (position and orientation), follow paths (according to ultrasonic, tactile or visual information, e.g.), or other aspects involving machines (detection of vibrations, e.g.) or unexpected phenomena in hostile or unaccessible environments (presence of gas, e.g.)

[4] A sensor is a device that measures some attribute of the world

[5] Actuators are the motors responsible for motion in the robot acting at joints.

Nevertheless the different nature of signals, the Fourier analysis provides a common formalism which is independent of the dimension and range of signals. The availability of direct and inverse Fourier transforms allows to reference all available information to a spatial domain, where the fusion of information is performed. In other words, the existence of a geometric support for the whole architecture and the environment provides a support to integrate the information arising from 1D, 2D and 3D signals. This simple feature will be exploited along the second module (mobile platforms) to provide an overall representation of the environment to ease the interaction.

1.1.3. Main types of mechanical architectures

The most important types are the following ones:

- *sequential robot*, as an articulated arm, e.g.;
- *parallel robot* as flight simulators, e.g.;
- *multibody robot* as multilegged robots, e.g.:
- *hybrid robot* as wheeled and legged robots, e.g.

Beyond the almost trivial 2R or 3R planar robots, the simplest mobile robots are based on wheeled platforms. In presence of irregular terrains or when the robot must cross through environments with obstacles (pipes, stairs, e.g.) it is necessary to consider legged-based robots. This kind of robots are specially useful in hazardous, toxic or unaccessible environments for human beings.

The most complex ones are the humanoid robots, because they incorporate the character of multilegged robots with inherent unstability along complex tasks linked to grasping-manipulation of weighted objects and/or complex locomotion tasks (including walking, running, jumping, e.g.) or even the interaction with other (human or artificial) agents. The simulation of like-insect and like-mammals robots poses interesting challenges for simulation and interaction which are being exploited by the multimedia industry.

1.1.4. Configurations and working spaces

By making an abstraction of specific mechanical architectures, a basic distinction to describe the mechanics is given by configurations (or articular or joints) space $\mathcal{C}$ and working space $\mathcal{W}$ of a robot $\mathcal{R}$.

- The *configurations space* $\mathcal{C}$ describes the state of joints located at control points of the robot architecture. The most usual joints are prismatic, rotational and spherical joints, in correspondence with translations and (planar or spatial) rotations which generate the euclidian group $SE(n)$ for $n = 2$ or $n = 3$. All of them are bounded by the allowable values which can take in terms of interval lengths and allowed angles. The allowed planar rotations describe a subset (in fact a semi-analytic variety) of the special orthogonal group $SO(2)$; furthermore, each spatial rotation given by an element of $SO(3)$ can be decomposed in a product of three planar rotations. Hence, $\mathcal{C}$ is a boundary subvariety of $\mathbb{R}^{\ell} \times SO(2)^r$ where ℓ denotes the number of *prismatic* joints (modifying the *length* of arms, e.g.) and r denotes the number of rotational joints (after decomposing spherical joints in product of three rotational joints).

- The *working space* $\mathcal{W}$ describes the state of the *end-effector* $\mathbf{e}(C^{\alpha})$ for each kinematic chain C^{α} of the robot. Each kinematic chain is composed by a collection of prismatic and rotational joints. Hence, the localization of $e(C^{\alpha})$ can be described by means of an element of the euclidian group $SE(2)$ (planar case) or $SE(3)$ (spatial case) which is given by the product of rotations and translations. The physical constraints about mechanisms imply that working space $\mathcal{W}$ for an anchored robot is given initially by a bounded region of a finite number of copies of $\mathbb{R}^3$ (one per kinematic chain). Hence, this subset is parameterized by a subvariety of the product of a finite number of copies of $SE(n)$

Each small motion at robot joints generates a movement on the end-effector $e(C^{\alpha})$ which is represented by a product of matrices. We shall label as a *transference map* (also called, "transfer" map) to this transformation which will be denoted by

$$\tau : \mathcal{C} \to \mathcal{W}$$

As each configuration or working spaces can be parameterized by a subset of a product of classical groups, one can describe both spaces in terms of "locally symmetric spaces" X, i.e. spaces such that each $x \in X$ has a neighborhood U whose points $x' \in U$ are achieved from x by using local symmetries [6]. This description is very useful for design principles, to identify first integrals for motion (invariant w.r.t. group actions), to generate

[6] A more formal definition is given in the module 1

and manage dynamical symmetries (action-reaction principles, first integrals for variational principles), optimization procedures (defined on Lie groups), control issues (extensible to singular configurations, also), and so on.

1.2. A hierarchical approach

Following a biological inspiration, hierarchies involve to shape and function. Roughly speaking, morphological aspects are modeled in terms of Analytical Mechanics involving to tasks to be performed, e.g. Functional aspects involve to higher order reasoning including data clustering or unsupervised learning, e.g. Along this subsection, the focus is put on the former aspects, leaving functional aspects till subsections §3,4 and §4,4, because they require additional elements of expert systems (the old AI).

Classical Analytical Mechanics introduces a basic hierarchy (which is well known from the very beginning of the 19th century) with three basic steps involving to Geometric, Kinematic and Dynamic aspects, by following an increasing complexity. In generalized coordinates this hierarchy involves to

1. *characteristic features* (position, orientation, e.g.) of meaningful points which are managed in terms of generalized coordinates (Lagrange, Legendre) with their contact structure;

2. *kinematic tracking* of control points along trajectories including velocities and accelerations, which are optimized by using corresponding functionals (work, energy, etc) by following Hamilton-Jacobi approach; and

3. *evaluation of dynamic aspects* (forces and moments) of "weighted" elements which are meaningful for a dynamical approach to motion analysis which includes forces (Euler-Lagrange) and momenta in an extended Symplectic framework (Moment Map).

In absence of external forces, *ideal motion laws* for rigid or articulated mechanisms are generally described in terms of a Hamilton-Jacobi structural model which is *preserved by the action of the symplectic group*. From the analytical viewpoint, there is a more complete structural model (Newton-Euler-Lagrange formulation) which incorporates forces.

Dynamic approach is based on the minimization of an action integral linked to a lagrangian functional (the total energy, e.g.) which can be modified by incorporating "small perturbations". By minimizing such functional one obtains the Hamilton-Jacobi equations; in the smooth case both formulations (Hamilton-Jacobi vs Lagrange-Euler are equivalent between them.

Anyway, the whole nature of Mechanics (in their geometric, kinematic, dynamic componentes) and different kinds of “events” (modeled as singularities of fields) induce different kinds of hierarchies involving not only mechanical aspects, but sensors and actuators integration in a common Perception-Action Cycle (PAC), also. In particular, hierarchies are present to organize different steps for a structured task (in terms of different kinds of control) or, more generally, to making decisions in presence of different options (relative to Optimization Criteria).

A non-trivial problem is how to organize hierarchies described in the above paragraphs. An intuitive idea consists of using different kinds of (scalar, vector, tensor) fields which is transversal to different matters B_i of these notes. More specifically, if one has in account materials developed along the first module (Computational Geometry) of the matter B_1 (Computational Mechanics of Continuous Media), it is easy to see that Voronoi diagrams provide a general framework for weighted optimization issues involving scalar fields defined on a “stratified” scene.

The novelty consists of this scheme can be extended to Kinematic and Dynamic issues; in fact this remark is implicit if one interprets Voronoi diagrams in terms of a gradient vector field where sites are attractors, edges support saddle points and vertices are repulsors for a “conservative” approach. The incorporation of different kinds of interaction between componentes or with (other agents in) the environment is translated in terms of coupling between systems. Obviously, for more than two systems, most couplings give non-integrable systems (in an exact way); in this case our strategy consists of finding “enough near” integrable systems (with foliations as partial solutions) with “weak coupling” linked to ordinary tangency conditions.

From a mechanical viewpoint, the accumulative effect of the described hierarchy for articulated mechanisms poses problems involving to

- *initial global equilibrium* at static configurations which involves to initialization of dynamical systems (locally given by ODEs);
- different kinds of *stability* , which involves to decoupling and packing kinematic effects along motion trajectories; and
- *response capability* for the whole mechanisms, including anticipatory and compensatory controlled movements to maintain a dynamic sta-

bility of the whole mechanism. Mathematical modeling is performed in terms of boundary conditions (typical in PDEs)

Algebraic description given in terms of (submanifolds of) classical groups for configurations $\mathcal{C}$ and working $\mathcal{W}$ spaces provides a support for (de)coupling behaviors for robot kinematics and dynamics in a locally homogeneous framework which is labeled as a "symmetric" variety [7]. To start with, in this section we shall restrict ourselves to mechanical issues, leaving a more general treatment in terms of the Perception-Action Cycle (PAC) for the second section.

As a first conclusion in the Analytical Mechanics context for Robotics, hierarchies appear along *three main axes* relative to Lie actions, transference map between configurations and working spaces, and multivector approach to Mechanics.

- *Lie actions* involve to classical groups (for planning and executing motions), their infinitesimal version in terms of their Lie algebras (for estimation, optimization and control issues), and their topological extensions in regard to topological or infinitesimal actions linked to the study and resolution of non-linear systems of (algebraic, differential, analytical) equations. *Algebraic* hierarchies between groups and algebras induce natural hierarchies between invariants and relations between covariants. Breaking of symmetries linked to bifurcation problems provides new insights to ease the control of robots along phase transitions. [8]

- Small motions at joints are transferred into movements at control points (such as c.o.g. or end-effector for each kinematic chain, e.g.). This transference is represented by the transfer map $\tau : \mathcal{C} \to \mathcal{W}$ between configurations $\mathcal{C}$ and working $\mathcal{W}$. A small motion on a joint of the configurations space is initially represented by a path $\gamma : I \to \mathcal{C}$ with $I = [0, 1]$, whereas each constraint on the working space is represented by a function $\mathcal{W} \to \mathbb{R}$; several simultaneous motions are represented by a multipath $\Gamma : I^n \to \mathcal{C}$ given by $(\gamma_1, \ldots, \gamma_n)$. Hence small motions

[7] There are different kinds of discrete, continuous and infinitesimal symmetries.

[8] I have developed this approach along late nineties and it is strongly influenced by the work of J.Burdick and J.Ostrowski. Some subsequent developments were integrated for locomotion tasks in cooperation with G.Belforte (Politecnico di Torino, Italy) for reciprocators (assistance to paraplegic persons), and P.Gorce (Ecole Polytechnique, Cachan, France) for control models in regard to humanoid robots.

involving n joints submitted to p constraints are represented by means of a composition of maps

$$I^n \to \mathcal{C} \to \mathcal{W} \to \mathbb{R}^p$$

where kinematic and dynamic effects are interpreted as k-th formal *analytical* prolongations defined on the k-jets spaces. The analytic support allows to incorporate contact structure, extend the analysis from regular to singular behavior, provides natural stratifications for spaces and maps, including a simultaneous treatment of equivariant phenomena which are crucial for optimization and control issues. [9]

- A unified and more compact treatment is performed in terms of multivector calculus in the Geometric Algebra framework, which is an extension of quaternionic computations which is well known in Mechanics. The resurgence of Geometric Algebra from the mid eighties (under the leadership of D.Hestenes in USA and the Cambridge group in Europe) has been extended to Robotics and Computer Vision at late nineties. It provides an intrinsic framework for cumbersome *differential* formulations including motion laws, contact structure, transmission and propagation phenomena, natural incorporation of non-preserved quantities (energy or momentum) linked to the apparition of non-holonomic constraints, etc. [10]

The triple (algebraic, analytical, differential) distinction is ubiquitous, and it crosses transversally all the six modules of these notes. The aforementioned approaches display a complementary character from the algebraic, analytical and differential viewpoint, with a predominance of local, infinitesimal and global character, respectively. They provide multiple articulations at different levels, and shed a new insight on some central issues involving coordination, control and optimization which will be explored along these notes. In the next paragraphs we give some snapshots of these ideas.

[9] I have developed this approach in regard to the invitation by Prof. González-Sprinberg to give a course about these topics in the Fourier Institute (Grenoble, France) along March'2000; applications to eye-hand coordination have been motivated by joint developments with the team of P.Dario (SSSA, Pisa, Italy)

[10] A preliminary version inspired by locomotion models developed with Prof. Belforte (Politecnico ti Torino, Italy) was presented in AGACSE'01 (Cambridge, UK)

1.2.1. Generalized coordinates and bivectors

In the lagrangian and/or legendrian framework, geometric, kinematic and dynamic aspects are modeled in terms of *generalized coordinate* $\underline{q} = (q_i)_{i \in I}$ and their first order and second order variations.

By following the Lagrangian notation, q_i are used to denote generalized coordinates, p_i for first order variation and r_i for second order variation of generalized coordinates. An adaptation of Legendre's approach motivates the *structural contact relations* given by 1-forms:

$$p_i dt - dq_i \quad , \quad r_i dt - dp_i$$

for first and second order variations of generalized coordinates. These relations provide *structural contact constraints* which must be verified by extended functionals defined on the successive prolongations of the ambient space. Thus, they are used to describe kinematics and dynamics in terms of generalized coordinates $\underline{q}$, $\underline{p}$ and $\underline{r}$, respectively.

There is also a multivector version of generalized coordinates which is expressed in terms of bivectors corresponding to screws $\mathbf{s}$, twists $\mathbf{t}$ and wrenches $\mathbf{w}$, playing a similar role to generalized coordinates $\underline{q}$, $\underline{p}$ and $\underline{r}$, respectively. In this framework, the differential operator is the natural extension of the exterior differential of usual Exterior Calculus to the Geometric Algebra framework (see below).

1.2.2. Transfer map and its extensions

This viewpoint is extended in a natural way to articular, joints or configurations space $\mathcal{C}$ (a product of classical groups, usually) and working space $\mathcal{C}$ (a cartesian space k-dimensional for $2 \leq k \leq 4$, usually) which are meaningful for Robotics. These structural constraints are compatible with transformations acting on both spaces and operators in charge of control and/or optimization of trajectories.

Definition.- We shall denote by means $\tau : \mathcal{C} \to \mathcal{W}$ the *transference map* which transforms small impulses at joints in motions of end effector along the set of kinematic chains and the behavior of central components of the whole multibody.

Each path in articular or configurations space (small motions at joints) is topologically represented as a map $\gamma : I \to \mathcal{C}$ on the configurations space $\mathcal{C}$, which are naturally extended to trajectories $\tau \circ \gamma : I \to \mathcal{W}$ for each control point (an end-effector, e.g.). For hyperredundant robot arms or for multilegged robots with n control points one must consider multipaths $\gamma : I^n \to \mathcal{W}$ representing feasible trajectories at control points belonging to $\mathcal{W}$ which must be lifted to multipaths on the configurations space $\mathcal{C}$. This analysis is more involved and requires to analyze the homotopy linked to the topological fibration $\tau : \mathcal{C} \to \mathcal{W}$ linked to the transference map.

Every function $f : \mathcal{W} \to \mathbb{R}$ on the working space $\mathcal{W}$b (corresponding to a constraint, e.g.) induces a function $f^* := f \circ \tau \ : \mathcal{C} \to \mathbb{R}$ on the configurations space $\mathcal{C}$; typical examples are linked to control and optimization issues. This construction is naturally extended to successive formal prolongations or k-jets of any function f defined on both spaces or the transference map, even.

The k-th jet $j^k f$ of a function (germ) is the formal Taylor polynomial of f truncated at order k, which can be represented (up to coefficients) at each point by means of

$$(\underline{x}\ ,\ f(\underline{x})\ ,\ f(\underline{x})\ ,\ \ldots\ ,\ d^k f(\underline{x})) \qquad \text{for } k \geq 0$$

Hence, given a function $f : \mathcal{W} \to \mathbb{R}$ and by using the chain's rule, their k-jets $j^k f$ induce $\tau^*(j^k f) := j^k(f \circ \tau)$. This construction can be encapsulated as

$$j^k \tau^* : J^k \mathcal{W} \to J^k \mathcal{C} \qquad k \geq 0$$

where successive prolongations $j^k \tau^*$ of transference map τ are formally represented in terms of k-jets $j^k(f \circ \tau)$ of sections corresponding to "structural sheaves" $\mathcal{O}$ for configurations $\mathcal{C}$ and working $\mathcal{W}$ spaces. Thus, $J^k \mathcal{W}$ or $J^k \mathcal{C}$ is an abuse of notation, because $J^k \mathcal{E}$ involves to the k-jets os sections of a fiber bundle or, more generally, a sheaf $\mathcal{F}$. However, one has preserved the original notation because it has a more intuitive character.

An unexpected advantage of this notation consists of any ODE or PDE used to describe kinematic or dynamic models, can be reformulated as a multilinear form (with variable coefficients in a coordinate ring) in the space of k-jets, which provides a formal simplification of the original context. Usually, coefficients are algebraic functions and thus, such functions are represented by algebraic equations in an abstract space $J^k \mathcal{E}$ which provides the support for geometric, kinematic and dynamic aspects in a common framework.

So, in particular if we work with a planar robot with local coordinates (x, y) for the end-efector, and we wish to represent its motion in the working space $\mathcal{W}$, then every PDE can be represented as an algebraic relation between the coordinates corresponding to k-jets of a function $f : \mathcal{W} = \mathbb{R}^2 \to \mathbb{R}$ given by

$$(x, y\ ,\ f(x,y)\ ,\ f_x, f_y\ ,\ f_{xx}, f_{xy}, f_{yy}\ ,\ f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy}\ ,\ \ldots)$$

where generalized coordinates are submitted to structural constraints in terms of contact geometry, also [11].

As conclusion, morphisms between k-th Jet Bundle of Fiber Bundles $\xi = (E; \pi, X)$ or more generally a sheaf $\mathcal{F}$ provides a general framework to represent any kind of Kinematic or Dynamic phenomena linked to the Robot Mechanics. Locally, it is given by maps between k-th order jets of sections of fiber bundles or, more generally, sheaves.

More generally, let us denote by $\mathcal{O}_{\mathcal{X}}$ the sheaf of C^r-regular functions on the base space $\mathcal{X}$ representing the configurations $\mathcal{C}$ or the working $\mathcal{W}$ space. Then, several systems (inter)acting in a simultaneous way with a robot $\mathcal{R}$ can be represented by means of $J^k(\mathcal{E})$ where $\mathcal{E}$ denotes a formal product (locally given by tensor fields) representing distributions of fields or systems of differential forms with coefficients belonging to the "structural sheaf" $\mathcal{O}_{\mathcal{X}}$.

In particular, this notation provides a compact notation for an analytic representation of inverse geometric, kinematic and dynamical aspects involving the mechanics of robotic devices. Obviously, this notation can be applied also to the analytic spaces representing the Perception $\mathcal{P}$ and Action $\mathcal{A}$ spaces linked to the Perception-Action Cycle, because all information about these issues is represented in terms of functions, and their k-th order "changes" are represented by vector or tensor fields defined on $J^k\mathcal{X}$ for $\mathcal{X} = \mathcal{P}$ or $\mathcal{X} = \mathcal{A}$, respectively.

According to the general approach performed above [12], from a formal viewpoint, $J^k(E)$ provide a general framework for a geometric reformulation of analytical aspects concerning to any kind of differential equations, optimization procedures and all kinds of control. This construction is extended in a natural way to any kind of morphisms (i.e. maps not everywhere defined), incorporates different kinds of hierarchies and is compatible with the presence of singularities involving spaces and morphisms between spaces $\mathcal{X}$.

In regard to robotic applications, the global management conditions relative to control of trajectories and incorporation of constraints is performed in terms of the Main Analytical Diagram (MAD). More formally,

[11] Let us remark that the 0-jet of a function f represents the graph Γ_f of f, and the 1-jet can be represented by the gradient vector field onto the graph

[12] For additional details see the module 4 (Computational Kinematics) of the matter B_1 (Computational Mechanics for Continuous Media)

it is constructed by adding to the transfer map τ the information relative to paths or trajectories $\gamma_i(t)$ for each control point (including end-effectors) and different kinds of constraints g_k defined on ambient spaces $\mathcal{X}$ representing the configurations space $\mathcal{C}$ or the working space $\mathcal{W}$. The 0-jet of a (scalar or vector) function $f : X \to Y$ is by definition the graph $\Gamma_f := \{(x, y) \in X \times Y \mid y = f(x)\}$ of the function f; one has two canonical projections onto the first and second component.

To fix ideas, let us denote by $j^0\tau$ the 0-th extension of $\tau : \mathcal{C} \to \mathcal{W}$ representing the graph which provides a "structural constraint" linking small movements at joints and eventually large motions at end-effectors (obtained as composition). Their k-th extensions given by successive differentials provide "structural contact constraints" corresponding to relations which are represented by multilinear forms associated to variation rates of independent and dependent variables linked by the components of τ; in particular, the local version for $k = 1$ is given by the Jacobian matrix. In this framework vector and covector fields are represented by local sections of successive prolongations; their direct and inverse images are computed in terms of push-down and pull-back procedures according to usual rules for composing maps. In this way, one obtains a natural representation for usual ODEs and PDEs, which can be extended to the global case. All this information can summarized in the *Extended Main Analytical Diagram* (EMAD) which can be written (abuse of notation) as

$$\begin{array}{ccccccc}
J^0 I^n & \to & J^0\mathcal{C} & \to & J^0\mathcal{W} & \to & J^0\mathbb{R}^p \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
J^1 I^n & \to & J^1\mathcal{C} & \to & J^1\mathcal{W} & \to & J^1\mathbb{R}^p \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
J^2 I^n & \to & J^2\mathcal{C} & \to & J^2\mathcal{W} & \to & J^2\mathbb{R}^p
\end{array}$$

The EMAD provides a geometrical (for $k = 0$), kinematic (for $k = 1$) and dynamical (for $k = 2$) representation for Robot Mechanics in terms of k-th prolongations $j^k\tau$ of the transfer map τ. An advantage of this representations consists of the incorporation of a contact structure (Legendre) to each level, which provides structural constraints for integrating motion equations. In the left part, we have included the collection of n paths to be performed at each joint and the number p of control points (end effectors and center of gravity, at least) which are associated to tasks to be performed and constraints according to different kinds of functions or functionals.

From an analytic viewpoint, all vertical and horizontal maps are stratified maps by the rank of its differential. In other words, local triviality of stratified maps of the EMAD (only true in generic open sets) allows to construct "descent strategies" by means of integration along the fiber (for each locally fibration), according to general principles of regular fibrations (Ehresmann, Thom). Their extension to the singular case displays some problems which require additional mathematical tools [13]

The use of an extended geometric framework allows not only a more compact treatment of systems of differential equations, but the introduction of geometric tools for a geometric treatment of solutions (involving Lagrangian or Legendrian subvarieties, e.g.) and constraints involving Kinematics and Dynamics. Nevertheless, not all functions defined on enlarged spaces (a Lagrangian for kinematics, a Lyapunov control function for dynamic control, e.g.) arise from functions defined on the ambient space. Additionally, one can consider n paths performed simultaneously on n control points of configurations space, or p functions acting simultaneously on each meaningful point of the working space.

An important advantage of the EMAD is the natural extension of geometric constraints to kinematic constraints (by using 1-jets), and the natural extension of kinematic constraints to dynamic constraints (by using 2-jets). All dynamical systems involving such extensions are re-formulated in terms of algebraic subvarieties of the corresponding k-jets spaces. Furthermore, the existence of a natural "contact structure" on any k-jets space provides additional constraints for an ideal mechanics. In particular, kinematic and dynamic aspects of eventually changing trajectories and constraints can be incorporated in terms of subvarieties which can read in terms of vertical and horizontal arrows of the EMAD (considered as fibrations), without performing additional hypotheses about an unlikelihood linear character.

Additionally, *non-holonomic constraints* appearing in Robot Mechanics are incorporated in a natural way to the above scheme, in terms of non-integrable subvarieties in the successive extensions. In other words, they appear as subvarieties in $J^k\mathcal{X}$ which do not descend to give subvarieties in $J^{k-1}\mathcal{X}$ and which require a special treatment in terms of the enlarged spaces[14] From a more down-to-earth approach they appear in kinematic spaces (resp. dynamical spaces) from constraints which do not arise from constraints in the geometric spaces (resp. kinematic spaces).

The simplest integrable example is given by contact constraints invol-

[13]See the module A_{45} (Stratifications) of the matter A_4 (Topología Diferencial) for details and references.

[14]This phenomenon was well known in the 19th century and has bee treated in terms of the Phase Space P (Poincaré) or its co-tangent bundle.

ving to "new variables" which live in the cotangent space; however, they give integrability conditions for Legendrian subvarieties which are maximal dimension subvarieties for contact constraints. Additionally, there are a lot of kinematic and dynamic effects which are non-integrable, in regard to the geometry of the space (non-holonomic effects) or the kinematics of actions (sliding effects, e.g.) or unexpected dynamic effects (friction and corresponding drift effects, e.g.). All of them can be characterized as non-integrable varieties in the total space representing the kinematics or the dynamics. Mathematical models are known from the last years of the 19th century, but they have been incorporated to mathematical models of Robotics along early years of the 21st century. Obstructions to lift paths or to integrate distributions can be read directly on the EMAD in topological terms. [15]

1.2.3. A multivector approach

From the topological viewpoint, a *multipath* is given by a finite collection of paths involving small motions at joints in $\mathcal{C}$ or simultaneous evolution of at least two control points in $\mathcal{W}$. Hence, the analytical description given at the precedent paragraph can be embedded in a global treatment in terms of multivector calculus involving the subspace generated by the control points of mechanism, in regard to the a finite references of the ambient space. The resulting multivector information can be managed in terms of Clifford algebras.

If we adopt a (multi)vectorial framework, then geometric, kinematic and dynamic aspects of meaningful elements for a robot $\mathcal{R}$ can be described in terms of finite collections of points $\mathbf{P}_i \in \mathcal{R} \subset \mathcal{W}$, segments $< \mathbf{P}_i, \mathbf{P}_j >$ and planar elements $< \mathbf{P}_i, \mathbf{P}_j, \mathbf{P}_k >$ which are describes as simplices. Their management can be described in terms of points $\mathbf{P}_i$, lines ℓ_{ij} and planes π_{ijk} verifying *incidence conditions* such as

$$\mathbf{P}_i \in \ell_{ij} \quad , \quad \ell_{ij} \subset \pi_{ijk}$$

Every action on a point $\mathbf{P}_i \in \mathcal{R}$ has effects on segments s_{ij} and planar elements π_{ijk} to which $\mathbf{P}_i$ belongs. Incidence conditions are projectively invariant and characterize "flags" which provide "universal" models for all kind of hierarchies between geometric elements (including curved ones).

A *complete flag* in a 3D space is given by a a collection of subspaces $(p, \ell, \pi, \ldots)$ verifying incidence conditions $p \in \ell \subset \pi \subset \ldots$. There exists a

[15]These aspects will be developed in the module 3 of these notes.

synthetic description of flag variety in terms of quotients of classical group; to obtain an intrinsic representation (independent of observer's viewpoint), it is convenient to adopt a projective notation. This description is valid not only for the geometric framework, but for any kind of eventually infinite-dimensional vector "subspaces" such those appearing in jets spaces. To begin with, we shall restrict ourselves to the geometric case.

The*simplest case* is given by the complete flag variety in the plane given by $\{(p,\ell) \mid p \in \ell\}$. It provides a support fo configurations of planar robots given by points and lines verifying incidence or tangency conditions (projectively invariant and dual conditions) corresponding to pass through a point (linked to position-force based control) or be tangent to a line (linked to impedance-based control) for trajectories. Nevertheless its simplicity, the geometry of planar flags is non-linear: it is a hyperplane section of the image of the Segre map $s_{2,2} : \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^7$. The compactification of feasible trajectories w.r.t. to the above conditions provides a support for all planar motions which can be alternately described in multivector terms [16]

A more involved case corresponds to complete flags in $\mathbb{P}^3$, i.e. triplets of points p, lines ℓ and planes π in a projective 3-space $\mathbb{P}^3$ verifying incidence conditions $p \in \ell \subset \pi$; an advanced application concerns to the description of configurations for Stewart platforms (in regard to flight simulators, e.g.) whose configurations space is not well understood, still. In this case, our hierarchical approach develops some aspects of flag bundles for the analysis of special configurations (including generic singularities) and control issues. A general understanding of singularities arises from describing the flag manifold as the projectivized quotient G/B of $G = GL(4;\mathbb{R})$ by the stabilizer B of a complete flag. The product of 4 copies of $GL(1;\mathbb{R}) \simeq \mathbb{R}^*$ (a 4-dimensional torus) parameterizes the hierarchies between closures of orbits (involving to phase transitions), which is given in this case by the permutations group S_4 of 4 elements [17].

The above arguments are easily extended to other flags verifying additional conditions (orthogonal, unitary, symplectic, etc) and also to "incomplete" flags corresponding to robots with external constraints [18].

[16]See last chapters of the first module for details and applications

[17]In general the orbits of the *complete* flag variety of a vector space are parameterized by the permutations group of $n+1$ elements, where $n+1$ is the dimension of the ordinary space. Thus, hierarchies can be easily described even the global geometry is much more involved.

[18]An incomplete flag is a collection of nested subspaces not necessarily of consecutive dimensions. Any Grassmannian provides the simplest example of a variety of incomplete flags.

The multivector-based approach is common to different architectures (serial manipulators vs parallel platforms, e.g.), and is more intrinsic than the more usual Lagrangian-Legendrian approach [19]. Furthermore, it provides a natural geometric support for position-, tasks- and force-based control in working space (by using points, segments and planar elements), and a natural feedback between all of them; the use of representations based on flag varieties provides very simple descriptions for feedback which are compatible with constraints intrinsically given in terms of cycles in flag varieties.

To begin with, one must specify a general framework to develop this approach. The leit-motiv is that a multivector reformulation of the Computational Geometry provides the support for Computational Kinematics and Dynamics of Robots. The most intuitive multivector reformulation of Computational Geometry is given by exterior calculus for static configurations and differential forms to describe dynamic aspects. A little bit more sophisticated approach is given by Clifford Algebra and Clifford Analysis. Both of them provide a support for non-linear modeling and adaptive control issues. With more detail:

1. *Computational Geometry* (module B_{11}) provides models, databases and algorithms for a semi-automatic treatment of problems which admit a planar or spatial representation. It uses configurations of simple geometric 2D (points, segments, triangles, circles) or 3D entities (simplices, cuboids, spheres, generalized cylinders) which can be grouped in more complex geometric entities and related between them by simple transformations. By patching together these pieces, it is possible to develop the study of geometric configurations which can be rigid ones (most vehicles, mobile platforms), and serial vs parallel manipulators (arms and legs, vs artificial hands or hips, e.g.). Furthermore, Computational Geometry provides modeling tools for 3D objects which can be labeled as structured (triangular vs quadrangular deformable meshes, e.g.) or non-structured (based on dense or sparse clouds of points, e.g.). It uses linear transformations to represent real or virtual displacements at joints or motions in the ambient space. Computational Geometry provides a support for robust control methods involving position and/or orientation, such that reachability and accessibility issues for end-effectors. Their extension to Piecewise-Linear case (meshes) is performed in the module B_{12} (Computational Algebraic Topology or CAT), whereas its extension to the smooth case (differential manifolds)

[19] Intrinsic character avoids to update coordinates w.r.t. an initial reference, and avoids typical errors linked to evaluate localization and information updating

is performed in the module B_{13} (Computational Differential Topology or CDT)

2. *Computational Kinematics* (module B_{14}) is focused towards evaluation and tracking along time of planar or spatial configurations in the ambient space and how can be generated from trains of impulses at joints. Thus, is is the natural extension of Computational Geometry in configurations and working spaces. In particular, Computational Kinematics provides a structural framework for (eventually non-linear) oriented tasks programming of robots; each task is represented as a trajectory in the space-time whose kinematic characteristics (location, velocity and acceleration at control points) is necessary evaluate and control, without losing the stability. The simplest strategy consists of controlling the center of mass around a stable trajectory of the whole robot. More involved strategies consist of task-oriented programming which are linked not only to evolution and tracking of a control point (end-effector, e.g.) as for Computational Geometry, but a finite collection of segments $s_{ij} :=< \mathbf{P}_i, \mathbf{P}_j >$ which are connected between them in a skeletal structure. Hence, in this multivector framework Computational Kinematics and tasks-based control are the natural extension of Computational Geometry and position-based control.

3. *Computational Dynamics* (module B_{15}) adds the effects linked to massive objects in terms of forces and moments which act on the above described configurations of control points. It is evaluated by reinterpreting second order derivatives, having in account inertial phenomena, and anticipatory/compensatory effects. They are expressed in terms of differences of first and second order involving the generalized coordinates. Forces acting on the mechanism are not isolated, and usually are presented in terms of pairs; furthermore (pairs of) forces, one has torques acting on articulated mechanisms. All of them determine a collection of planes along which we must evaluate their dynamical effects. The geometric hierarchy based on points-lines-planes verifying incidence conditions $p \in \ell \subset \pi$ provide a feedback between position-tasks- and force-based control for complex motions such as locomotion, and/or grasping and manipulation tasks. Hence, Computational Dynamics is the natural extension of Computational Kinematics which has been presented above. A novelty of our approach consists of incorporating a kinematic-based control which closes the scheme of position-force control, by contributing to stable configurations and

avoiding vibrations near to singular configurations.

Hence, Computational Geometry provides a first (static) step with successive prolongations given by Computational Kinematics and Dynamics, which provide a support to different tasks and interactions with environment [20]. This claim is extended in a natural way to the study of transformations (which is modeled in terms of Lie Groups G and Lie Algebras $\mathfrak{g} := T_eG$ as natural extension of rigid motions in the space), and control and optimization issues (involving the execution of motions and tasks).

The approach performed in terms of Lie Groups G is justified by the G-invariance of structural equations w.r.t. the action of a Lie group G and by the preservation of functionals under ideal conditions. Typical "examples" for Mechanics of Rigid Solid are given by the Moment Map, but this approach has been extended to more general situations involving articulated mechanisms and the mechanics of deformable media in terms of the Moment Map. This unifying approach can be extended, also, to several aspects of Computer Vision concerning to 3D Reconstruction and Motion Estimation in an extended SLAM framework. [21]

Thus, invariant functionals and structural equations provide the starting point to be modified from the interaction with environments and the energy expenditure, e.g.. Such modifications can be understood as "deformations" in a stratified equivariant framework, where the algebraic action can be understood in algebraic, differential or infinitesimal terms. All of them are related in a more general topological framework. Hurewicz fibrations provide general models for their interrelations.

In fact, the Lie-based approach has been exploited from the early nineties giving a motivation for the reformulation of Robot Mechanics in terms of group actions (or their infinitesimal version) and, more generally, in terms of Principal Bundles; it has been exploited by J.Burdick, J. Ostrowski and his collaborators, between others. On the other hand, Geometric Algebra provides a framework for unifying different aspects arising from Robot Mechanics; a high-level approach can be read in [Bay01] [22].

[20] A Spanish version of Computational Mechanics as an extension of Computational Geometry is available in my web site

[21] See the introduction to the module 4 (Computational Kinematics) of the matter B_1 (Computational Mechanics of Continuous Media) for details and references.

[22] E.Bayro-Corrochano: *Geometric Computing for Perception-Action Systems*, Springer-Verlag, 2001.

1.2.4. The role of singularities

Roughly speaking, singularities appear in any context as "events" of some static or evolving process. If one models such process in terms of (scalar, vector, tensor) fields. Thus, events can be characterized as singularities of fields. If fields were "isolated", then the classification problem of fields would be "easy" (already performed from the local viewpoint for the integrable case).

Unfortunately, things are a little bit more complicated because interacting fields (linked to coupling conditions, e.g.) generate (dis)aparition phenomena linked to evolving singularities. In this matter, one does not intend to solve the general problem, but to "illustrate" some troubles appearing in Robotics and give some hints about how to solve them. [23]

Usual approach to Robot Mechanics is usually restricted to regular regions. However, singularities are present at the three levels for Mechanics of Robots:

1. *Geometric level*: They involve to singular points of functions or more generally functionals involving joints at configurations space $\mathcal{C}$ or control points at working space $\mathcal{W}$. They are responsible of shape changes (of paramount importance in Biomimetics approach for living beings, e.g.).

2. *Kinematic level*: They involve to a rank deficiency for the Jacobian matrix representing the forward kinematics of the robot. They are responsible of phase changes in the (co)tangent space.

3. *Dynamic level*: They involve to (de)coupling between external and/or internal forces or moments acting onto components or the whole mechanism. They are responsible of sudden or qualitative changes in dynamics, represented by limit cycles acting as organizers of dynamics.

"Intrinsic singularities" are linked to robot architecture, and appear typically at the boundary of configurations or working spaces; additionally, "extrinsic singularities" are linked to rank deficiency locus of maps linked to tasks, and they can appear inside regular regions also. All of them give a "segmentation" (decomposition in a disjoint union of regular regions) of the ambient space $\mathcal{X}$ for the Robot Mechanics.

[23] A more general approach can be read in several modules of the matter A_4 (Topología Diferencial).

Hence, singularities are present at the three levels of Robots Mechanics, involving geometric, kinematic and dynamic aspects. Furthermore, the natural stratification linked to the transfer map $\tau : \mathcal{C} \to \mathcal{W}$ (in terms of jets spaces) shows how singularities at each level are propagated to other levels (by means sections of fibrations, e.g.).

Usually, singularities are avoided in Robotics because (following a very extended belief) traditional methods for control and optimization are no longer valid outside the regular region. Hence, in the classical regular framework, it is necessary to design and implement different control methods for different "modes" of complex robots, and switching procedures to exchange such control methods. This approach is not natural, because not only terrestrial mammals but insects have solved this problem in a more natural way passing through singularities.

Key features for solving control issues near to singularities are related to the management of symmetries and their evolution according to breaking procedures. Nature has solved this problem from several millions of years by developing symmetric architectures which are controlled by components with opposite functionalities to balance their global effect and restore the equilibrium or, more generally, the stability along the execution of movements. Some advanced examples are linked to anticipatory and compensatory movements along locomotion tasks, which involve to a combination of geometric, kinematic and dynamic aspects, with their corresponding position-, impedance- and force-based control devices. Along the modules 5 and 6, one introduces a more detailed presentation.

In addition of these geometric (axial or central) symmetries, one can find symmetries involving kinematic aspects (involving behavior of trajectories performed by control points, e.g.) and dynamic aspects (involving qualitative changes in transference maps $j^2\tau$). They can be represented in terms of infinitesimal symmetries involving "generic" deformations around each local type of singularities . This strategy is supported from the mathematical viewpoint by the introduction of ordinary and infinitesimal symmetries on the k-th extensions $j^k : J^k\mathcal{C} \to J^k\mathcal{W}$ of the transfer map $j^0\tau = \tau$.

The key result in our approach is the proof of an equivariant stratified structure for $j^k\tau$ which provides the support for equivariant bifurcation linked to the mechanics of articulated mechanisms. The general framework is introduced in module 3 (Kinematics) and their treatment involving equivariant bifurcation is developed in the fourth module (Dynamics and Control). Most difficult problems concern to non-linear constraints linked to non-integrable distributions on the total space for kinematics (the cotangente space in the classical terminology) and the dynamics spaces.

1.3. Physical-mathematical models

Along the 19th century, Symplectic (Lagrange) and Contact (Legendre) Geometries have provided the support for the Hamilton-Jacobi formulation of Classical Mechanics, and geometric-topological properties of solutions for structural equations of Classical Mechanics. In the smooth case, Symplectic Geometry provides the support to prove the equivalence between differential approach (Hamilton-Jacobi) and integral approach (Euler-Newton). A modern reformulation of lagrangian and legendrian Mechanics with simplectification and contactification structures on the phase space has been performed till the mid eighties by V.I.Arnold and his school (Moscow, Leningrad).

On the other hand, the *symplectic geometry of the moment map* provides an invariant decomposition w.r.t. ordinary symmetries in regard to kinematic and dynamic aspects of robot motions. A general and very complete reformulation for the smooth case can be read in [Mar99] [24]. The extension of this approach to eventually singular varieties and stratified maps linked to the transference map τ is the next step to be accomplished.

In the *stratified case*, the first goal is to obtain an equivariant decomposition linked initially to functions (or more generally functionals for optimization issues) defined on eventually singular varieties [25]. All results for the regular case are still valid in the open general stratum of any stratification.

However, these results can be extended to intrinsic or extrinsic singularities . A natural topological context for this development is initially given by Morse Stratified Theory (Goresky and Macpherson, 1986) where one replaces isolated critical points by "singular cells" which are obtained by matching Tangential and Normal Morse Data; roughly speaking they support smooth (TMD) and singular (NMD) information about the system evolution.

An intrinsic formulation from the topological viewpoint requires some cohomology theory which allows to patch together functions, (distributions of) vector fields or (systems of) differential forms which are defined locally, only. However, the mathematical formalism is harder than usual to be de-

[24] J.E.Marsden and T.Ratiu: *Introduction to Mechanics and Symmetry* (2nd ed), Springer-Verlag, 1999

[25] Mathematical foundations are developed along the matters A_3 (Geometría Algebraica) and A_4 (Topología Diferencial) in my web site

veloped in an applied field as the Robotics [26].

Thus, along this introductory chapter one adopts a more down-to-earth approach which involves to matching locally defined functions or functionals f linked to Optimization and Control issues. In this framework, the main tool is the analysis of critical locus of f and the topology linked to such locus.

Differential calculus on manifolds (regular loci of varieties and maps) is commonly used to identify behaviors, to evaluate the “transport” of geometric quantities (tensors) in terms of connections, and to identify (curvature) invariants linked to such connections. Furthermore the explicit resolution of motion equations, integral calculus on manifolds provides an accumulative measure of “geometric quantities” linked to the evaluation of functionals on data distributions.

It is necessary to extend the above functionalities to non-regular loci for varieties or singularities of maps between manifolds. Our strategy consists of combining some general principles of stratifications and algebraic actions. Roughly speaking, algebraic actions are introduced to represent different kinds of (algebraic, topological, infinitesimal) interaction; but this introduction modifies the original (differential) stratification. Thus, the notions of G-equivariant stratification (an extension of locally symmetric space) and Equivariant Bifurcations play a fundamental role.

From a complementary viewpoint in order to improve the applicability of models arising from advanced geometric models, they must be compatible with metrical aspects linked to uncertainty about measures, errors in the execution and unexpected reactions arising from interaction with the environment. In other words, one must adopt an extension of Riemannian Geometry as general framework. First of all, there are several sources for uncertainty which arise from sensors, information processing and analysis, re-projection on a common space for representation, decision making near to singular cases, and errors propagation along articulated mechanisms, when commands are executed.

Some hard mechanical problems to be solved concern to damping of vibrations and correcting errors which are propagated along articulated mechanisms. Usual formulations (based on Lyapunov functions, e.g.) are not enough to solve this problem. It is required a global analysis based on (distri-

[26] Some elements are presented in my notes for Algebraic Geometry (for cohomology of distributions issues), Differential Topology (for Stratified Morse Theory) and Computational Mechanics (including advanced visualization issues), whose introductions are available in my web site (currently in Spanish language, only)

butions of) fields involving eventually non-linear effects on the whole mechanism. Non-linear phenomena are a source of unstability which is necessary to control and correct by evaluating local differences between the current and expected observations.

Proportional-derivative Control is inspired in the differential and integral calculus for solving ODE linked to stabilization procedures. An identification of non-holonomic constraints (linked to real or virtual displacements of geometric quantities or friction effects, e.g.) allows to design strategies to correct undesirable effects in order to improve the efficiency of robots.

1.3.1. Applied Classical Mechanics

Two general approaches to Classical Mechanics are based on the method of forces and moments (Euler, Newton) or the verification of a system of PDE (Lagrange, Hamilton, Jacobi) for a Hamiltonian function $H : TM \to \mathbb{R}$ (initially given by the total energy of a system). Both of them are used in classical books in Mechanics. In the context of symplectic geometry, both formulations are equivalent between them in the regular case. Roughly speaking and in a modern language, Euler-Newton follows an integral approach which is based on minimizing an action functional (called Lagrangian); resulting equations arising from variational principles satisfy the Hamilton-Jacobi. Inversely, if H is the total energy of a system, its minimization gives the classical Euler-Newton formulation.

The above description can be easily extended to systems including the presence of external forces, including generalized Lagrangians. This formalism provides a general framework for Differential Kinematics and Dynamics of Robots, and their applications to Trajectory Planning, Motion Planning and different kinds of control. A good reference covering all the above topics is [Sci00] [27]

The introduction of Lie groups and their corresponding infinitesimal version given by Lie algebras provide a general framework to incorporate ordinary symmetries linked to movements of end-effectors, invariance of structural equations for kinematics and dynamics, and infinitesimal symmetries giving integrant factors for structural equations, linked to preservation of "geometric quantities". Unfortunately, this scheme is too ideal for more advanced applications, where equations are non-linear, systems are not conservative and there appear non-integrable systems with constraints which

[27] L.Sciavicco and B.Siciliano: *Modelling and Control of Robot Manipulators* (2nd ed), Springer-Verlag, 2000.

do not arise from the geometry of the base space (configurations or working space).

A relevant topic which is not covered by the classical approach concerns to the apparition of holonomy non-trivial, which is linked to the existence of constraints which do not arise from the geometry of configurations or working spaces. Some typical examples are the following ones:

- *kinematic effects* linked to drift or sliding effects, or more recently the self-motion of a hand-held camera for environmental reconstruction of 3D scenes;
- *dynamic effects* linked to different kinds of frictions and non-linear mechanical vibrations.

Both of them can be represented by means of a non-integrable subbundle which generates distortions at the level of expected trajectories or forces-and-moments distributions. Their combination increase the troubles for an efficient control of robots in complex tasks involving the navigation, at first instance and/or, grasping and handling tasks which involve to more complex interactions with the environment.

1.3.2. A functional approach

Description of tasks as trajectories and optimization in the space of paths requires some additional elements arising from Topology; the identification of non-contractible paths in surfaces and/or "tunnels" in 3D representations are important to identify which kind of interpolations can be performed by preserving the topology [28] .

Furthermore, all feasible paths performed by the end-effector of any kinematic chain must be given by a composition of rotations and translations, which imposes additional constraints for feasible motions. Hence, smooth ideal trajectories must be approached by Piecewise Linear (PL) or Piecewise Quadratic (PQ) trajectories compatible with mechanical constraints, even if these objects are not dense in the working space. Thus, it is necessary to balance the need of accuracy with the real performance of mechatronic devices, and this goal includes the errors evaluation and their tracking along the tasks execution.

[28] A tunnel is a 2-dimensional non-contractible cycle; is the natural extension of the notion of non-contractible path whose homotopy classes generate the fundamental group $\pi_1(X)$ of a surface X.

Functional aspects involve to dynamical optimization (relative to trajectories to be performed, e.g.) and control issues (relativa to different kinds of stability, e.g.). Both of them can be developed in a joint way by using variational principles which are ubiquitous in structural models for Robotics. The classical integro-differential formulation is labeled as Newton-Euler approach, and minimizes a Lagrangian action functional (such as the total energy of the system, e.g.).

A recent extension based on the minimization of a D'Alembert action functional allows to incorporate non-holonomic constraints which generate drift or sliding effects in Robotic mechanisms. Anyway, one has a collection of infinitesimal symmetries which provide integrant factors, even if the distribution is not completely integrable; Emmy Noether was the first one in studying this type of phenomena at the end of the 19th century, which now are included as gauge transformations in variational problems.

1.3.3. Differential and integral aspects

Differential aspects involve to multiple aspects related to the support for the kinematics or the dynamics of a robot (given initially by a manifold), transformations at joints or in the working space (given initially by Lie groups, i.e. groups with a differentiable structure), tasks (given by integral curves of vector fields), evaluation of performance (given by differential forms on evolving geometric quantities), and geometric grouping of data (given by tensors which are transformed accordingly to covariant transformations and can include non-linear effects, also).

Smooth boundary manifolds $(M, \partial M)$ and their superimposed structures (fiber bundles or principal bundles, e.g.) provide initial models, but smoothness conditions can not be globally fulfilled due to the apparition of singularities (components alignment, rank default for Jacobian or matrices linked to Dynamics), discontinuities at interaction (including relocation for grasping, contact or friction issues for handling, jumping or running, e.g.) . Along these notes, we develop a *semi-analytic extension* of smooth framework where differential and integral calculus are still valid, even the notation is a little bit more complicated.

Some crucial problems to solve from the *differential viewpoint* involving to dynamic aspects concern to

- *Optimization issues* given initially by the minimization of a Lagrangian action functional, which provides a unification of differential (Hamilton-

Jacobi) and integral (Newton-Euler) approaches for smooth varieties in the framework of symplectic and/or contact geometries.

- *Control issues* (controllability, stability, reachability). Vector formulation of these issues can be easily reformulated in a more intrinsic terms by using the Geometric Algebra language.

Some developments from numerical analysis holding along eighties and nineties have provided methods and solutions for particular problems in Robotics, but approximation methods are of a limited applicability in absence of a geometric or, more generally, topological approach. Unfortunately, the scarce development of discrete approaches to Differential Geometry has acted as a barrier till the beginning of the 21st century. Recent developments of discrete versions for (distributions of) vector fields and (systems of) differential forms are providing a more versatile approach from the computational viewpoint [29]

Integral aspects concern primarily to solvability criteria and numerical solutions of functionals linked to structural motion equations or discrete versions for the information fusion. More meaningful are the issues related to integrability of distributions $\mathcal{D}$ of vector fields $\xi_1, \ldots, \xi_k$ or systems $\mathcal{S}$ of differential forms $\omega_1, \ldots, \omega_\ell$ linked to evolving systems superimposed to a variety. A crucial issue is the *integrability* of the system which, in the smooth case, is represented by means of a vector subbundle E of the (co)tangent bundle of M.

Most realistic systems are not globally integrable and one must take care with local regions of integrability and the detection of obstructions for global exact integrability. A source for the non-integrability of distributions or systems is linked to the existence of non-holonomic constraints on the (co)tangent bundle, i.e. which do not arise from geometric constraints in the base space. Typical examples are given by the drift (for kinematics) or different types of friction (for dynamics).

Partial integrability of systems provides local infinitesimal symmetries (in the sense of E.Noether) linked to D'Alembertian action functionals which must be modeled, by adding terms to the original formulation of the dynamics, which allow to control the behavior even for non-linear phenomena. Their estimation and tracking poses additional changes which are revised in the next paragraph.

[29] See my Course on *Mecánica Computacional* (in Spanish language) for additional details and references.

1.3.4. Managing the uncertainty

Uncertainty is pervasive throughout robotics; it arises from incomplete information linked to sensors, high ratio for noise/signal, outliers arising from information processing and analysis, oscillating behaviors or more general unstable phenomena, unexpected vibrations, et cetera. The diversity of sources and the need of providing robust or adaptive responses (depending on the context), suggests the development of a hierarchy including mechatronic devices and logical procedures for managing the above problems in terms of a large diversity of patterns.

Distributed systems provide an articulation between mechatronic devices and logical procedures. So, in a similar way to superior mammals and depending on stimuli, some components can react by following reflex mechanisms, whereas other components require more advanced making decision procedures which involve to different kinds of logical procedures. From a methodological viewpoint, one can distinguish several tools linked to logic of classes, propositional logic and descriptive logic:

1. *Logic of classes* is applied for modeling *reflex motions*, which require only appropriate thresholds for decision making relative to the information provided by sensors. From the biological viewpoint they correspond to the lowest level involving the reflex system (useful for like-insect robots, microbots, etc) and some automated tasks performed by cerebellum in the case of superior mammals. Optimization involves to the geometrical design for identifying the most efficient architectures to perform such reflex motions.

2. *Propositional logic* must be translated in a planned concatenation of motions involving different subsystems whose signals must be reinterpreted in terms of their accumulated effects. It is applied for modeling complex tasks involving planning, anticipation and/or compensation of effects. Optimization involves to the kinematical design for identifying the most efficient architectures control mechanisms under complete information arising from the scene or the internal mechanisms. Tasks can be performed in an unsupervised way, in a similar way to the programming based on logic of classes.

3. *Descriptive logic* is applied to making decisions in presence of uncertainty. It is the highest level for programming languages, and requires fault tolerance solutions to solve problems under incomplete information. It can include a dynamical feedback arising from the interaction

with the environment, and it is applied to modeling the Perception-Action Cycle (PAC) in complex tasks. Optimization issues involves a dynamical design to accomplish the fusion of information arising from different sensors, decision making, and real-time supervised control of performed tasks. Tasks can be performed in an supervised way, including the possibility of self-adapting parameters in terms of the environmental response.

Nevertheless the existence and efficiency of randomized versions of algorithms in Computational Geometry and Kinematics [30], most existing motion planning and control algorithms in robotics does not take uncertainty into account. This viewpoint is not compatible with reasoning patterns of superior mammals which evaluate different choices in their daily behavior.

From the artificial viewpoint, there are a lot of hard problems for which only NP algorithms are known, which can not be solved by using logic of classes and/or propositional logic, because last ones analyze all possibilities before taking a decision. It is necessary to combine deterministic with probabilistic approaches in hybrid models, and to design a feedback between different models for decision making in a fault-tolerant way.

The *probabilistic approach to Computational Geometry* has provided a lot of algorithms which are much more efficient than their deterministic version. This approach is particularly useful for modeling complex tasks related to PAC (Perception-Action Cycle). In other words, a proper handling of uncertainty will almost certainly lead to significantly more robust systems, along with a better understanding on how to perceive and act in the physical world. Besides the obvious military applications, the importance of uncertainty will increase as robots move away from factory floors into increasingly unstructured environments, such as consumption or home robotics, or more advanced animats or humanoid robots.

There are a lot of problems which are characterized by meaningful uncertainty levels, and hence would benefit from better mathematical and computational tools to make decisions under uncertainty. Next, we provide a list of *meaningful examples* which are ordered by following an increasing order of difficulty:

1. *Industrial production* involving manufacture and quality control, which must detect the presence of small imperfections and evaluate its acceptance or rejection according to previously known standards.

[30] See my notes on Computational Dynamics for details and references

2. *Civil Engineering* in tasks related to drilling, tunneling, maintenance of hazardous installations and/or consolidation of slopes with climbing robots, e.g.

3. *Robotic exploration* in non-accessible for humans (Mars, e.g), hazardous (nuclear central or repositories, e.g.) or difficult environments (undersea or bathimetric examination, e.g.)

4. *Military applications* including detection and evaluation of possible goals for neutralization or elimination, collaboration with human teams, etc

5. *Surgical systems* with laparoscopy for internal organs, replacement of components, Prosthethics and Orthopaedical Surgery, or the assistance to the most complex operations involving human heart or brain.

Currently, a wide range of complimentary frameworks exist for representing uncertainty: Probabilistic methods (which include parametric and non-parametric representations), binary representations, Dempster-Shafer logic, fuzzy set theory, and others. The choice of the representation influences the difficulty of crafting models and the computational efficiency of using these models. Additionally, a range of different problems can be attacked under uncertainty, such as: prediction vs. planning vs. control; worst case vs. average case; and correctness vs. optimality.

At early nineties several scientific questions that warrant research were introduced as challenges to be developed along next decades. We include some general comments relative to each one of these issues:

- *How can uncertain information be propagated through process models, and what type of bounds can be obtained?* Processes Oriented Architectures (POA) provide a "natural" solution which arises by combining Software Oriented Architectures (SOA) with two general models (labeled as Orchestration and Choreography). It is necessary to identify thresholds from which different sequences of services are concatenated between them, and this requires a combination of Expert Systems and Statistical Learning, which is an active research line.

- *How can we devise systems that can reason about when and what to sense?* Classical solutions have been developed from Artificial Neural Networks (ANN) and Fuzzy Systems, by superimposing additional structures given by Genetic Algorithms (GAs), Evolutionary Programming (EP) or Self-Organized Maps (SOM)[31]. Unfortunately, the interaction with Fuzzy Systems is low, and it is necessary to go in depth about their mutual relations with sensitivity analysis, by identifying critical values for functionals from which there appear qualitative changes in regard to interaction with the environment.

- *How can we develop contingency plans that interleave sensing and control?* This issue involves to the coupling with sensors and commands in the Perception-Action Cycle (PAC). There are some important contributions from different viewpoints going from Gestalt approaches, till those based on dynamical systems or Clifford Algebras; however, most of them ignore basic control models. It is necessary to reformulate the coupling with robust or adaptive models by profiting the advances performed in hardware components.

- *How can we reduce the complexity of probabilistic propagation and planning?* Propagation phenomena involves to continuous or discrete differential models linked to kinematics and/or dynamics. Different approaches have been developed along nineties which include Markov fields and Gibbs distributions (probabilistic modeling of usual vector fields in Differential Geometry), more general Hidden Markov Models (in presence of high uncertainty), Boltzman distributions (for thermodynamical inspired models), structural patterns (for statistical learning, or feedback with expert systems) etc. The interplay between differential and probabilistic aspects of the problem is a permanent source of inspiration for adapting hierarchies involving the reorganization of information. Partial observability and/or incomplete information suggest complementary approaches involving Markov Distributions and/or equilibria in environments with incomplete information, which can be solved by using different probabilistic and/or differential approaches on granulated or quasi-homogeneous spaces [32].

- *How can we approximate uncertainty and devise bounds for those approximations?* Furthermore Markov models described in the above

[31] SOM are specially appropriate for multistage, multilevel and hierarchical approaches; see specific comments in [Koh97], p.285

[32] Typical spaces for modeling incomplete information in homogeneous spaces are given by Grassmannians or, more generally, Flag Varieties. It is necessary to develop Markov Decision Processes in these spaces in a compatible way with structural PDEs (Riccati equations, typically) for solving planning under uncertainty conditions

item, a modern reformulation of entropy is given by Shannon Information models which have been revealed as very useful for recognition systems involving different digital formats. Their application for Robotics issues involves to a knowledge representations which includes semantic, graphic and symbolic aspects. The interplay between them requires advances in inter-operability issues which can be referred to common geometrical, kinematic or dynamic supports. In a complementary way, this approach must incorporate issues related to sensibility w.r.t. initial and/or boundary conditions involving temporal and/or spatial propagation phenomena. These issues concern to solve differential operators in charge of representing tasks which are linked to optimization problems in some of usual differential (Lagrange) or integral (Newton) frameworks

- *What problems can be solved in closed form, and which solutions can be computed efficiently?* Non-linear character of most problems has suggested initially an algebraic approach for which several tools of computer algebra for modeling objects were developed along the nineties. This approach is not enough flexible to be applied to noised data or under incomplete information conditions. A differential framework is much more flexible, including linear approaches to objects (by means of tangent spaces to manifolds representing working or configurations spaces, e.g.), maps (by means their differentials, including stratified approaches), distributions of vector fields (for modeling processes in terms of eventually random fields), and their dual versions (more natural because they provide a measure of observable effects in terms of differential forms), etc. All of them are particular cases of tensors (involving a finite number of vector fields and differential forms), where the uncertainty is managed in terms of voting procedures which have been adapted to Computer Vision and, subsequently, to Robotics [33].

- *How can we best represent geometric uncertainty, or shape uncertainty?* Classical solutions are given by deformations of original models for which there are a lot of models and tools in Differential, Algebraic and Analytic Geometries. Variational Calculus provide a general framework to solve Optimization issues involving any kind of functionals; unfortunately, there is no a closed form for their solutions. Even worse: the global structure of the space of deformations is unknown in

[33]For details about the use of Computer Vision techniques for Navigation and Recognition issues see the modules 3 and 4 of the CEViC.

most cases. Its description is necessary to design and implement Optimization procedures having in account the proximity between shapes. In fact, from a pure mathematical viewpoint, only the proximity between plane "simple" curves can be measured (Mumford-Shah) [34]. It would be desirable to have a similar result for the 2-dimensional case, but the management of proximity issues between "simple surfaces" is much harder than for the 1-dimensional case even if we avoid "exotic spheres".

- *How can we devise planners that employ feedback mechanisms for reducing uncertainty?* From the mathematical viewpoint, planning is a problem in a paths space which is superimposed to an eventually weighted collection of isolated points (symbolic approach), a planar/volumetric representation of the working space $\mathcal{W}$, or more general representations of the tasks or the configurations spaces $\mathcal{C}$. Neither of these spaces has a trivial structure, because it depends on the robot architecture (with the corresponding constraints involving geometric, kinematic and dynamic aspects, e.g.), the complexity of the scene (including obstacles, e.g.) or the task to be performed. From the topological viewpoint, cellular decompositions provide solutions for the geometric support, but it is necessary to develop mathematical models (extensions of Morse Theory) and software tools for solving functionals linked to tasks to be performed by robots. [35]

- *How can we, on solid mathematical grounds, ascertain which uncertainties can be ignored?.* A naive analytical answer could be given in terms of null measure sets. However, this answer is wrong, because very often the solution is unique only under unrealistic solutions. It is necessary to work with multivalued functions (with only local unicity conditions), to consider not only isolated equilibrium solutions but stable regions (with different stability criteria involving Geometrical, Kinematic and Dynamical issues), evaluate the behavior of functionals under deformations and identify stable regions where the control can be efficiently implemented [36]

Nevertheless the advances performed along more than twenty years, most

[34] A plane curve C is *simple curve* if is connexe and has no self-intersections; hence it is topologically equivalent to the circle $\mathbb{S}^1$

[35] A visually-based approach to semi-automatic navigation is developed in the module 3 of the CEViC.

[36] For more details, see my notes on Differential Topology.

of them have only partial responses which are only valid for structured environments. In practice, it is quite impossible to treat all these problems in an introductory Course. Furthermore, the integration of data in a common framework requires an objective representation which can be provided by geometry and their natural extensions to kinematics and dynamics. Jets spaces for maps provides a language for developing and representing these extensions in a natural way. Computer Vision provides a very general knowledge domain for estimating, modeling and adapting information relative to the above aspects. Thus, it provides a feedback with Robotics has been present from the early years of both scientific and technological areas.

An important inconvenient is the lack of developments from the probabilistic and/or statistical viewpoint in regard to jets space. Nevertheless, along the nineties there appear several developments relating the uncertainty of measures in these knowledge areas with some basic aspects of Vector Bundles and Learning Theory. The geometric version of statistical learning theory is focused towards "learning varieties" which can be associated to different tasks such as localization, grasping and handling objects, locomotion, etc. Typical strategies consist of reducing the complexity of objects by means linearization procedures (it involves to a discrete version of co-tangent bundles). The extension of this quasi-linear approach to manifolds to linked kinematics, requires an extrapolation of well known concepts of differential geometry which is performed in terms of jets spaces with their structural constraints.

1.4. An outline of topics

The whole Robotics Course is focused towards the development of mathematical methods and applications to Design and Development of tasks to be performed by the robot. Hence, it does not include very important aspects which are related with electromechanical devices, sensors and actuators, or the information fusion arising from different sensors. Both of them are very dependent of the current state of involved technologies and related results become obsolete very quickly. The adopted approach is geometric in a broad sense, i.e., includes topological aspects [37] and extensions to Kinematics/Dynamics as first/second order approach to the Geometric approach.

[37]The Topology can be understood as a Geometry with an infinite-dimensional group of bijective and bicontinuous transformations (homeomorphisms)

1.4.1. A general overview

Materials of these notes are organized around six modules, which can be given along two complete Courses. A preliminary version of some parts have been explained in several centers of France (Fourier Institute of Grenoble, LASC of Metz) and Italy (SSSA of Pisa and Politecnico di Torino. The selection of materials depends on the background of students. First module has an introductory character, but the other ones are partially independent between them; this organization eases the selection of a basic module and a more advanced one for a typical semester. Next, one includes a sketch of contents, jointly with some relevant motivations for each module.

1. **Planar and Anchored Robots** including serial mechanisms (mechanisms of bars, typically) and parallel platforms (such as Stewart platforms). Some typical problems concern to forward and inverse kinematics of end-effectors for serial manipulators; furthermore the study of optimal configurations along tasks execution, one must evaluate dynamical effects corresponding to inertial motions performed by the robot. Resulting dynamical effects are linked to (real or virtual) displacements of the whole architecture; it is absolutely necessary to avoid those motions which can "unbalance" the robot, by generating damages on the electro-mechanical architecture. Thus, it is necessary to design and implement strategies combining anticipatory and compensatory effects. So, we intend to develop a counter-cyclic approach which minimizes adverse effects linked to inertial phenomena. The most advanced part of this module concerns to grasping and manipulation tasks to be performed by the robot.

2. **Mobile Platforms** with Free-Collision Motion Planning and semiautomatic Navigation as central topics. To simplify, each mobile platform is represented by means of a material point (including mass for inertial phenomena, e.g.) which represents its center of gravity (c.o.g. in the successive). This representations allows to simplify issues related with planning, tracking and control of motions. The developed strategies can be extrapolated to the other three modules. An advanced topic concerns to the simultaneous management of several mobile platforms with their corresponding multi-agent systems; from the mid nineties, these systems have been developed for automatic driving of vehicles in highways. Some more advanced applications include hybrid systems which use different kinds of embarked devices for semi-automatic navigation from the feedback between terrestrial and aerial information.

The interplay between related information require more advanced tools for kinematic fusion of information arising from different sensors which is only developed for simplified updatable 2.5D representations.

3. **Kinematics of Robots** with a special regard to multibody robots composed by one or more articulated bodies with several devices (wheels, legs) for displacements or locomotion tasks. The description of the transfer map $\tau : \mathcal{C} \rightarrow \mathcal{W}$ provides the starting point to introduce a natural hierarchy between geometric, kinematic and dynamical issues. This formulation is compatible with quite different architectures including redundant wheeled large mechanisms for Civil Engineering (tunneling or railway maintenance, e.g.) or hybrid (multilegged and wheeled) systems to perform tasks in hostile or unaccessible environments (planetary exploration, forestry exploitation, undersea research, nuclear maintenance). Two classical problems concern to Forward and Inverse Kinematics which is developed in terms of the 1-jet of the transference map.

4. **Robot Dynamics** including modeling and simulation of dynamic behaviors having in account inertial effects and how to compensate them. Some important problems concern to how generated, track and control forward and inverse dynamics. Again, (a) local and infinitesimal symmetries provide organizing criteria for motions execution according to symmetrical principles involving robot mechatronics; (b) hierarchies between tasks and involved constraints can be reformulated in terms of a semi-analytic approach; and (c) basic control issues (accessibility, reachability, controllability) are easier to understand in terms of multivector calculus.

5. **Humanoid Robots** which simulate the behavior of some aspects of the muscle-skeletal system of human body. In this module we include some advanced biomechanical aspects and pattern recognition of human gestures and their imitation by artificial systems, including some aspects of Computational Vision. Furthermore applications to multimedia production, the imitation-based approach has a growing interest for applications related to human-machine interaction, including non-verbal communication. Some elements corresponding to simulation of body gestures will be displayed in regard to the generation of virtual characters for real actors. An extension of these topics is displayed in the fifth module (Dynamical Stereo Vision) of the on-line "Curso de Especialista en Vision por Computador" (CEViC).

6. **Animats** A special attention is paid to robots with a like-insect architecture and to robots which simulate the behavior of animals, which are generically called *animats*. Both of them use advanced resources of Computer Graphics and are related to applications linked to animation and special effects inside the production of multimedia contents. Thus, we include some simulation and computer graphics components. The most advanced part of this module corresponds to the description and simulation of biomechanical models for animats related mainly with locomotion tasks. The increasing availability of biomechanical simulations for motions of animals makes more accessible this subject with a lot of applications to multimedia industry in strong connection with Computer Graphics and Advanced Visualization tools.

1.4.2. Computational Mechanics as a general framework

Along these notes, *Computational Mechanics* is understood as a reformulation of Classical Mechanics of articulated objects in computational terms. Hence, it has three main components relative to

- *Mechanical models* with the above mentioned levels including geometric, kinematic and dynamic aspects, by following an increasing difficulty order. Furthermore, motion laws and interaction with environment, it includes Design, Optimization and Control issues for each one of the three levels.
- *Data structures* including the capture, processing and analysis of exterior and propioceptive information captured from sensors. Furthermore, it includes the conversion of signals, their transduction and representation on a common space to ease decision making in regard to commands which are acted by several kinds of electromechanical devices (acoustic, electromagnetic, electronic devices for servomotors).
- *Algorithms* which interpret the information in regard to the tasks to be developed. They are implemented to ease a hierarchical management, and to provide internal and external representations for optimal making decision processes.

The above scheme is quite general for any kind of robots, but it does not cover the scope of the whole Computational Mechanics. In particular, some important topics concerning to computational mechanics of continuous media (including the mechanics of fluids, e.g.) or the mechanics of particles are not considered here, [38]

From a methodological viewpoint, a space-temporal representation of the above three components allows to provide a common support to re-organize tasks according to different weighted constraints (including variable weights). Static case is formulated in terms of a reformulation of weighted Voronoi diagrams which allow to reformulate constraints according to different hierarchies going from ordinal to metric issues. Their extensions to kinematic and dynamic cases are a challenge to be solved.

1.4.3. Advantages of the geometric approach

The main advantage of geometric approaches is the robustness of models, and its intrinsic character w.r.t. the observer's localization. Hence, a geometric formulation for Kinematics and Dynamics has been developed from the beginning along the 19th century. In presence of complex interactions with changing environments, a more flexible approach is needed. In this case, the robustness can be weakened by extending Geometry to Topology (as the study of properties which remain invariant by the action of a subgroups of homeomorphisms). Hence Topology provides a structural framework for a more adaptive approach to eventually changing constraints.

To minimize the casuistic arising from a so large diversity of topics, the extended geometric approach is performed in a transversal way along the whole Course. In this way, we intend to provide several threads which intend to provide a global overview of useful tools arising from geometric and topological models, and their extension to dynamical systems in Robotics.

To achieve these goals, we use several abstract concepts linked to Differential and Integral Calculus on Manifolds (including Lie groups and Lie Algebras), their extension to stratified varieties, and a collection of superimposed structures to describe the Robot Kinematics and Dynamics. Main geometric superimposed structures are given by vector bundles (to represent the simplest aspects of kinematics), principal bundles (to incorporate structural models for symmetries management) and, more generally, topological fibrations (for the stratified case including singularities).

Basically, a topological variety M is obtained by matching "pieces" which are equivalent to cartesian spaces $\mathbb{R}^n$; in the smooth case, one considers

[38] For an introduction, one can see my notes on *Computational Mechanics*. An introduction in Spanish language is included in my web page, also.

a (pseudo-)riemannian metric, whereas in the algebraic case, one matches pieces which are algebraically equivalent to affine spaces $\mathbb{A}^n$. There is a natural hierarchy between C^r-categories; in particular topological or C^0-category is the coarsest one; it provides the support for smooth or C^∞-category, the analytic or C^ω-category, with the algebraic or C^{rat}-category as a more operative case where functions are described by polynomials or, more generally, rational functions.

The choice of the most appropriate category is exploited to design coarse-to-fine strategies depends strongly on the problem to be solved and the available software tools; often, one can start with simpler PL-structures and add"more structure" depending on the support and the allowed maps. So for example, in the category of sets, equivalences are given by bijective maps, but in the topological category, equivalences are given by homeomorphisms, i.e. bijective and bi-continuous (continuous and with continuous inverse) maps.

Thus, some special attention must be paid to topological issues because they provide the support for successive hierarchies and the integration of different aspects. The introduction of intermediate C^r-structures provides additional tools which allows to apply structural results linked to different geometries as support for a more complex dynamics relative to the environmental interaction. The key is to use any kind of Geometry as the support for Kinematics or, more generally, Dynamics involving all kinds of interaction with the environment.

By following the usual hierarchy for Mechanics, an additional trouble arises from the apparition of non-holonomic constraints linked to robot kinematics (drift effects, e.g.) or dynamics (friction cones, e.g.) linked to navigation or grasping and handling issues. Their geometric characterization is given by non-integrable subbundles of the (co)tangent bundle linked to systems of structural equations. Nevertheless their non-linear and non-integrable character, they can be managed in terms of (metric, affine, Ehresmann) connections which are defined on the "linearization" of configurations and working spaces.

From a computational viewpoint, it is very important to perform a discrete approach to the above disciplines. Discrete approaches are well known for design and for solution of structural equations for differential (Hamilton-Jacobi) or integral (Newton-Euler-Lagrange) formulation of Mechanics. More recently, the development of discrete approaches to Differential Geometry has provided a new stimulus for a unified treatment according to the hierarchical approach developed along these notes.

Finally, the availability of algorithms arising from Computational Geometry (determinist and randomized versions) has provided the initial support to advance in their extensions to Kinematics and Dynamics, according to the above scheme. In particular, the use of Lie-based representations for configurations and working spaces provides a locally symmetric support for Optimal Control in terms of Sub- or Semi-Riemannian Geometry. A typical example is obtained by replacing lines of ordinary space by geodesics on subvarieties of products of Lie groups which reproduce in a more faithful way motions to be performed by robots.

1.4.4. Some selected applications

Main applications developed by several researchers of the MoBiVAP Group along the late nineties and the first years of 21st century concern to visually guided manipulation (grasping and handling), visually guided navigation of mobile platforms and locomotion tasks for humanoid robots. All these applications have been focused towards the assistance to disabled persons including tetraplegic persons (Pisa, Italy), semiautomatic navigation at home for cerebral palsy patients (Metz, France) and reciprocator-based devices for paraplegic persons (Paris and Torino). Nevertheless the interest of all of them, the lack of financial support for all these activities is the main motivation for my abandon of these research lines.

Most materials are originally dispersed in several talks (given in several RTD centers of the EU), and specialized Courses given at Spain (Cartagena, Murcia), France (Fourier Institute in Grenoble, LASC in Metz) and Italy (SSSA in Pisa, Politechnic Institute in Torino). First versions arise as an extension to Computational Kinematics of Computational Geometry methods (given at High Schools of Industrial Engineering and Informatics Engineering) from 1996. Several aspects have been included in a collection of invited conferences and/or courses given in different Robotic Labs. Thus, most ideas arise from discussions and talks with different experts working in different RTD Labs. Some of the most meaningful are the following ones:

- Robotics and Vision Lab of the University of Oxford with Prof. Andrew Zisserman as coordinator of a British-Spanish Integrated Action, and Prof. Steve Maybank as researcher for the development of 3D reconstruction of scenes from a mobile camera by adapting SLAM (Simultaneous Localization and Mapping). These activities have obtained financial support from a British-Spanish Integrated Action along 1996-97.

- Department of Mechanical Engineering of CalTech (Pasadena, CA, USA) with Prof. Joel Burdick and J.Ostrowski as main researcher. Their invitation and their encouragement has allowed a first exposition of methods in autumn 1998 which is based on control based on the use of ordinary and infinitesimal symmetries for a hierarchical control of mechatronic devices.

- CREARE Lab (Paris VI) with Prof. Yves Bournod as director, where some modules for simulating learning tasks and motor control by human brain have been developed along 1998-99 in regard to some neural deficiencies. The framework for this (non-published) research is an extension of the tensor approach performed by Pellionisz at eighties in the SOM framework.

- ARTS Lab (SSSA, Pisa) with Prof. Paolo Dario as director, where several aspects concerning to the feedback between visual information and grasping have been developed for robotics assistance to tetraplegic persons. These activities have obtained financial support from an Italian-Spanish Integrated Action along 1999-01.

- Cachan Lab with Prof. Philippe Gorce as director for control modeling of humanoid biped robots (shared with Paris), including the modeling of kinematic control near and through singularities. These activities have been performed along 2001-03 in the IFRATH framework with some contributions to the French Handicap Network with MoBiVAP as temporal external partner.

- Mechanical Engineering Department of the Istituto Politecnico di Torino (Italy) with Prof. Guido Belforte as director, including the study and development of artificial muscles (SMA: Shape Memory Alloys) to be inserted in reciprocators inside a Program of Robotics Assistance to paraplegic persons. These activities have obtained financial support from an Italian-Spanish Integrated Action on Algebraic Geometry between 2001-03 with Prof. Sylvio Greco as responsible by the Italian side.

- Institute of Industrial Automatics (IAI, CSIC) with Manuel Armada and Pablo Gonzalez-Santos as main researchers for the development of a hexapod robot between 2001-03, which has been designed to operate in hazardous or unaccesible environments (inspection tasks in a nuclear central). Unfortunately, the robot was never constructed.

- Laboratoire de Systemes Cooperatifs (LASC, Univ. of Metz, France) with Prof. Alain Prusky as director. The joint activity was focused to improve the semi-automatic indoor navigation of a wheelchair to be used by persons with degenerative illness in brain. The main contribution along 2001-03 concerns to the adaptation of a variant of Kalman filtering (IEKF) to the real-time generation of quadrilaterals maps generated from a mobile camera which is embedded in a wheelchair.

- Fourier Institute and INRIA Rhone-Alpes with Profs. Gerard Gonzalez-Sprinberg and Radu Horaud as the main researchers, respectively. The financial support of Fourier Institute is acknowledged and has allowed to perform a cycle of conferences about Geometric Methods in Robotics, where a first version of these notes has been presented. The encouragement and ideas of Prof. Radu Horaud have been a source of inspiration for the integration of methods based in the integration of Robotics and Computer Vision inside the Lie theory.

Anyway, the responsibility for misunderstanding and mistakes in these notes is only mine. Their encouragement, patience and generosity for sharing ideas and expend their time around these topics are gratefully acknowledged, nevertheless sometimes adverse circumstances.

Final remark: After performing the preliminary versions of this Course between 1998 and 2000, competences linked to the current Robotics Course have been assumed by other Department. Thus, and due to the lack of academic assignation in this domain, these notes have no continuity from the docent viewpoint from 2002. Nevertheless and after several comments of some colleagues which encourage to me for a larger availability, perhaps some of these materials have some interest for theoretical aspects of Robotics.

2. The mechatronic support

The mechatronic support involves to mechanical architecture, electronic components, control devices and expert systems in charge of managing all of them. Their integration must guarantee the information exchange and the compatibility between components relative to perception and action to provide a feedback able of efficient interaction with the environment.

The integration must be able of managing proprioceptive and external components in the PAC (Perception-Action Cycle) which characterizes the behavior of all live beings. Several millions of evolution have generated a lot of very efficient architectures at different scales which are adapted to quite different functionalities and environments.

The very quick technological progress able of integrating complex tasks in hardware devices (which previously have been developed via software) makes very difficult to perform an update of mechatronic devices. Thus, instead of a catalogue of sensors and actuators, we shall develop along this section a *functional approach* which is biologically inspired. Obviously, we are still far from achieving the efficiency of specialized mechatronic architectures, but biological inspiration provides meaningful keys to be modeled and developed.

Roughly speaking, from a biological viewpoint a classical distinction involves to "shape" and "function", which can be modeled in geometric and analytic terms, respectively. Microscopic organisms are able of modifying their shape by using chemical reactions, and to perform different functionalities going from eating to locomotion tasks; in this case, the kinetic energy is generated from different potential levels.

In more complex mechanisms, going from insects till vertebrate animals, some key facts arise from reflex or voluntary movements which allow to generate forces and moments from shape changes linked to articulated components. This approach takes advantage of *symmetrical principles* in geometric design and asynchronous movements between simplest models (such as oscillators, elastic springs and their mutual interconnections on progressively complex structures), with their corresponding algebraic (space movements) and infinitesimal versions (going from pluricellular till dense representations).

From a mathematical viewpoint, the integration of the above models is performed at different levels which involve to

1. a *geometrically inspired Design* as the support for all components in regard to the tasks to pe performed;

2. an *algebraic approach to Statics* organized around equilibrium configurations as attractors for the dynamics;

3. a *kinematic analysis for tasks* to warrant the adjustment to stable trajectories along complex motions; and

4. a *dynamic integration* including optimization and control issues linked to structural equations of motion, controllability and stability of solutions.

Along next subsections we will provide some additional details for each one of the four items.

2.1. Design. A geometric insight

Design includes geometric aspects, i.,e. relative to the whole mechanism and a representation of the scene, and kinematic aspects, i.e. relative to the tasks to be performed. Along this subsection, the main attention is paid to *geometric design*; in the subsection §2,3, we shall display some aspects of *kinematic design* with an additional support arising from computational topology and dynamical systems which are natural extensions of the geometric approach.

Geometric principles provide *robust models*, i.e. which are stable under small perturbations or under uncertainty conditions relative to capture, tracking or propagation phenomena. Thus, robust models for robotics inheritate incorporate basic geometric principles from the beginning. This choice is motivated by the geometric nature of simplest transformations such as rotations and translations in low-dimensional cartesian spaces.

Geometric Design can be understood in several ways. Assisted Design has two paradigms which are labeled as CAD/CAM (Computer Aided Design/Manufacture) and they have a geometric nature from the beginning including different kinds of

- linear and non-linear k-dimensional *geometric primitives*;
- *linear transformations* usually represented by matrices belonging to a classical group and topological operations [39]

[39] Quaternions provides a much more precise (multi)vector approach to linear transformations which is included in the Geometric Algebra framework

- *topological operations* relative to cutting or culling pieces and/or matching together basic pieces to compose more complex objects.

Thus, a geometric design is not an original claim. The pioneering work of Pellionisz and his collaborators along the early eighties introduces tensor calculus to model advanced functionalities for coordination tasks performed by the cerebellum. The intrinsic character and covariance properties of tensor representations provide an almost universal support for functional aspects of any subject, which are still valid independently of environmental conditions.

An inconvenient of this approach involves to its formulation in matricial terms which makes some cumbersome the understanding of underlying ideas; in particular, matricial calculus must be replaced by a more intrinsic version of geometric calculus (in the framework of Geometric Algebra), and "translational effects" of geometric entities must be replaced by the use of (metric, affine, Ehresmann) connections.

From the mid nineties, the use of varieties (as extension of manifolds) and superimposed structures (fiber bundles, principal bundles, fibrations) with their corresponding transformations, is becoming more and more common. This approach is justified by the apparition of singular configurations or the acquisition of singularities along the execution of some tasks. Furthermore, the increasing role of Lie groups (and its infinitesimal version given by Lie Algebras) has provided a powerful tool to integrate geometric design with kinematic and dynamic aspects, including a geometric re-formulation control and optimization issues.

All of them are a natural extension of the original geometric approach to the design, because manifolds M and varieties X are obtained as the result of matching together pieces which are equivalent to open sets of the cartesian space. By taking appropriate embeddings or immersions of such objects in cartesian spaces, one can consider the restriction of actions to each object. Furthermore, differential or integral functionals defined on manifolds or varieties are the natural extension of systems of ODE/PDE or variational calculus defined originally on objects embedded in cartesian spaces.

2.1.1. A general approach

Symmetries are ubiquitous in any mechanical system, even at the lowest level of action-reaction principles. Thus, it is natural search some type of structure for symmetries and their interpretation in different (discrete, algebraic, analytic) contexts. The simplest structure is given by a group of transformations which can be discrete (the group of permutations, e.g.) or continuous; last ones can have finite dimension (some subgroup or quotient group of regular matrices, e.g.) or infinite dimension (some subgroup of the group of homeomorphisms, e.g.)

To fix ideas, we shall restrict ourselves to continuous finite-dimensional groups. Following the Erlangen's Program of F.Klein (1873), the structural group G of a Linear Geometry is usually given by a classical group, i.e. a subgroup or a quotient group of the general lineal group which preserves a quadratic or a bilinear form. The initial example in Robotics is given by the Euclidian Geometry which is characterized by transformations leaving invariant the Euclidian metric, i.e. the (semidirect) product of rotations and translations in low dimension.

The description of Linear Geometries in terms of subgroups or quotient groups of Classical Groups can be extended to a more general topological framework with different superimposed structures which allow to incorporate usual tools of Analysis. To achieve this goal, it is necessary to replace

- the ordinary cartesian space $\mathbb{R}^m$ by a base space B given by a (locally symmetric) manifold M or a variety X, which is obtained by matching together pieces which are equivalente to open sets of the cartesian space; and
- the classical group (given as a subgroup or a quotient of $G = GL(m, \mathbb{R})$ of regular linear transformations), by local symmetries (having a regular inverse) along paths (preferably geodesics) which extend elementary transformation of the Linear Algebra defined on lines.

General Topology is a natural extension of Geometry which is obtained by replacing the structural linear finite-dimensional group G linked to a group G of *linear regular* transformations by a topological infinite-dimensional group of bi-continuous regular transformations. In particular, General Topology is characterized by the set of properties (dimensionality, separation, connectedness, compactness, etc) which remain invariant by the action of the group of homeomorphisms [40].

For each additional structure (differentiable, algebraic, analytic) on the topological space X, one obtains a C^r-equivalence between varieties where $r = \infty$ (differentiable clase), $r = rat$ (algebraic case) or $r = \omega$ (analytic case). Superimposed structures on these C^r-varieties must verify compatibility conditions between local data which are essentially the same as the

[40] A homeomorphism between two topological spaces X and $Y$is a bijective and bi-continuous map $f : X \to Y$, i.e. f and f^{-1} are continuous

given for the base space B; so, it is possible to lift the C^r-structure of the base space to the total space E of the superimposed structure in a natural way. The main structures which are useful in Robotics are given by fiber bundles, principal bundles and topological fibrations (Ehresmann).

Some almost obvious advantages of the topological approach consists of the following features:

- It allows to incorporate *regular deformations* to the geometric models which are initially rigid ones; so, one obtains a quite general support for *adaptive strategies* relative to paths (linked to trajectories, e.g.), functions (crucial to evaluate critical phenomena linked to phase transitions, e.g.) or functionals (variational principles linked to action functionals or optimization issues, e.g.).

- It provides a support to consider different structures in a simultaneous way on the same original base space B (in terms of tensorial products, e.g.), including different behaviors (covariant and contravariant, e.g.).

- It integrates propioceptive and stereoceptive behaviors in terms of intrinsic data (relative to the variety or its tangent fiber bundle, e.g.) and extrinsic data (relative to normal bundle, e.g.).

- It allows to represent any kind of transport models (in terms of different types of connections) and propagation phenomena, including integrable and non-integrable behaviors on the same support.

- It provides quite general models for hierarchies appearing w.r.t. the mechatronic architecture (fusion of information) and eventually changing environments, in terms of flag bundles, as universal model for any stratified map.

Almost all are well known and they appear in several references which have appeared from the early nineties. Perhaps, the less known is the last one, because it involves to the geometry of flags. Roughly speaking, a *flag* is a collection

$$L^0 = (0) \subset L^{k_1} \subset \dots L^{k_i} \subset L^{k_{i+1}} \subset \dots V$$

of nested linear k_i-dimensional subspaces L^{k_i} of a vector space V. A priori, the vector space can have arbitrary (even infinite) dimension as it occurs in Hilbert spaces, e.g. However, to fix ideas, we shall restrict to a finite collection of k_i-dimensional subspaces of a n-dimensional vector space V

over $\mathbb{R}$. If we denote by means $W^{\ell_i} = L^{k_i}/L^{k_{i-1}}$ the quotient subspaces, one has a decomposition of V in a direct sum of W^{ℓ_i} where $\ell_i = k_i - k_{i-1}$ represents a partition of $n := dim(V)$. The action of the general linear group $GL(n; V)$ on V induces an action on the flags of "nationality" $(\ell_1, \ldots, \ell_r)$ where r is the number of meaningful subspaces. In particular for $r = 1$ one obtains the projective space, whereas for $r = 2$ one obtains a Grassmann manifold.

In our case, nested subspaces $W_{k_i} \subset W_{k_j}$ provide a simplified *linear representation of hierarchies* involving the fusion of internal (propioceptive) and external (relative to environmental) information which is included in an ideal space V to represent the whole *Perception-Action Cycle* (PAC):

- The main goal for *Perception* is to perform the fusion of the relevant information in terms of sensor fusion which is performed on a geometric representation of the whole mechatronic architecture (similar to the performed by the cerebellum in the human case).

- The main goal for *Action* is to organize, coordinate and execute actions by following hierarchies subordinated to the task, which represent an interaction with the environment.

In next two paragraphs, one displays a first approach to both issues and some snapshots for their integration in a common PAC.

2.1.2. Sensors for Perception

Perception is the ability to sense the internal state (propioceptive sensing) and the environment (extereoceptive sensing) and to interpret these data. Capture is performed from sensors, whereas interpretation is performed by some kind of "embedded intelligence", which can be of reflex type (already present in insects, e.g.) with partially distributed or, alternately, centralized architectures. The most difficult part concerns to the design and implementation of expert systems for different kinds of embedded intelligence, which are introduced in the subsection §3,4. In this paragraph, one considers only some general aspects relative to the capture and the fusion of information from sensors.

Neverthelss the advances of last years, we are still far from achieving a sensorial system as complete as of invertebrates. To fix ideas, we shall concentrate our attention in sensors relative to the internal state of the robot. In [Eve95] one can find a very explicit description of sensors for robots which are commonly used in advanced manipulators and/or mobile platforms, and which have been extended and improved along next twenty years. The most appropriate description of sensors for our approach makes a mechanical description involving to localization (position and orientation) transducers, (linear and angular) velocity transducers and force sensors (see [Sic01], chapter 8, e.g.). This description provides a unified treatment in terms of propioceptive information which is updated in terms of motions or interaction with the environment.

The frequency domain for any kind of 1D/2D/3D signals can be transformed in the spatial domain with a similar formalism. Hence, nevertheless the initial nature of signals, all the available information can be reformulated in mechanical terms with involve to the kinematic and dynamic extensions of the original geometry. The paradigm for this kind of transformations is given by the two-fold approach to Computer Vision which can be formulated in frequency- and domain-based terms. Thus, the Fourier analysis provides the general framework to convert different kinds of signals in a representation of image from which a scene representation can be obtained.

The first module of CEViC (Curso de Especialista en Vision por Computador) is devoted to Image Processing and Analysis; in particular, chapter 6 shows this equivalence and chapter 7 displays some more recent developments in terms of wavelets. Moreover, 3D Reconstruction tools provide a representation of the ambient space (module 2 of CEViC), Motion estimation and related topics (image and scene flows) provide a representation for kinematics (module 3 of CEViC) and the Recognition tools (module 4 of CEViC) provide the key for interaction in complex environments.

Classical approach to flows is based on visual information, where flows appear associated to vector fields linked to real or apparent motion. The need of integrating information arising from scalar fields or more general tensor fields involving quite different information (arising from different sensors e.g.), suggests the development of tensor flows with their corresponding computational tools for an efficient management. This goal requires an interaction with more advanced Visualization tools which are presented in the module 6 of Computational Mechanics. [41]

[41] In https://www.tensorflow.org/ one can find related models and software tools.

2.1.3. Actuators for Action

All mechanical devices produced by the man can be represented as a combination of rigid transformations involving rotations and translations. However, the appearance is distorted by sensors, the transmission generated instabilities and vibrations, and it is necessary to correct trajectories along the tasks execution by balancing forces to be applied and their generated moments.

Biomechanical models provide a permanent source of inspiration to improve the pipeline associated to the above remarks, and it is exploited in an intensive way along the module 5 (Humanoid Robots) and module 6 (Animats) of this matter. However, all biomechanical models are redundant: several muscles can be combined to perform the same task and each muscle can contribute to different motions. Any redundant mechanism requires the design and implementation of advanced optimization procedures for tasks execution.

To avoid troubles linked to optimization procedures, we shall suppose that mechanisms are not redundant ones. In this case, usual approach to actuators in Robotics is performed in terms of different kinds of motors acting on joints to generate small motions, which are propagated along the kinematic chains to produce the desired movement at the end-effector of each component. The main elements for joint actuator systems must consider transmission mechanisms, servomotors (including electric and hydraulic ones), power amplifiers and power supplies [42]. Some practical problems to be solved concern to correct localization, tracking, adapting behavior and correcting vibrations.

From an algebraic viewpoint, the spatial representation of possible configurations can be geometrically represented in terms of subvarieties of a product of Lie groups; kinematic and dynamic effects are reformulated in terms of subbundles of their tangent spaces. Lie groups are parallelizable and, consequently, their tangent space (Lie algebra) are trivial ones. However, mechanical (geometric, kinematic and dynamic) constraints introduce subvarieties of the product of Lie groups which are far from being trivial ones. In particular, there appear non-integrable subbundles which are linked to non-holonomic constraints for kinematics (drift effects, e.g.) and dynamics (friction cones, e.g.). Thus an extended geometric approach is necessary to understand the whole mechanical representation for Robotics.

Above, in the paragraph §1,3,3 we have reasoned the need of replacing

[42] See Chapter 8 of [Sic01] for more details

usual smooth framework by a semianalytic framework to include a collection of behaviors for legged robots. By the same reason, it is necessary to extend usual control devices linked to the feedback between sensors and actuators. Geometric control devices are organized by following the typical hierarchy in Advanced visualization which involves to scalar, vector and tensor fields in correspondence) with position (Lyapunov-Kalman), impedance (Poincaré-Thom-Brocket) and force-based (Newton-Euler-Sussman) control.

The *main novelty* of the approach performed along these notes in regard to these issues is linked to the use of symmetries (algebraic, infinitesimal, dynamic) to design and implement a feedback between all of them. Let us remark that algebraic hierarchies linked to symmetries can be extended to the integration of information arising from sensors in a shared representation of the evolving environment. In particular and in regard to Computer Vision techniques (which the most relevant information for navigation of autonomous robots), (a) scalar fields involve to errors minimization w.r.t. a goal function (for any kind of distance or min-max optimization procedures); (b) vector fields involve to motion estimation (concerning to flow image and flow scene, e.g.); and (c) tensor fields involve to the estimation of complex representations relative to mechanical quantities (energy, work, etc) linked to a finite number of vectors and co-vectors.

2.1.4. Smart devices for the PAC

Beyond mechatronics architecture, it is necessary to incorporate a combination of robust and adaptive models for managing expert systems. The lowest level does correspond to reflex motions which are managed in a decentralized way and are located along some kinematic chains. We are more interested in voluntary movements which depend on centralized architectures, where some tasks have been automatized along several millions of evolution years by following a hierarchy going from reflex to voluntary behaviors. The incorporation of uncertainty relative to signals and commands and their feedback, requires to introduce a probabilistic approach.

At least from the eighties, Artificial Neural Networks (ANN in the successive) are commonly used as support for modeling different functionalities linked to expert systems under uncertainty conditions. Roughly speaking, ANNs have a multilayered structure with at least three (input, hidden, output) layers and a system of variable weights for nodes or "neurons". They are organized by following a regular planar distribution which are interconnected by following regular patterns; learning procedures modify weights according to the tasks to be developed. Convergence rate (linked to learning procedures) to a task and the lack of adaptability are two serious inconvenients which require additional structures. [43]

Design and implementation of Expert Systems require to introduce a stratification between different kinds of logic which allow to manage the responses. A typical scheme is given by three successive steps corresponding to logic of classes, logic of (first and second order) predicates and descriptive logic (managed by Fuzzy Algorithms), by following an increasing order of difficulty. Its computer implementation eases an interaction corresponding to reflex motions, learned behaviors and reflective procedures linked to decision making in complex environments.

To ease more complex interactions with humans, it is convenient to consider the above three levels and adopt a biological inspiration based on Central Nervous Systems (CNS in the successive) of higher mammals. Furthermore the spinal chord, the CNS has three main components corresponding to basal ganglia, cerebellum and brain which constitute the encephalous:

- *Basal ganglia* are in charge of generating an appropriate balance of neurotransmitters to avoid unstability, maintaining equilibrium and stability along trajectories.
- The *Cerebellum* displays an almost perfectly homogenous structure: All cells are "identical" between them and the relative disposition is designed on a cubical web. This reticular structure reproduces at a symbolic level the relative distribution of different organs whose behavior is automatized along the gestation or the first stages of life. It plays a role as coordinator and regulator of complex tasks which, after learning, are performed in an automatic way.
- The *Brain* is the most complex organ because it is in charge of processing, analyzing and matching the information arising from different sensors; evaluation of fusion, planning and executing tasks in a voluntary way. It has five layers (with jumps in their interconnections), with a lot of distinct cells whose functionalities are unknown, still.

Currently, it is not viable to try of imitating the hug morphological and functional complexity of brain. Thus, the main challenge concerns to the

[43] The most relevant solutions are given by Genetic Algorithms (GAs), Evolutionary Programming (EP) and Self-Organized Maps (SOM) which are commented in the §3,4

brain modeling from a functional viewpoint [44]. To accomplish this goal, at the late eighties M.Pellionisz introduces an almost forgotten geometric structure which is based on tensor calculus.

The justification for the use of tensors is very simple: If we wish to perform a simultaneous tracking of evolving vector quantities and an evaluation of their effects on different components, we need to introduce s vector fields for tracking and r differential forms to evaluate its joint effects, which are formally represented by a (r, s)-tensor on a variety supporting the multiply ramified cerebral structure. A multivector reformulation has been recovered in the Geometric Algebra framework. Differential aspects are linked to different kinds of connections which are explored in the third module.

2.2. Objects and transformations for Statics

A geometric approach provides a general framework for static issues involving to the scene or initial configurations of each robot. Its main advantage is given by the objective character of geometry, independently of the use of coordinates systems (cartesian, spherical, cylindrical, e.g.) or vectorial representation (ordinary vectors, multivectors, Geometric Algebra). The management of each geometry is performed in terms of groups of transformations (given usually by groups) and operators defined on geometric objects (for optimization issues, e.g.).

There are two complementary strategies to describe a geometry which are labeled as top-down and bottom-up:

- *Top-down approach* which supposes the geometric framework is already given. It is appropriate for representing structured scenes, known configurations and pre-determined motions to be accomplished by the robot. It uses known geometric models which have been previously described from the design phase. Usual motions are a composition of linear transformation involving the robot mechanisms or, alternately, a representation of the scene linked to a virtual camera representing an ideal observer.
- *Bottom approach* with a higher uncertainty level for the scene or the task to be developed. It is appropriate for exploration, navigation in

[44]Morphological aspects are too complicated, and one ignores the role played by internal layers, different cells and irregular distribution of connections between non-necessarily consecutive layers

open scenes, feedback under uncertainty. It uses different types of varieties which must be constructed by matching together small pieces of increasing degree. So, we have PL, PQ or more generally PS-varieties (Piecewise Linear, Quadratic or Smooth varieties) which provide approaches to more general and ideal solutions given by algebraic or analytic varieties.

Hybrid methods are based on a feedback between both approaches, and in non-structured scenarios are the most efficient ones. One needs to introduce symmetrical principles going from the simplest reactive models to the most complex reconfigurable behaviors, including self-adaptation to chainging environments. Along this subsection we insert some general comments about (linear vs non-linear) geometries, and functional aspects with two illustrations concerning to Optimization issues and algebraic inequalities.

2.2.1. Linear Geometries

Linear Geometries are characterized by a linear group of transformations, i.e. subgroups G of the general linear group $GL(n;\mathbb{R})$ which preserve some bilinear or quadratic form defined on the ambient space or the tangent space. In some cases, these groups are considered up to scale. i.e. we consider their superimposed projective structure [45].

According to the Erlangen's Program of Klein, there exists a natural hierarchy between groups which is translated to a natural hierarchy between Linear Geometries. This remark is very useful because it allows to design coarse-to-fine approaches by increasing the number of parameters to be estimated depending on the task to be performed by the robot. The most common geometries used in Robotics correspond to the following groups

- *Special Orthogonal Group* $SO(m) = \{A \in GL(m;\mathbb{R}) \mid {}^{T}A.A = I_m\}$ which corresponds to rotations in the ordinary space for $m = 2$ or $m = 3$. Its semidirect product with the translations group $\mathbb{R}^m$ provides the *euclidian group* $G_{\mathbb{E}}$ which is characterized by the preservation of euclidian distance and allows to consider the euclidian space $\mathbb{E}^m$ as the cartesian space $\mathbb{R}^m$ with the euclidian norm as the distance.
- The *group of similarities* is the quotient of the euclidian group by the homotheties, i.e., the multiplicative group of units $\mathbb{R}^* := \{\lambda I_m \mid \lambda \neq 0\}$

[45]Typical applications concern to the application of Computer Vision techniques for navigation, visual servoing and interaction in complex environments

where I_m is the diagonal identity matrix.

- The *affine group* which is the semidirect product of the general linear group $GL(m;\mathbb{R})$ by the group of translations $\mathbb{R}^m$, i.e. it is represented by $(m+1)\times(m+1)$ whose first $m\times m$ box is an element of $GL(m;\mathbb{R})$, last column is equivalente to a translation vector $(\mathbf{v},1)^T$ las last row is the vector $(0,\ldots,0,1)$. Some important subcases correspond to replace the general linear group by the special linear group or the conformal group:
  - The *Special linear group* which preserves the volume form.
  - The *Conformal group* which preserves angles between lines.
- The *symplectic group* on $\mathbb{R}^{2m}$ which is characterized by the preservation of the anti-involutive matrix J verifying $J^2=-I$ (the natural extension of the condition $i^2=-1$ in matricial terms). It preserves the structural equations of the Analytical Mechanics in the formulation given by Hamilton. The *contact geometry* is the analogue for odd dimension, and it is very appropriate to analyze Analytical Mechanics linked to each constraint appearing in Optimization or control issues.

Each structural group is a subgroup of the general linear group of regular transformations. Furthermore, it has a natural differential structure, i.e. is a Lie group whose smooth structure is compatible with the group action and the inverse (as continuous operations defined onto the group). Its tangent space T_eG at the neutral element e (the identity matrix in the above cases) defines the Lie algebra which is denoted as $\mathfrak{g}$. The direct estimation of group transformations is a veri difficult problem; however, the estimation of generators of Lie algebras is very easy. Hence, the natural strategy consists of estimating generators of Lie algebra, and compute its exponential by the map $\exp:\mathfrak{g}\to G$ which is a local diffeomorphism.

2.2.2. Nonlinear Geometries

Nonlinear Geometries involve to curved objects -given by inequalities defined by a finite number of non-linear functions $f_i(\underline{x})$- and their transformations -given $\mathcal{C}^r$-equivalences-. The "simplest" case corresponds to functions given by polynomials $p_i(\underline{x})$ defined usually on $\mathbf{R}$ or $\mathbb{C}$; one can consider different types of C^r-equivalences including the following cases:

- the *Algebraic equivalence* (also called birrational) $r = alg$ corresponds to take transformations given by the group of birational functions (to warrant their invertibility) $f_i(\underline{x}) = p_i(\underline{x})/q_i(\underline{x})$ which are not everywhere defined (the vanishing of denominator gives an indeterminacy region); usually, we shall take $p_i(\underline{x})$ and $g_i(\underline{x})$ as rational functions, with snakes and splines as typical examples;

- the *Differential equivalence* (also called smooth) $r = \infty$ corresponds to take transformations given by the group of (local) diffeomorphisms, i.e. homeomorphisms $h(\underline{x})$ which are differentiable or smooth with inverse differentiable. Let us remember that a homeomorphism is a bijective and bicontinuous map, whereas a smooth map has all derivatives of any order continuous and derivable. The generation of homeomorphism is very easy: It suffices to integrate an ODE or, from a more global viewpoint, a vector field. The condition of being a diffeomorphism is much more difficult to verify. Thus, very often we shall work with different kinds of (scalar, vector, tensor) fields, by incorporating only a piecewise smoothness (PS) condition for practical solutions.

- the *Analytical equivalence* (also called holomorphic in the complex case) $r = \omega$ corresponds to take transformations given by the group of (local) bianalytic functions. Let us remember that an analytic function at a point $\underline{x}$ is characterized by a convergent series development at such point. Their incorporation allows to treat any kind of singularities, including non-isolated singularities.

Each type of C^r-transformations is used for different proposals, accordingly with the chosen geometric framework. All of them have relevant *applications in Robotics* including the following ones:

- *Algebraic Geometry* is the natural extension of the Geometry of low degree curves and surfaces (quadrics, cubics, quartics). They have been applied to solve kinematic problems in the joints or configurations space $\mathcal{C}$ (by using symbolic algebra for trigonometric functions, e.g.), accessibility issues for the end-effector of each kinematic chain in the working space $\mathcal{W}$, effective computations for the composition of movements under low-degree constraints, resolution of algebraic transforms of differential systems (by means the Laplace transform, typically), or to identify singular configurations [46].

[46]Geometric singularities are linked to components alignement, whereas kinematic sin-

- *Differential Geometry* including the study of any kind of (scalar, vector, tensor) fields and their solutions on differentiable manifolds M; hence, they can be applied to any Lie group G and its Lie algebra $\mathfrak{g} = T_eG$. In particular, this kind of tools eases the tracking of control points, the integrability of dynamical systems (given by distributions of vector fields or systems of differential forms), piecewise smooth interpolations (between varieties or functions, including their applications to Lie groups), non-holonomic effects in the non-integrable case (linked to kinematic drifts or dynamic frictions, e.g.), optimization linked to different kinds of functionals (energy, curvature) appearing in the resolution of variational problems, robust vs adaptive control of any kind of devices, modeling of cooperative (realization of a task, e.g.) vs competitive (occupied space, propagation phenomena, e.g.) for multiagent systems, etc.

- *Analytic Geometry* for prolongation issues involving paths or trajectories performed by control points, local analysis near to (geometric, kinematic, dynamic) singularities, stratifications of configurations $\mathcal{C}$ or working $\mathcal{W}$ spaces linked to a task to be performed (represented by means a transfer function $\tau : \mathcal{C} \to \mathcal{W}$) and their prolongations (en terms of k-jets spaces), information fusion on an eventually singular support (by means different kinds of C^r-transformations, preserving or not symplectic or contact structures), regularization of signals (in terms of activation-inhibition patterns depending on thresholds, p.e.) in regard with the sensor fusion, stabilization in presence of eventually non-linear vibrations, etc.

Hence, one can find a very large range of applications which is necessary to reorganize in terms of Global Analysis involving manifolds as support for non-linear geometries and non-linear operators or systems defined onto them.

The use of *non-linear geometries* is not new in Robotics [47], but there is no a systematic treatment from the geometric viewpoint including an efficient management of singularities. To organize the large range of applications, we shall consider methods of Algebraic Geometry and Geometric Algebra (please, do not mix them) along the first part including the two

gularities are linked to default rank of Jacobian matrices and their dynamical extensions. Anyway, all of the are linked to default rank matrices, and this description eases an unified treatment in terms of Geometric Algebra

[47] Some explicit mentions to Algebraic and Differential Geometry appears in [Sel95]

first modules. In a complementary way, methods of (semi-)Analytic Geometry will be developed in the third module (Robot Kinematics), whereas methods of (an extension of classical) Differential Geometry will be developed in the fourth module (Robot Dynamics).Their integration for the most advanced cases are developed in fifth (Humanoid Robots) and sixth modules (Animats), including some connections with external areas.

The performed *stratified approach* does not intend to develop an illustration of geometric techniques. It is just the contrary. My goal is to prove how the geometries provide the natural language for a stratified approach to the Robotics, where stratifications represent the different kinds of hierarchies between systems accordingly to the tasks to be developed. Lie actions provide algebraic support for a more compact presentation which eases propagation phenomena, whereas Geometric Algebra provides an intrinsic representation which allows to unify precedent approaches.

2.2.3. Optimization algorithms

Robotic system design and many problems in robot task planning can be formulated as optimization problems. In most cases, optimization consists of minimizing a functional or a set of functionals; more precisely, a minimization problem is defined by a pair $(\mathcal{L}, f)$ where $\mathcal{L}$ is a set of *feasible solutions* and $f : \mathcal{L} \to \mathbb{R}$ is a *cost function* where $\mathbf{x}^* \in \mathcal{L}$ is an *optimal solution* if $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{L}$. There are a lot of strategies for optimization issues; usual prerequisites in Robotics are linked to real-time execution and their robust vs adaptive behavior.

Linear programming use convex hulls for solving linear optimization problems which are the simplest ones with convex regions where optimal solutions are usually located at the boundaries. *Quadratic programming* (or more generally, Dynamic Programming) poses additional problems, because regions are not necessarily convex ones, nor optimal solutions are necessarily unique. In general, one introduces (eventually augmented) lagrangians which minimize the difference linked to desirable equalities, and incorporate additional constraints linked to the minimization of additional constraints linked to momenta, energy, work or any kind of mechanical constraints.

From a general viewpoint, the presence of a large number of non-linear constraints for the set $\mathcal{L}$ of feasible solutions makes harder optimization problems in Robotics and require more advanced tools. *Constraints* for optimization problems can be relative to external or internal conditions; they can involve to geometric, kinematic or dynamic aspects, in terms of (scalar, vector, tensor) fields by following an increasing difficulty. Jets spaces provide a general framework to manage hybrid constraints which can involve to several "levels" of k-jets extension.

Furthermore, in some cases one has *non-holonomic constraints*, i.e. they involve to kinematic (resp. dynamical) variables or functionals which do not arise from geometric (resp. kinematic) properties. Two typical examples concern to drift effects in Dynamics or friction effects in Dynamic. In the classical framework (fiber bundles or principal bundles), non-holonomic constraints introduces non-integrability along subbundles which must be modeled and solved at least in an approximate way.

Some tractable cases can be described in terms of default rank matrices linked to kinematics and dynamics which can be formulated in terms of Geometric Algebra.

Some extremal (discrete vs smooth, respectively) cases concern to

- *Optimization on graphs* linked to paths which minimize some scalar (such as length, e.g.) or maximize the cardinality of matching (in isomorphism subgraph problem, e.g.).
- *Optimization* of fields or functionals, including several subcases such as:
    - *Optimization of scalar fields* linked to isolated functions defined on a manifold such as height, depth, distance, etc. It is solved by computing critical values of functions.
    - *Optimization of vector fields* such as flow maximization, which is solved in terms of flow box, e.g.
    - *Optimization of differential forms* (or covectors) such as the work performed by a system, forces to be applied or torques at joints, e.g.
    - *Optimization linked to variational problems* defined by an action functional (Lagrangian) such as the total energy of a system, e.g.

If constraints verify simple conditions (linearity or convexity, typically), one has deterministic or randomized algorithms (in presence of huge data) for finding solutions to all of them. However, the apparition of several non-linear or even non-smooth constraints for the boundary of regions –even for the simplest cases described above– poses problems which are much harder to solve.

A typical strategy for managing multiple constraints which can appear in a sequential way corresponds to the reformulation of the optimization problem in terms of *weighted Voronoi diagrams*. In the *geometric case*, we are looking for elements which minimize locally the distance w.r.t. a collection of nodes which are labeled as Voronoi sites [48] From a much more interesting dynamical viewpoint, Voronoi sites act as attractors and nodes as repulsors, with saddle points located at the intersections of Voronoi edges with bisectors.

The *radial vector field* for each Voronoi site represents the simplest propagation model linked to (an eventually normalized version of) the gradient vector field. In this case, each node can represent a milestone to be achieved by an evolving system which is in charge of managing a hierachised system. The lower envelope of local distances w.r.t. a finite collection of functions $f_1, \ldots, f_n$ provides the support for global functional to minimize giving a minimization diagram for the resulting functional representation. Furthermore, the introduction of indices linked to the lower envelop provides ordering criteria (managed by sorting lists, e.g.) even in presence of non-linear constraints eventually linked to modifiable weights.

The problem becomes a little bit more complicated in presence of *mobile data*, where the motion can be "absolute" (external to the robot) or relative (linked to egomotion of a mobile platform). In this case, metric constraints are much more difficult to evaluate in real-time,, and a hierarchy going from topological, functional, and metric aspects must be designed with their corresponding feedback procedures, in presence of uncertainty. Again, optimization procedures on symbolic representations (different kinds of graphs and trees) provide a first solution for this problem. Some typical problems which are considered along the module 2 are the following ones:

- Development of more general models (beyond the regular case) involving *Duality between Optimization and Control issues* in presence of multi-objective criteria and different control modes.

- *Optimal control and decision making policies* in poorly structured or mobile environments (traffic scenes, e.g.) under changing environmental conditions for an individual agent .

- Algorithms for *simultaneous tracking and scheduling processes*, allocating resources (energy, distribution of forces), and decomposition of

[48] See the first module of my notes about *Computational Mechanics* (in Spanish) for details and references.

tasks and organization of teams of robots in a wide range of applications.

- *Feedback between robust and adaptive methods* arising from relations between fine geometric and coarser topological approaches.

Most these techniques have an almost universal character and they have been applied to optimal resources allocation relative in static environments; a typical solution can be developed in terms of weighted Voronoi diagrams [49] and/or some problems in Economic Theory [50]

Their applications in Robotics are a little bit more complicated and interesting, because the incorporation of changing environmental conditions requires a more dynamical approach to Optimization and Control issues in regard to dynamical systems under evolving initial and boundary conditions. A first approach can be designed in terms of constraints given by differential inequalities. To fix ideas, we restrict ourselves to the algebraic case, which is considered in the following paragraph from a theoretical viewpoint.

2.2.4. Differential algebraic inequalities

Constraints can involve not only to geometrical issues, but to "kinematic quantities" which are expressed in terms of vector fields, differential forms, tensor fields or differential operators in general. However, we can reason in a geometric way, not only for (scalar, co-vector or tensor) fields, but for differential operators, also. Indeed, (the orbit of) any k-th order ODEs can be represented as a subset of the k-jets space. Hence, differential (in)equalities can be interpreted as constraints in the jets space. A more classical approach uses the Laplace transform which allows to interpret differential equations in terms of the linked algebraic symbol given by a polynomial. The interplay between classical geometric invariant and differential invariant theory is a research field with a lot of open problems to be solved.

From a more applied viewpoint, discrete and continuum mechanics play an important role in the modeling of multibody systems in contact, which in turn are central to robot manipulation. Traditionally, these systems are governed by differential algebraic equations (DAEs). In order to be able to model unilateral obstacle constraints and/or more general environmental changing constraints, these DAEs have to be augmented by inequalities.

[49] See references in my web page and my publications available in the web.

[50] See my notes about *Métodos Diferenciales en Teoría Económica* (in Spanish language).

Again, the Semi-analytic Geometry [51] provides the right framework for solving this kind of problems.

The resulting differential algebraic inequalities are special instances of differential inclusions between subsets verifying additional algebraic properties (ideals of a ring, e.g.). A typical well-known example is given by Frobenius integrability conditions for differential systems [52]. However, the apparition of non-holonomic constraints and their global characterization (as non-integrable subbundles), requires additional theoretical and numerical developments going beyond the classical integrability conditions. In particular, some required developments involve to

- Qualitative behavior which is useful for modeling robustness issues around stable solutions. The interplay between differential topology and dynamical systems is an old topic going back to Poincaré, with important developments from sixties (Arnold and Smale, independently), but their discrete versions are waiting for more computable solutions.
- Simulation under uncertainty with self-adaptive or resilient behaviors; this issue is related with computational dynamics, and/or dynamic modeling based on materials with some kind of memory for the recovery of original shape, e.g.
- Non-smooth analysis including not only the analytical case which has been described above, but discrete approaches to differential methods, also. Discrete or PL-versions for vector fields and discrete differential forms provide two areas with relevant developments from the late nineties.

The development of Optimization methods in the PS-framework with adaptive constraints represented by differential ideals has a lot of open problems waiting to be solved. In particular, connections with Differential Theory of Invariants (Olver) display several connections with the main topics of the approach developed along these notes which is transversally crossed by Lie groups and algebras, semi-analytic stratifications, Geometric Algebra for jets spaces

[51] Semi-Analytic (resp. Algebraic) Geometry is locally defined by a finite set of analytic (resp. algebraic) inequalities $f_i(\underline{x}) \leq a_i$

[52] See the module 4 (Differential Forms) of my notes on Differential Geometry and references therein, e.g.

2.3. Tasks. A kinematic approach

A basic taxonomy for topological models concerns to the continuous nature of models vs discrete nature of data. Discrete aspects involve to information capture and generation of signals and commands; whereas continuous models involve to sets and functionals. This remark is valid for the three steps of Mechanics (Geometry, Kinematics, Dynamics), but in this subsection we shall paid a special attention to kinematic issues related to the execution of tasks.

Traditionally, one starts with continuous kinematic models in terms of (scalar, vector, tensor) fields, and then introduces a discrete version which allows to connect with the information capture, fusion and tracking. The simplest continuous model is performed in terms of cartesian space $\mathbb{R}^n$ which provides the support for different *linear* structures labeled as euclidian $\mathbb{E}^n$, affine $\mathbb{A}^n$ or projective $\mathbb{P}^n$ structures. Other interesting structures for Mechanics which are linked to the preservation of geometric amounts are given by conformal geometry (preservation of angles), symplectic geometry (preservation of Hamilton-Jacobi equations or the associated 2-form) or contact geometry (preservation of contact 1-form).

More generally, one can match together such basic pieces given by $\mathbb{R}^n$ plus an additional structure by means of regular (algebraic, differential, analytic) functions [53] to get non-linear algebraic, differential and/or analytic varieties X. More realistic models are given by Piecewise Linear/Quadratic (PL/PQ) or, more generally, Piecewise Algebraic (PA), Piecewise Smooth (PS) structures. Embedded submanifolds of each geometric structure on a space inheritate the structural properties of the ambient geometry. In particular, C^r-constraints require specific developments of each geometric framework to understand basic issues relatve to the considered C^r-structure.

Matching of any kind of additional structures is performed by using regular applications which preserve some kind of (metric, affine, projective) linear structure for the support. For arbitrary regular (algebraic, smooth, analytic) functions one obtains a non-linear structure. Thus, it is necessary to understand how matching can be performed for basic pieces corresponding to euclidian (i.e. locally equivalente to $\mathbb{E}^n$ spaces) or affine (i.e. locally equivalente to $\mathbb{A}^n$ spaces) pieces.

et-theoretical topology is usually modeled in terms of sets U of a topological space X verifying "good properties" for set-theoretical operations (union and intersection), and "regular" functions defined on such open sets.

[53]Regular functions are invertible with inverse belonging to the same category

Roughly speaking, sets U represent data collections to be matched, whereas regular functions represent the evaluation of functionals on such data which allow to compare, group and classify them. Very often we have only a partial information about the surrounding space or functions (or transformations or operators) acting on a topological space. Some typical cases correspond to regularization of signals, kinematic tracking of trajectories, or optimization and control issues, where only fragmentary or discontinuous information is available, usually.

In practice, the robot must take decisions by matching together such partial information, even when data can not be matched in an exact way (due to noise or deformations, e.g.) or in presence of uncertainty (partial or skewed information, biased data, e.g.). Thus, furthermore well-defined information (which can be described in terms of "deterministic" models), it is necessary to incorporate a less-defined (which can be described in terms of "probabilistic" models). Thus, the feedback between deterministic and probabilistic modeling is present from the beginning, even for the design and implementation of algorithms and the devices in charge of managing any kind of kinematic information. In particular, the statistical approach to Mechanics requires a randomized version of fields (Markov).

Mechanical matching of common (continuous or discrete) data is performed at different levels. The lowest level can be described in terms of set-theoretical topology which provides mathematical tools for matching data for constructing models relative to sets and operators with any kind of structure; initially, one supposes that such data are continuous, but discrete data can be also matched to construct models. Next step consists of introducing additional (algebraic, analytic, differential) structures which allow to guarantee stronger properties for more sophisticated problems involving feedback procedures appearing in optimization and control issues, e.g. To solve this problem we have a large diversity of tools. The nearest ones to the approach performed to Robotics along these notes are the following ones:

- *Combinatorial topology* involving configurations of linear and/or quadratic constraints (geometrically represented as arrangements, e.g.) which induce a tessellation of spaces which are meaningful for planning, tracking and executing tasks.

- *Algebraic topology* involving to paths, cell decompositions and functionals defined on them to identify geometric and topological changes (relative to different kinds of scalar, vector or tensor fields) from invariants.

- *Differential topology* involving to vector fields for trajectories, differential forms to represent forces, connections to represent displacements of geometric quantities, and their invariants on configurations and working spaces. All of them include discrete versions which allow to connect with the precedent items.
- *Dynamical Systems Theory* involving the resolution of structural equations linked to robot kinematic and dynamics, with a special regard to optimization and control issues.

2.3.1. Combinatorial and Discrete Topology

Discrete topology involves not only to (sparse or dense) clouds of isolated elements (usually points), but to mutual relations between them. The most common representation for such relations is given by symbolic representations (graphs, e.g.) which are superimposed to the original clouds. From the 18th century (Euler), symbolic representations are represented by graphs $\mathcal{G}$ (including trees $\mathcal{T}$ as graphs without cycles), or more generally forests $\mathcal{F}$ given by a finite union of trees. Basic dynamics is represented not only by weighted symbolic entities (graphs, trees or forests), but by activation/inhibition phenomena which modify the original state of each symbolic entity; its formulation is given in terms of symbolic dynamics, also. Evolving configurations give planar or spatial configurations where edges or faces are regrouped to give new shapes.

A computational management of evolving configurations requires to identify "persistent configurations" to be reinforced, and their topological invariants. Homology Theories provide a support for managing such superimposed structures, and compute their invariants by using combinatorial properties of the associated configurations. Usually, combinatorial topology is focused towards the estimation of topological invariants linked to discrete or sparse information which can be irregularly distributed. In particular and in regard to some topics sketched above, graphs provide a general tool for general configurations involving the whole structure or the external space for the robot, in terms of configurations and working spaces. Very often (in presence of non-structured data, e.g.) it is necessary to explore different possibilities, including the worse case associated to all possible combinations.

Data organization requires introduce different optimization criteria involving to ordering criteria (sorting algorithms, e.g.), minimize or at least bound some geometric amount (distance or area, e.g.) or kinematic amount (energy expenditure, work to be performed, e.g.). The presence of local symmetries (arising from mechanisms or the external space) simplifies the data treatment and, consequently, optimization strategies. Thus, it is convenient to design strategies on locally symmetric spaces to minimize casuistics, and to reinforce the robustness of algorithms.

The linked low-level strategies are described in terms of combinatorial topology on the simplest homogeneous (cartesian, euclidian, affine, projective) spaces, which considers local combinations of basic elements to generate intermediate primitives. The static approach is developed along the first module (anchored robots), whereas the dynamic approach (evolving graphs) is developed along the second module (mobile platforms). This reasoning scheme can be applied to configurations of $0D$ elements (points, vertices, intensity maxima, e.g.), $1D$ elements (segments, arcs, linear or non-linear arcs) or higher dimensional entities. Independently of dimensions and shapes (even for curved primitives), the combinatorial arguments are always the same. Some typical examples concern to

- States of a system composed by several eventually mobile agents representing the current state of mobile elements (cars, control points, etc) of one or several robots.
- Configurations of 1D elements (segments/lines or curves, and their grouping in polygonals/polyarchs or surfaces).
- Evaluation of efforts performed by superficial elements (kinematic chains contained in a plane), including generation of moments and their effects on the whole structure.

The support for internal (resp. external) spaces can be represented by a graph whose nodes are points of control (resp. focus of attention, e.g.), whose edges are links (meaningful segments or lines in the scene, e.g.) and whose planar elements are planar kinematic chains or plats (resp. Some meaningful problems involve to influence regions for each element (extensions of Voronoi diagrams, e.g.) or their relative localization (in terms of arrangements, e.g.) relative to PL or PQ elements.

There is no a unique strategy for solving all of them. Furthermore, geometric elements can display different weights depending on their relative importance, or the role which is played along the representation of a task as a pipeline of subtasks. Thus, it is convenient to develop several combinatorially based models for solving this kind of problems. Some typical applications concern to:

- *Modular robots* which are built of many identical modules, which can be replicated in a semi-automatic way. Such devices can reconfigure themselves to suit their environments. The configuration space of such robots is inherently discrete.

- *Discrete actuators:* Smoothly actuated robotic arms (and other manipulators) can be replaced by a cheaper network of discrete actuators (devices that extend or contract into only two positions). Cellular automata provide some interesting connections which are able of incorporating a distributed intelligence.

- *Discrete sensing:* Instead of determining robot position and configuration by a raster image (essentially, a continuous image of the environment), faster and cheaper robot localization can be achieved by using a modest number of discrete sensors. Their integration in complex networks eases the search and finding of patterns w.r.t. multiple criteria.

Anyway, symbolic representations given by graphs (including trees or, more generally, forests) play an important role to organize tasks, recombine components and manage information which can be reused in different ways.

2.3.2. Algebraic Topology in Robotics

The main goals of Algebraic Topology are the characterization and classification of topological spaces by using algebraic properties. To achieve these goals there are two strategies which consist of

- describing *classes of paths* (corresponding to trajectories to be performed by control points, .e.g) or more precisely loops, i.e. paths which are not contractible to a base point (due to the presence of obstacles, e.g.);

- *superposing additional PL- or PS-structures* (triangular or quadrangular meshes, e.g.) and study properties of functionals defined on them; this kind of methods eases the design of mechanisms (in terms of finite element methods, deformable plates based on B-splines, e.e.) or the automatic generation of spatial representations (Digital Elevation/Terrain Maps) from the discrete information arising from sensors, e.g.;

- combining the two approaches in terms of generalized loops on complexes; in other words, try of extending basic constructions based on closed loops (topologically equivalent to the circle $\mathbb{S}^1$), to higher dimensional spheres in regard to metric constraints, propagation phenomena and resolution of conflicts linked to the mutual influence between components of PL or PS-superimposed structures.

Some meaningful applications of Algebraic Topology are related to

- *Motion planning* from loop spaces for tasks requiring some kind of displacement in presence of eventually mobile obstacles.
- *Interpolation models* for arbitrary open paths and closed loops for trajectories and mechanisms associated to kinematic chains and platforms including several kinds of joints and links.
- *Simplified representations* by means of (cubical or simplicial) complexes linked to complicated scenes or objects (mobile entities, .e.g) with a sparse collection of meaningful key points and/or segments to ease an optimal control.
- *Symbolic representations* in terms of cellular decomposition of configurations and working spaces compatible with internal constraints (linked to mechanisms) and external constraints (linked to obstacles contained in the scene).

2.3.3. Differential Topology in Robotics

Differential Topology studies PS-objects (PS: Piecewise Smooth) X and differentiable maps $f : X \to Y$ relating them from the differential viewpoint. The condition for a map or a manifold of being smooth is very strong, but the condition of being piecewise smooth is very commonly fulfilled. It allows to perform local optimization procedures and, consequently, provides the support for robust and/or adaptive control.

Differential Topology provides a support to apply the differential and integral machinery developed on cartesian spaces, including differential analysis of functions, behavior of vector fields (to describe Kinematics on a variety, e.g.), differential forms (to represent forces and the work performed by components in terms of the cotangent space, e.g.), or any kind of geometric amounts which are transformed in a covariant way on a manifold, or more generally, a variety. To accomplish these tasks, it uses *different kinds of* (scalar, vector and tensor) *fields*.

The simplest case does correspond to *scalar fields* which can be geometrically represented in terms of level (hyper)surfaces linked to functions $f : M \to \mathbb{R}$ with "good properties" such as having only non-degenerate critical points; for a smooth compact variety M, this kind of functions are dense and they are called Morse functions. Their use allows to reconstruct M by adjoining "cells" whose dimension is determined in terms of algebraic invariants of the Hessian matrix $Hess(f)(\underline{p})$ at each critical point $\underline{p} \in M$. Typical Morse functions are given by height, depth, scalar energy, distance intensity of a signal, etc. Critical levels correspond to local topological changes which are expressed in terms of adjunction of cells.

Typical examples for *vector fields* on manifolds are given by gradient, *curl*, divergence, which provide local versions for the (dual of) exterior differential. Main related differential operators are given by Laplacian and their generalizations. The classification of quadratic differential operators provides the starting point not only for classification of systems, but for topological analysis for their solutions. Typical *tensor fields* are linked to different kinds of curvature, which can be reinterpreted in terms of the external space or the "troubles to perform a task".

Along these notes, the approach based in (scalar, vector, and tensor) fields has been extended to consider more general constraints

- defined on eventually singular varieties in the Stratified Morse Theory framework for configurations and working spaces in Robotics, and their feedback given by the transference map τ ;
- given by dynamical systems in terms of PS-flows (and their integration given by homeomorphisms, e.g.) with advanced control systems by following Conley-Morse Theory.
- functionals involving different kinds of curvature-energy functionals or generalized forces acting on meaningful elements;

From a practical viewpoint, it is necessary to unify optimal control strategies for mechatronic devices, to design and implement efficient algorithms, and optimal management of information. The development of specific sensors, integration of multisensors on each body, and the development of distributed intelligence between different agents provide solutions with increasing performance.

- Distributed sensing and actuation systems, such as those made possible with MEMS (Micro-Electronic Mechanical Systems) and nanotechnology;
- High dimension design problems, such as intelligent vehicle-highway, sea- and air-traffic management systems;
- Adaptive and friendly interaction with humans in less-structured environments by using supervised learning techniques, e.g..

2.3.4. Dynamical Systems aspects

Furthermore Control and Optimization issues which have been considered above, some areas of increasing interplay between experts in Dynamical Systems and Robotics involve to high complex tasks of humanoid robots such as locomotion (including walking, running, jumping) and manipulation tasks (including grasping and handling). Both of them require a feedback between anticipatory and compensatory phases to stabilize and to improve their performance.

The apparition of different phases (including open and closed loops, relocalization, adaptation to changing environments) requires the development of complex representations which can be recombined and reorganized depending on variable weights. A typical motivation is given by networks of cooperating robots which can be modeled as dynamical systems on graphs. Symbolic Dynamics provides some interesting tools going beyond usual Dynamical Systems Theory. [54]

In all cases, the discrete nature of signals and commands, motivates the development of discrete versions of traditional approaches corresponding to the smooth case. From the late nineties, there is an increasing interest in developing discrete versions of Differential Calculus, including vector fields (interesting for trajectories, e.g.) and differential forms (interesting for dynamical aspects where forces or the work to be performed is modeled in terms of differential forms, e.g.). However, some additional developments are required, specially in regard to the reformulation of classical Morse Theory for multiple nodes interacting between them.

The changing environmental conditions requires a combination of robust and adaptive strategies which can be organized around central (attractor vs

[54] See B.P.Kitchens: *Symbolic Dynamics*, Springer-Verlag, 1998, for an introduction

repulsor centers e.g.), organizers for dynamics (saddle points and their extensions), or structurally stable behaviors (around limit cycles, e.g.). Due to the uncertainty about initial conditions and the need of self-adaptive strategies, it is necessary identify invariants of dynamical systems w.r.t. deformations. The Morse-Conley theory provides a support for these additional requirements.

In the final report which summarizes some conclusions of the Workshop about *The Interplay between Mathematics and Robotics* [55] the authors call the attention about the "increasing awareness of the hybrid nature of artificially engineered (and naturally occurring systems) and the fact that discrete and continuous dynamics interact in complex ways. The mathematics of such hybrid systems is not well developed. Issues like robustness to modeling uncertainty and to noise, and sensitivity analysis, are not well understood. There is no systematic approach to developing abstractions of dynamical systems and to decompose a dynamical system into a multi-scale hierarchy of subsystems."

The approach performed along these notes introduce a strategy which combines some aspects of an extension of Morse Theory and some others arising from weighted Voronoi diagrams:

- *Weighted Voronoi Diagrams* allow to manage different organizers for the dynamics according to criteria going from (different kinds of) metric to ordinal criteria. It provides a spatial decomposition which is primarily applied to working space, and next to the configurations space for solving inverse kinematics and dynamic problems. This approach allows to coordinate different strategies, and eases a framework for the communications between different subsystems in terms of propagation phenomena.

- *Morse Theory* allows to manage critical behaviors linked to the presence of different functions (in terms of lower envelopes, e.g.) which act in a sequential way in the working space. Environmental conditions can be interpreted as different constraints represented by functions. Their changes are represented by vector fields whose tangential and normal conditions are controlled in terms of propagation phenomena, again.

Some applications of both issues for interacting robots are displayed in the modules 5 (Humanoid Robots) and 6 (Animats) of these notes.

[55] National Science Foundation, Arlington, Virginia, May 2000

2.4. Towards an dynamic integration

A *leit-motiv* which is transversal to the six modules of these notes concerns to the role of symmetries along all the mentioned topics. Symmetries are ubiquitous in Geometry (classical groups, principal bundles, e.g.), Topology (C^r-equivalences, deformations, e.g.) and Dynamical Systems (automatic generation of homeomorphisms, first integrals, e.g.). More recently, they have been used to integrate different viewpoints in regard to Differential Invariant Theory (Olver et al), Equivariante Stratifications beyond the Moment Map (Marsden et al) and Equivariant Bifurcation Theory (Golubitsky et al), between others.

The application of Lie groups to the geometry of the Moment Map is not new (see [Gui90] for a survey), neither its adaptation to Mechanics of Rigid Bodes [Sat86] or to more complex systems including deformable objects [Mar94]. In fact, Lie groups and Lie algebras make part of the core of classical books in Robotics such as [Mur94]. However, there is no a general reference able of extending the above approaches to the stratified case in the analytic or multivectorial framework. This absence justifies the development of these notes, whose preliminary versions have been exposed in different centers (Caltech in Pasadena, SSSA in Pisa, Politecnico di Torino, LASC de Metz, Institut Fourier de Grenoble).

Classical Lie Groups are increasingly used for motion planning, control and optimization issues in Robotics. However, topological symmetries are less developed and are even more crucial for autonomous robots, because they involve to different kinds of interction with non-structured or open environments. A *local* topological approach allows to model transitions between robust and adaptive behaviors, e.g.; indeed, it suffices to take the compact-open topology which is the natural framework for proximity issues in jets space [56].

2.4.1. The role of symmetries

There are quite different kinds of symmetries which very often generate a group G; the most common ones are the following ones:

- *Finite groups* which involve to internal or external configurations of

[56] Let us remember that, following our approach, jets space provides the general framework for a hierarchical approach to Mechanics of Robots (see below); this approach is developed along the third module

elements, including their applications to tessellations, graphs or replication at different LoD.

- *Classical groups* which involve to the preservation of a "geometric amount" which is represented by a quadratic form (the distance or the kinetic energy, e.g.), a bilinear form (the symplectic or the contact form relative to the moment map, e.g.) or more general tesnors (volume forms, e.g.). These groups are the structural groups for the Linear Geometries (Euclidian, Affine, Projective) and their extensions (Special Linear, Conformal, Symplectic or Contact structures)

- *Infinite groups* which involve to the different kinds of Non-linear Geometries (Algebraic, Differential, Analytic) for non-linear objects.

All of them have an infinitesimal version in the Lie algebra $\mathfrak{g} := T_eG$ which is very useful for estimation issues; the reason is very clear: computations in a vector space is much easier in a vector space, rather a topological space. The exponential map provides a local homeomorphism with a neighborhood of the neutral element, and it allows to recover local properties of the group.

Furthermore, the *inclusion relations between groups* induce different hierarchies between linked Geometries which are translated to structural constraints involving the representation of the environment and different ways of acting onto such space.

On the other hand, all groups of non-linear geometries are subgroups of the group of homeomorphisms and the first order component of each homeomorphism is a general linear group $GL(n;\mathbb{R})$ for some $n > 0$. As each classical group is a subgroup of $GL(n;\mathbb{R})$, one recovers a structural connection between Non-linear Geometries and Linear Geometries by means the linearization of the group action of C^r-equivalences between manifolds.

This approach was extended to dynamical systems, where each solution (a trajectory through a point with a fixed velocity, e.g.) can be interpreted of a local homeomorphism acting on an integral curve. Hence, to solve a dynamical system it suffices to find a complete system of first integrals which are interpreted as a collection of first integrals with their corresponding Lie-Poisson structures.

Unfortunately, things are not so simple, and most interesting systems are not fully integrable. Nor even the operation of car parking is an integrable system. However, the lack of complete integrability has some meaningful advantages because it allows to act from an external viewpoint to modify the original system and to re-drive according to the planned task (car parking, e.g.). This lack of complete integrability is labeled as non-holonomic constraint which is developed along the fourth module of the Course.

2.4.2. Breaking symmetries

The simplest industrial robots develop a repetitive task for which a robust control can be easily implemented. Up to this case, most robots display more complex behaviors which require an adaptation to changing environmental conditions. Furthermore, hierarchies between tasks introduce different kinds of symmetries which can be grouped or break-down depending on the phase. These phenomena involves to geometric, kinematic and dynamic aspects.

Luckily, one has an almost complete dictionary which allows to exchange information between different aspects involving Geometries, Topologies and dynamical systems [57]. This local approach has been applied to some aspects of smooth manifolds, but contributions to stratified maps and relations between symmetries involving the natural hierarchy for the Mechanics is less explored.

Some crucial aspects to be developed consist of identifying which kind of vector fields can be lifted from the geometry to the kinematics or, even more, the dynamics involving configurations and working spaces. For simple singularities, the solution for the local problem is an easy consequence of some developments performed in the Doctoral Dissertation of T.Perez (non published, unfortunately). However, the general problem is quite open.

From a global viewpoint, liftable fields are strongly related to the existence of non-holonomic constraints which have been extensively studied in [Blo03] [58] and it is extensively commented along the fourth module. However, connections between local and global aspects are unknown, mostly. Along modules 5 and 6, we shall explore several aspects in regard to locomotion tasks in a stratified framework which extends the original approach performed along the late nineties by I.Stewart et al.

[57] [Gol88] M.Golubitsky, I.Stewart, D.Schaeffer: *Singularities and Groups in Bifurcation Theory. Volume II*, Springer-Verlag, 1988

[58] A.M. Bloch: *Nonholonomic Mechanics and Control*, Springer-Verlag, 2003.

2.4.3. Perception-Action Cycle

The foundation for the hierarchical approach to Mechanics is a feedback between impulses at joints (performed by torques) and re-localization of end-effector to improve the execution of tasks. The feedback is represented by an iteration of the transferencia map $\tau : \mathcal{C} \to \mathcal{W}$ and some pseudo-inverse. Beyond the particular matricial expression of the pseudo-inverse, it existence is warranted by the structure of τ as stratified map (locally trivial away from the singular case, i.e. giving a finite set of local sections which allow to find local inverses). The iteration of τ and its pseudo-inverse is written as

$$\ldots \to \mathcal{C} \to \mathcal{W} \to \mathcal{C} \to \mathcal{W} \to \ldots$$

Their extensions to jets spaces provides a representation for more advanced issues concerning to manipulatibility and dexterity in grasping and handling tasks, or for analyzing the feedback between anticipatory and compensatory strategies to balance the human body along complex locomotion tasks (including jumping and running, e.g.).

More generally, the geometric representation of the information provided by multisensorial perception allows to construct semi-analytic spaces to represent different kinds of signals and commands, which provide the inputs for Perception and Action modules. Usual description of signals and commands is given in terms of 1D/2D/3D frequencies which can be translated to a spatial representation by means Fourier transform going from the frequency to the spatial domain.

Thus, if we denote by means $\mathcal{P}$ (resp. $\mathcal{A}$) the common space for the geometric representation of signals linked to different sensors (resp. actuators), one has a feedback between both spaces

$$\ldots \to \mathcal{P} \to \mathcal{A} \to \mathcal{P} \to \mathcal{A} \to \ldots$$

which extends the above mechanical feedback, but from a higher viewpoint which can be formulated in terms of the (spatial version of) Fourier analysis.

2.4.4. Geometric Algebra approach

Roughly speaking, Geometric Algebra approach can be understood as a multilinear framework which allows to manage any kind of scalar, vector and multivector entities by using two kinds of products which extend the well-known scalar and cross-products [59]

From a geometric viewpoint, one can use bivectors for Mechanics which are described by screws, twists and wrenches involving to the usual mechanical hierarchy in geometric, kinematic and dynamic terms. Each bivector represents an oriented area element which can evolve along time, according to mechanical constraints.

The geometry of bivectors is essentially the same as the given by oriented lines in a three-dimensional projective space, which is classically describen by Grassmannians. Hence each space of mechanical bivectors (screws, twists and wrenches) can be describen in terms of oriented Grassmannians. This simple remark provides a geometric framework for a common treatment of their properties.

Furthermore the internal duality inside each Grassmannian, one has an "external duality" which involves to the exchange between position and force mechanical quantities (i.e. screws and wrenches), and self-dual twists. This "external duality" allows to develop a feedback between position-based and force-based control, which is commonly developed in Statics. Furthermore, the self-duality involving twists can be translated in terms of impedance-based control.

A reformulation of the Contact Geometry (involving incidence and tangency conditions, which are dual between them in their projective completions) provides the feedback between position-force based control and impedance-based control. In this way, one obtains a structural connection between Dynamics and Kinematics which is a geometric reformulation of structural contact constraints for k-jets spaces relative to the transference map $\tau : \mathcal{C} \to \mathcal{W}$

An effective control of these issues requires a geometric reformulation of basic functionals linked to work and energy which must be rewritten in multivector language. A first version of these ideas was presented in Agacse'01 (Cambridge, 2001) in regard to human locomotion, and it is developed with more detail in the fifth module (Humanoid Robotics). Their extension to much more complex evolving shapes (and their application to animats) is a challenge to be solved.

[59] More details are developed in several places along this Course in regard to kinematics and dynamics approach and their applications to different kinds of robots.

3. Determinism and uncertainty in Robotics

Data arising from external world are captured by sensors with a high uncertainty level. Information processing and fusion of *noisy discrete data* are initially reprojected on *robust continuous models*; to begin with, most of them have a geometric nature. Their kinematic variation along time and different kinds of dynamic interaction with other intelligent agents in increasingly complex environments requires the development of adaptive models able of managing feedforward strategies.

Most mechanical models have a continuous character, but real-time interaction requires the development of hybrid models able of integrating discrete information arising from sensors. There are a lot of developments which integrate geometric or functional aspects with statistical or probabilistic models; however, there is no enough adaptive strategies for dynamic issues in presence of uncertainty for real domains.

Strategies to be developed follow a two-way road, with relaxation of deterministic models (by using Stochastic Differential Equations, e.g.); if one looks the problem from the other side, it is necessary to incorporate some kind of structure on sparse data (by using Monte Carlo Methods for clouds of particles, e.g.). However, to improve the sensitivity w.r.t. small variations and the capability of response, one needs finer models, able of incorporating low-order variations of tracked data which are represented by (discrete vs continuous) fields.

General frameworks based on (scalar, vector, tensor) fields suggests an approach which is based on Hidden Markov Models (as extension of Markov Fields), to be estimated by using Gibbs distributions, e.g.. Capability for taking decisions in presence of uncertainty can use observable Markov Decision Processes which provide a probabilistic approach to classical Optimization issues, and consequently for adaptive control issues under uncertainty conditions.

In absence of structural mechanical models, the above proposals seem at first sight “too generic”, and can give an excessive casuistic making impossible a real capability for self-adapting to evolving environments. Thus, the main emphasis of this section is put on the specification of structural models, leaving the development of probabilistic approaches for last chapters of modules 2 (Navigation), three (Kinematics) and 4 (Dynamics and Control) of this matter B_1 (Robotics. A hierarchical approach).

Initial “deterministic” mathematical models for mechanics try of identifying universal motion laws which are translated to rigid and articulated mechanisms. Ideally, one supposes ideal environmental conditions which are progressively relaxed, by replacing initial hypotheses relative to mechanical components (absence of drift, e.g.), sensors (white noise for signals, e.g.), actuators (linear vibrations, e.g.) or interaction with the environment (frictionless, e.g.) by other more realistic ones.

Along the 19th century there is a very fast development of mathematical models for the Classical Mechanics which is initially formulated under ideal conditions in the Lagrangian and Legendrian framework at the early decades. The framework for these developments makes part of the Differential Geometry, and it runs into the Hamilton-Jacobi formulation of the Analytical Mechanics. The introduction of a dynamics linked to the minimization of a Lagrangian (the total energy of a system, initially) contributes to the development of more general variational principles which are linked to the minimization of action functionals (called lagrangians in the literature), extending the original formulations given by Newton and Euler in the framework of Integral Geometry on Manifolds.

In the smooth case, the equivalence between differential (Hamilton-Jacobi) and integral (Newton-Euler-Lagrange) in the framework of the Symplectic Geometry was already known at the end of 19th century. Symplectic Geometry is characterized by the preservation of a bilinear form which is represented by a matrix J verifying $J^2 = -I$ (natural extension of the condition $i^2 = 1$ for the anti-involutive automorphism of complex numbers).

The use of methods arising from Differential Geometry is not new in Robotics. This use is natural, because Differential Geometry is directly linked to non-linear constraints involving descriptions of Configurations and Working Spaces in the whole Design phase, trajectories tracking for control points in Kinematics, identification and management of forces and moments in terms of systems of differential forms acting for each component. Furthermore, general principles of Classical Mechanics formulated as conservation laws (momentum, energy, work to be performed) are easily formulated in differential geometric terms as conserved quantities by a field which is linked to a variational principle in the lagrangian formulation.

The identification of infinitesimal symmetries linked to lagrangian functionals was started by E.Noether who underlined their role to find first integrals for motion. The introduction of the geometry of G-orbits for a classical group from the sixties provides a new insight in the framework of equivariant Differential Geometry which culminates in the equivariant analysis in terms of the moment map [Gui84].

The unification of the dynamics of articulated multibodies with some aspects of the Mechanics of Continuous Media was performed by Marsden and his collaborators at the late eighties [Mar94]. Along these years it begins a systematic treatment in terms of Lie actions and their applications to Robotics, with different applications including Optimization and Control issues. A feedback between kinematic and dynamic aspects in terms of Lie algebraic actions was started by J.Burdick and J.Ostrowski at the second half of nineties with very meaningful applications to snake-like robots and hybrid (wheeled and legged) mechanisms in regard to the Mars exploration.

In most publications one supposes that robot is working in smooth manifolds M, and one considers typical linear structures given by vector bundles (cotangent bundles with its natural symplectic structure) or principal bundles (when symmetries groups play a structural role), with their corresponding (metric, affine, Ehresman) connections. In some cases, if singularities are allowed, they are avoided because control issues are more difficult to solve near to the singular locus. However, this is not a natural approach from the biomechanical viewpoint, because most live beings take advantage of energy exchange with the environment through symmetries appearing at different (geometric, kinematic, dynamic) levels.

Our approach tries of taking advantage of passing through some simple singularities; this viewpoint is inspired by biomechanical behaviors, including complex operations (jumping, running) linked to human locomotion. The initial indeterminacy and inherent unstability linked to simple singularities are solved by introducing infinitesimal symmetries which are linked to entry trajectories w.r.t. the normal direction to the variety supporting the kinematics or the dynamics. Some fruitful discussions with J.Burdick and J.Ostrowski (Caltech) have allowed to shape it in regard to some applications to human locomotion for disabled persons in cooperation with G.Belforte (Torino) and P.Gorce (Cachan) and M.Guihard (Paris). The resulting model provides a higher efficiency to perform some complex motions, provided maintaining stability around the expected trajectories. Furthermore, it provides a natural feedback between control kinematic and dynamics, which can be restated in terms of impedance-based and force-position based control.

To implement this program, it is necessary extend some well known results of Differential Geometry of Manifolds and superimposed structures (vector bundles, principal bundles, connections) to the framework of Semi-analytic Geometry. The basic idea consists of using different kinds of (scalar, vector, tensor) fields to control (in terms of controlled submersions, e.g.) non-smooth scalar functionals defined on phase spaces such as the Lagrangian, PS-trajectories (integral curves of vector fields with corners, e.g.) and surfaces behavior (including deformations, non-linear vibrations, friction effects, e.g.). From an algebraic-geometric viewpoint, our *strategy* consists of going from infinitesimal aspects corresponding to small impulses in Configurations Space (modeled in terms of Lie algebras or first integrals of motion) to true motions in working Space (through an extension to the geometry of moment map) equipped with a natural equivariant structure linked to contact structures.

Last subsection is devoted to a very short presentation of Expert Systems in Robotics as a central part of Artificial Intelligence (AI) in Computer Science. This research area is very large, and by reasons of space we shall limit ourselves to some of the most known paradigms which are related with Artificial Neural Networks (ANNs) and their extensions to Genetic Algorithms (GAs), Evolutionary Programming (EP) and Self-Organized Maps (SOMs). To ease the connection of this area of AI with the geometric approach which has been displayed below, we reinterpret the above paradigms in geometric terms, by interpreting a planar ANN as (the discrete version of) a manifold, and the above paradigms as (discrete versions) of typical superimposed structures to manifolds (vector bundles, principal bundles, fibrations, e.g.). This approach provides a natural feedback between mechanical issues (where fiber bundles are ubiquitous) and the Perception-Action Cycle, which displays a similar structure from the structural viewpoint.

3.1. Modeling the coordination

Coordination tasks in robotics involves to mechanical aspects (relative to kinematic chains connected to mobile or eventually articulated platforms), integrated electronic devices (sensors and actuators, mainly) and expert systems (in charge of capturing, analyzing, interpreting and making decisions). Additionally, complex interactions with the environment require foresee the response and, consequently, anticipate and compensate their effects.

Relative to mechanical issues, two important cases for motions coordination are related to locomotion tasks by one side, and manipulation (including grasping and handling) tasks by the other side. Both of them requiere a good initialization (including re-positioning, e.g.), a feedback and integration of data arising from propioceptive and exteroceptive sensors, and an identification of hierarchies involving to optimization and control issues for the execution of motions.

Their organization makes part of the *Perception-Action Cycle* (PAC) which is a hard challenge for the most advanced topics in Robotics. To perform an efficient fusion of information arising from different sensors, it is commonly acknowledged that it is necessary to use a robust representation of space which can be adapted to appearances-based model along the tasks execution, and an adaptation of sensory-motor devices in terms of tasks to be performed. Hence, it is necessary to combine the robustness of the external world, with the adaptability of the mechatronic architecture which tries of mimifying some functionalities of superior mammals.

Understanding the environment is the hardest challenge and requires an intensive development of Expert Systems including Cellular Automata (CA), Artificial Neural Networks (ANN) and their improvements (Genetic Algorithms, Evolutionary Programming, Self-Organized Maps), and Fuzzy Systems. All of them reappear in different ways along different chapters, because they provide some of the most common options to incorporate some kind of "Artificial Intelligence" (AI following the old terminology), and consequently autonomy on mechatronic devices

Furthermore, the chosen representation for control, analysis and decision making, must be compatible with kinematics and dynamics. Last components of Mechanics can be modeled in different ways according to a modular structure which can ease the reuse of componentes and routines. Our strategy for mechanical issues consists of starting with symmetrical principles (represented by reflection or Lie groups and algebras, e.g.), stratified approaches (linked to an analytical presentation of tasks) and their synthesis in multivector terms (Clifford or Geometric Algebra). Surprisingly, all of them can be incorporated for the design of more efficient Expert Systems in Robotics. Nevertheless the advances from the nineties, this program is far from being completed, and there is room for a lot of research to be done.

To fix ideas, we shall restrict ourselves to some preliminary comments about the above topics , which include some applications for assistance robotics to disabled persons, which has been our main research topic between 1998 and 2003.

3.1.1. Towards an intrinsic representation

Recent formulations for the unification of different interactions in Physics make use of fields to explain and represent the observed effects. There are different kinds of fields which are labeled as scalar, vector or tensor fields, by following an increasing order or difficulty. Scalar fields are linked to functionals, vector fields provide a dynamical representation for trajectories and covector fields (also called differential forms) provide a numerical evaluation of effects linked to vector fields, such as the work performed by a system, e.g.. Tensor fields are the natural extension of scalar and (co)vector fields for giving a compact representation of phenomena involving different geometric entities or functionals. They are not invariant, but they are covariantly transformed, which eases their tracking along a manifold. However, the original hyper-matricial notation is very cumbersome, and several more compact and powerful alternatives haven been developed from the mid of the 20th century.

Furthermore, as we need to "translate" objects (including eventual modifications), it is necessary to have an intrinsic notion, compatible with abstract representations of configurations and working spaces as manifolds (in fact, as submanifolds of products of Lie groups). A *connection* can be described as a rule of covariant differentiation for any kind of tensors, and allows to represent the "transport" of "geometric entities" on a manifold. There are several types of connections which are labeled as

- *metric or Riemann connections* which preserves the metric and extends the Riemannian connection to any kind of pseudo-riemannian variety;
- *affine or Koszul connections* which allow apparent distortions linked to distortions produced by sensors or linked to relative motion (observer w.r.t the scene), e.g.; and
- *Ehresmann connections* which allow to incorporate strong non-linearity dynamical effects linked to the apparition of non-holonomic constraints on the velocities (drift effect. e.g.) or the accelerations (different types of frictions, e.g.)

Let us remark that connections allow to transform contravariant in covariant representations, which extend usual contraction (integration of differential forms along vector fields) or expansion (external product of differential forms or vector fields, e.g.) operations in a compact and intrinsic way.

From a complementary viewpoint, one can perform a multivector approach in terms of *Geometric Algebra* which can be introduced as an extension of quaternions introduced by Hamilton and extended by Clifford. The representation of spatial rotations in terms of quaternions is more intrinsic (i.e. coordinate-independent), avoids error propagation linked to trigonometric functions, is much more compact, understable and easily updatable than usual matricial representations. As the formalism of connections is more complex than the multivector formalism of Geometric Algebra, we adopt this framewrok for the following paragraphs.

3.1.2. Motor coordination

Biomechanical models are hyperredundant even for the simplest models of complex organisms such as mammals. In particular, even for a simple rotation around an axis one has different muscles which can act in an independent or joint way to produce different kinds of responses, or contrarily the same response in regard to the end-effector. Inversely, each muscle can contribute to different movements in other components.

From a theoretical viewpoint, the above remarks pose some optimization issues which are not quite elementary. One can have more forces acting on a joint than the number of degrees of freedom, and some muscles can contribute along different directions. In this way, the resulting biomechanical architecture can recombine robustness and adaptiveness in a very efficient way. and to prevent failures linked to unstable tasks. These aspects display a high complexity, and because of the first chapters of modules 5 and 6 are devoted to understand some biomechanical issues related to the main tasks (locomotion and grasping-handling) to be analyzed.

An intrinsic representation of motor coordination is performed in terms of the so-called “motors” inside the Geometric Algebra, which we shall describe next. The Geometric Algebra framework allows to adapt the usual description of motors to the cases appearing in the following paragraphs.

3.1.3. Interactive navigation

Interactive navigation must go from a source to a target point of a scene which we shall suppose already known. Navigation must avoid collisions with obstacles which we shall suppose initially fixed and known; in more advanced stages, one considers mobile obstacles such those appearing in real traffic scenes, e.g.

Usual models for real-time interactive navigation follow increasing complex models depending on the scene (structured vs free), available sensors and reactive behaviors of other agents eventually present in the scene. The most complete representations of the scene arises from the application of Computer Vision techniques which combine topics arising from 3D Reconstruction, Motion estimation and objects Recognition [60]

There are several types of representation for 3D scenes which are labeled as sparse and dense (sometimes, semidense is also considered). Roughly speaking,

- *Sparse representations* are represented by perspective representations which are described in terms of linear projection matrices $M(\pi_{\mathbf{C}(t)})$ linked to a central projection $\pi_{\mathbf{C}(t)}$ of the scene on a mobile camera's plane $\Pi_{\mathbf{C}(t)}$ associated to a central projection with center $\mathbf{C}(t)$. The persistence of structural elements (vanishing points, horizon lines) of the perspective model provides a robust model for an interactive navigation.

- *Dense representations* are represented by mobile clouds of points corresponding to homologue points along a video (temporal stereo vision) which are matched between them by using an estimation of Fundamental (affine model) or the Essential (Euclidian model) Matrix. The most important extensions of this method started along nineties include SLAM (Simultaneous Localization and Mapping) methods.

Sparse representations are the most appropriate solutions for structured environments (indoor scenes or outdoor scenes with beacons, e.g.), whereas dense representations are more appropriate for non-structured or even unknown environments. A real-time implementation able of incorporating recognition modules is a challenge of paramount importance for a lot of applications such as the automatic navigation of cars or trucks, e.g.

Usual approach for RT navigation in structured scenes is based on perspective maps which are managed in terms of space lines and their projections onto the image plane. An inconvenient of this approach is the indeterminacy of flow image along perspective lines when similar beacons appear along the video sequence. To overcome this trouble, we introduce quadrilateral maps which are are generated from the intersection of pencils of perspective lines. To fix ideas, let us consider the simplest and more usual perspective maps generated from a mobile camera in a structured scene:

- *frontal type* i.e. with a vanishing point at finite distance; in this case the intersection of perspective lines with horizontal or vertical lines gives a map of trapezoids;

[60] The above three topics make the core of modules 2, 3 and 4 of the CEViC (Curso de Especialista en Vision Computacional) in Spanish language which is coordinated and written by JF

- *angular type* i.e. with two vanishing points at finite distance; in this case the intersection of pencils of perspective lines with vertical lines gives two maps of quadrilaterals;
- *oblique type* i.e. with three vanishing points at finite distance; in this case the intersection of each pair of pencils of perspective lines gives a map of arbitrary quadrilaterals.

In fact, each perspective map can be visualized in terms of the contraction of a map of cuboids along an eventually changing perspective direction which corresponds to the third vanishing point. Visualization of structured scenes based on perspective lines is well known from the 15th century, but tracking based on quadrangular regions is less frequent and much more robust w.r.t partial occlusions and or sliding effects than lines-based model.

To shape this idea, one introduces a bivector representation $\mathbf{q}$ of a quadrilateral with edges $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ (counterclockwise sense) as a pair of bivectors $(\mathbf{a} \wedge \mathbf{b}, -\mathbf{c} \wedge \mathbf{d})$ which have an opposite orientation with a constraint relative to their extremes. A rectangle is characterized by the anti-diagonal. The set $\mathcal{Q}$ of quadrilaterals is a homogeneous space by the action of two copies of $SL(2)$ up to scale. Each quadrilateral can be propagated along two directions giving the same bivectors up to scale, in correspondence with each bivector appearing in the description of each quadrilateral $\mathbf{q}$. Their symmetrical replication provides a compact representation of the perspective map. In particular a planar (resp. volumetric) perspective representation can be represented as a pair (resp. triplet) of lines in the space $\mathcal{Q}$ of quadrilaterals.

A relevant feature consists of this representation provides a more compact and stable formulation for Kalman-based filtering in regard to updating and tracking trajectories linked to the apparent motion arising from cameras mounted on a mobile platform. Additional details of this formulation appear in the second module of these notes.

3.1.4. Eye-hand coordination

One of the most outstanding problems in Robotics concerns to the eye-hand coordination in manipulation tasks, including grasping and handling subtasks. Visual control of robotic devices needs not only very efficient algorithms for video processing, but a sensor fusion and their reprojection of anticipated kinematic representations of the scene, also. In particular, an efficient grasping requires usually a repositioning of the trunk in regard to the object, with a re-configuration between components and re-shaping the whole structure going from the forearm, wrist and fingers which must be involved along the first stages of this task. A natural hierarchy for this coordination is performed in terms of hand gestures which support visuo-mechanic information arising from propioceptive and exteroceptive sensors.

Tasks execution is supported by a representation of hand gestures for grasping and handling subtasks. Visual information is performed from video processing and analysis which is independently developed in the third module of the CEViC [61]. A *hand gesture* is a space-temporal sequence of hand postures. To ease their readaptation, a typical strategy consists of identifying *typical postures* linked to stable kinematic configurations and interpolate between them. In our case, we adapt some basic notions arising from the typical representation in Geometric Algebra for oriented oriented areas in 3D space given by screws, twists and wrenches.

Our experimental work concerns mainly to visual servoing for a three-fingered artificial hand (available at the SSSA of Pisa, Italy) which is mounted on a mobile platform equipped with a 6R kinematic chain for the arm. It has a thumb and two additional fingers which are represented by means three kinematic chains which are anchored on a plat; the base joint for the thumb is given by a constrained spherical joint to increase versatility and adaptability to the object. Thus, it is not exactly an anthropomorphic hand, but it has some basic functionalities of anthropomorphic hands.

1. *Grasping tasks* are represented by means of typical gestures which are adapted to each object, by identifying contact points, evaluating friction effects to avoid slipping effects, and correcting instabilities in apprehension phase. The *dexterity matrix* $J^{\dagger}J$ is the key to achieve an optimal grasping. [62]

2. *Handling tasks* require a careful planning for objects transport, by avoiding collisions with the augmented architecture (represented as a Minkowski sum, e.g.) and including a correction of possible instabilities (with anticipatory and compensatory strategies in presence of weighted objects, e.g.). The *manipulability matrix* $JJ^{\dagger}$ is the key to achieve an optimal handling.

A typical approach is based on updating *spatial arrangements of lines* as a linearization of appearances-based model which extend traditional skeletal

[61] Curso de Especialista en Vision Computacional (on line, in Spanish language, www.cevic.eu)

[62] Here, J is the Jacobian matrix representing the kinematics $\dot{\mathbf{x}} = \mathbf{J}\underline{\dot{q}}$ and $J^{\dagger}$ is the Moore-Penrose generalized inverse

approaches linked to the different components of each kinematic chain (arm, forearm, wrist, fingers). Visual feedback requires combine a structural approach (based on skeletal representations) with appearances-based approach (based on a linearization of visible boundaries).

Articulations at fingers, wrist and elbow generate subdivisions in simplified geometric representations which are treated in terms of quadrilateral maps for planar representations (similarly in terms of cuboidal maps for volumetric representations). Anyway, the curved nature of articulations is translated in non-linear algebraic representations on the quadrilaterals space $\mathcal{Q}$, where the simplest ones are given by "pieces" of conics on the space of quadrilaterals (similarly one would have "pieces" of cubics on the space of cuboidal representations). More generally, one has a B-spline of low degree (two for fingers, three for hand, forearm and arm) to describe each articulated mechanism. This representation provides a geometric support to describe configurations in terms of a very low number of low-degree rational curves which are defined on the space of quadrilaterals (resp. cuboidal) maps.

The combinatorics of configurations of low degree algebraic curves is easily reduced to a combinatorics of lines (it suffices to use "chords" passing through control points). Hence, the introduction of pieces of conics and cubics can be understood as a natural extension of geometric arrangements of lines. furthermore, kinematic effects (described in terms of twists) and dynamic effects (described in terms of wrenches) can be introduced onto the original geometric support (given by screws).

The approach performed in the fifth and sixth modules follows the Lagrangian scheme(formulated at the beginning of the 19th century), but adapted to a more intrinsic representation (Geometric Algebra tools) of mechanisms and eventually curved objects in the 3D scene which are described as low-degree curves defined on the space of quadrilaterals (for planar representations) or cuboidal maps (for volumetric representations). In the same way as in Classical Mechanics, it supports a natural contact structure which can be compared with the semi-analytic approach performed in terms of jets spaces linked to configurations and working spaces.

3.2. Elements of Robots Kinematics

The goal of Robots Kinematics is to provide models, develop tools for analysis of mobile data for control points of the robot, and implement algorithms for the information fusion which allows to estimate, track and predict robot behavior. According to the usual hierarchy it involves to low-order variation of localization data (position and orientation) including (linear and angular) velocities and accelerations. To fix ideas, we shall restrict ourselves to *serial manipulators* where it suffices to replace the current values for joint parameters for each link and propagate along each kinematic chain [63].

Most aspects of Robot Kinematics can be reinterpreted in terms of the EMAD which has been introduced in the §1,2,2. To fix ideas, we remember some basic concepts of Mechanics which are related to the description of the EMAD. The subsection is organized around the following topics:

1. *Forward kinematics* whose aim is to specify the joint parameters and compute the configuration of the chain by forward propagation along each kinematic chain. It can be represented as the differential of the transfer map $\tau : \mathcal{C} \to \mathcal{W}$ which is formally given by the 1-jet with a contact structure.

2. *Inverse kinematics* which computes the motions to be performed at joints in order to achieve a desired localization for the end-effector of each kinematic chain. Most robots are redundant ones; hence, matrices are not inversible and optimal solutions are not necessarily unique. In more formal terms, the goal is to compute a pseudo-inverse of $j^1\tau : J^1\mathcal{C} \to J^1\mathcal{W}$ whose last component is locally represented by the Jacobian matrix.

3. *Task oriented control* which involves to kinematic characteristics (velocities, accelerations) of trajectories to be performed by control points, with forward and inverse variants. It requires to identify integral curves which are solutions for distributions of liftable vector fields from the working to the configurations space. A typical approach uses impedance control, with a feedback arising from position-force based control appearing in Robots Dynamics.

4. *A geometric reformulation* can be given in terms of Geometric Algebra.

In all cases and to illustrate basic ideas, we shall restrict ourselves to serial manipulators due to the difficulty for parallel manipulators.

[63]Parallel manipulators (Stewart platforms, e.g. are much more complicated and it requires to solve a set of polynomial constraints involving parameters of the platform; see last chapter of module 1 for more details

3.2.1. Forward Kinematics

Roughly speaking, *Forward Kinematics* corresponds to the calculation of the velocity and acceleration response of key points (c.o.g., end-effectors, control points) of a given rigid-body system to a given collection of actions (forces, moments) at joints.

A simple spatial representation corresponds to a serial manipulator with prismatic and spherical joints, allowing translations and rotations, with a matricial representation standardized by J.Denavit and R.Hartenberg (1955) in terms of joint matrices $[Z]$ and link matrices $[X]$ to represent screw displacements on spatial linkages along the Z-axis. According to their notation, one has

$$[Z_i] \;=\; \mathbf{Trans}_{Z_i}(d_i)\mathbf{Rot}_{Z_i}(\theta_i) \;,$$

which positions the link frame given as a *screw displacement* along the X axis as

$$[X_i] \;=\; \mathbf{Trans}_{X_i}(a_{i,i+1})\mathbf{Rot}_{X_i}(\alpha_{i,i+1}) \;,$$

where θ_i, d_i, $a_{i,i+1}$ and $\alpha_{i,i+1}$ are called the *Denavit-Hartenberg parameters.* According to the above notation, the general motion of the i-th link is described as

$${}^{i-1}T_i \;:=\; [Z_i][X_i] \;=\; \mathbf{Trans}_{Z_i}(d_i)\mathbf{Rot}_{Z_i}(\theta_i)\mathbf{Trans}_{X_i}(a_{i,i+1})\mathbf{Rot}_{X_i}(\alpha_{i,i+1})$$

whose matricial expressions are described in the first module giving a cumbersome notation for the computer implementation. The use of vector representation in the Geometric Algebra framework simplifies the notation and avoids the use of arguments linked to "special position" for euclidian transformations (rotations and translations).

Beyond the analytical (in terms of trigonometric functions) or algebraic (in terms of multivectors) expressions, the ambient space is a subset S of a product of Lie groups which allow to reproduce locally symmetric movements in the working space $\mathcal{W}$ from small motions in the configurations space $\mathcal{C}$ (both of them are related through the transfer map τ). An advantage of this representation is the availability of a lot of information relative to differential and integral calculus on Lie groups. Unfortunately and due to the mechanical constraints in the design, the subset S is a boundary subvariety of the product of Lie groups representing both spaces; in particular, kinematic singularities (appearing in the boundary ∂S, usually) deserve a special treatment for control and optimization issues.

3.2.2. Inverse Kinematics

Roughly speaking, the computation of kinematic characteristics (velocities and accelerations) to be applied at joints to achieve the expected results along trajectories. From a matricial viewpoint, it requires to invert the Jacobian matrix representing the forward kinematics. As the Jacobian matrix is not a square matrix, it is necessary to use some kind of pseudo-inverse. Seemingly, the most robust is the Penrose-Moore pseudo-inverse. Additionally, the products $J^T.J$ and $J.J^T$ provide very useful information for dexterity and manipulability issues which are developed with more detail in the fifth module.

A more geometric approach consists of identifying the contribution of vector fields applied at joints in regard to real movements of end-effector for each kinematic chain. The transposed matrix J^T of the Jacobian provides a criterium for the dual map of $j^1\tau$ which is very useful for control issues.

Instead of using the classical approach based on Denavit-Hartenberg parameters, one can use a description of the tangent space to configurations and working spaces as subvarieties of a product of Lie groups. In the decoupled case, the Lie algebra of a direct product of Lie groups is the product of their Lie algebras, and the exponential map allows to recover true motions in the working space $\mathcal{W}$ from their infinitesimal version which is modeled in terms of infinitesimal symmetries in the configurations space $\mathcal{C}$ through the differential map $d\tau : T\mathcal{C} \to T\mathcal{W}$ at the neutral element (represented by the identity matrix for each component of the product of Lie groups).

The presence of eventually non-linear coupling between components or the interaction with the environment introduces some troubles, which do not affect to the general proposed scheme. Indeed, the transfer map $\tau : \mathcal{C} \to \mathcal{W}$ is generically a submersion; hence, one has that push-forward of vector fields defined on the configurations space $\mathcal{C}$ give vector fields on the working space $\mathcal{W}$ even in presence of singularities. Hence, it suffices to evaluate the contribution, including redundancy conditions. The evaluation of functionals is analyzed in the following paragraph.

In the generic case (i.e. away of kinematic singularities), due to the Jacobian regularity, one can find local inverses which are treated in terms of the tangent space to grassmannian manifolds. Degeneracy locus involves to lowering dimensions of meaningful subspaces, and requires the introduction of intermediate flags to control possible evolutions.

3.2.3. Task-oriented control

Control linked to tasks impose constraints linked to paths to be followed or the evaluation of mechanical quantities linked to geometry (allowed motions, e.g.), kinematics (performed work, energy to be supplied, e.g.) or dynamics (dissipative effects linked to friction, e.g.). The design of efficient control devices requires a coarse-to-fine approach which is formulated in topological-to-geometric terms. Along this paragraph, main attention is paid to topological aspects. To start with, one gives a mathematical description of a task:

A *task* can be considered in a two-fold way as a path γ defined on the configurations or working space, and in a complementary way, in terms of functions or functionals defined on some of such spaces which evaluate data along paths or trajectories in the ambient (configurations or working) space.

- A *path* $\gamma : I \to \mathcal{C}$ is a *continuous* map, not necessarily with continuous derivative, defined on the unit interval $I = [0, 1]$ with a source configuration $\underline{c}_0 := \gamma(0)$ and a target configuration $\underline{c}_1 := \gamma(1)$. Any path defined on the configurations space $\mathcal{C}$ induces a path defined on the working space $\mathcal{W}$ by composition $\tau \circ \gamma$ with the transfer map. To evaluate if a path on working space $\mathcal{W}$ can be lifted or not to a path on the configurations space $\mathcal{C}$ it is necessary to compute topological invariants of the fibration τ.

- A *function* $f : \mathcal{W} \to \mathbb{R}$ allows to evaluate different "mechanical quantities" on the working space which are linked to control points of each kinematic chain (the end-effector, typically). Any such function $f \in C^r(\mathcal{W}, \mathbb{R}$ for $r \geq 0$ can be lifted in a unique way to a a function $\tau \circ f \in C^r(\mathcal{C}, \mathbb{R}$. However, the inverse is not necessarily true, because the transfer map τ is a multiple covering; however, the "direct image" can be represented as a multiple-valued map. Again, the topology of the transfer map as a fibration plays a fundamental role.

From the mechanical viewpoint, the above approach is quite general and it reappears in different ways along the different modules in regard to the tasks to be performed. Some of the most important tasks to be modeled are the following ones;

- For anchored robots concern to pick up, perform welding or assembly operations, by following an increasing order of difficulty. Lyapunov functions provide a very common and simple strategy for design operational control.
- For mobile robots self-localization, transportation and autonomous guiding in partially known environments pose some of the most relevant issues.
- Main tasks to be modeled for humanoid concern to locomotion and grasping tasks.

All of them display a mixture of geometric, kinematic and dynamic aspects which require very often a feedback between position, impedance and force-based control. In some cases, it suffices with a robust control (typical for industrial robots in isolated environments, e.g.); in other cases, it is necessary to perform to sometimes changing environmental conditions which require an adaptive and more intelligent behavior, including reconfigurable systems (see below §3,4,4 for additional details).

To be effective, it is necessary to develop a multisensor fusion which provides additional keys for self-localization and autonomous guiding. Range sensors (ultrasonic, infrared, laser range finders) are very useful to avoid collisions, but an interpretation of the environment requires visual feedback. Thus, it is necessary to perform a coupling between mechanical and visual aspects which is initially performed at kinematic level. An introduction to this coupling at kinematic level is performed in the second module of these notes (Navigation of Mobile Platforms); additional details about visual kinematics are developed in the third module of the CEViC.

3.2.4. A geometric reformulation

At least from the mid of nineties it is very common to reformulate the Mechanics in geometric terms. This formulation involves not only to the design of mechanisms, but to the design of advanced tasks such as dexterity or manipulability, e.g.. In particular, *dexterity* means the capability of change the localization (position and orientation) of the manipulated object from a given reference configuration to a different one, arbitrarily chosen within the hand workspace. [64]

[64] [Bic96] A.Bicchi: "Hands for Dexterous Manipulation and Powerful Grasping. A difficult road towards simplicity", in [Gir96], 2-15

A non-trivial issue concerns to the discontinuities between contact elements along the tasks to be performed. Some reasons for discontinuous regrasping are linked to non-optimal relative localization of platform/robotic arm or the object irregularities which impose external constraints for grasping and manipulating objects. In particular, it is necessary to perform a control of contact conditions and friction cones along eventually rolling contacts, to avoid instabilities along the execution of tasks.

Geometric reformulation is the lowest level for understanding mechanics on a variety V representing the configurations $\mathcal{C}$ or the working $\mathcal{W}$ space. Nevertheless, a geometric description in terms of a variety V provides the support for constructing the kinematics on the fiber tangent τ_V or cotangent bundle τ_V^* (dual of the tangent bundle τ_V) which is more usually denoted as Ω_V^1.

In particular, the differential of the transference map $\tau : \mathcal{C} \to \mathcal{W}$ is locally represented by the jacobian map which allows to describe the forward kinematics and identify the most meaningful kinematic relations in terms of the jacobian matrix. The structure of this fiber map induces a natural hierarchy between both tangent bundles which can be reformulated in dual terms to provide a better understanding of the inverse kinematics.

The basic formalism developed in the precedent paragraph for paths and functions can be extended immediately to any kind of vector fields $\xi \in \Theta_{\mathcal{C},c}$ defined on the configurations space $\mathcal{C}$ or to any kind of 1-differential forms $\omega \in \Omega^1_{\mathcal{W},w}$ (i.e. covector fields) defined on the working space $\mathcal{W}$, respectively. Hence, they are valid for distributions $\mathcal{D}$ of vector fields defined on $\mathcal{C}$ or for systems $\mathcal{S}$ of differential forms defined on $\mathcal{W}$. Thus, ordinary and exterior differential calculus can be extended in a natural way, at least for the regular part of each ambient space.

3.3. Elements of Robots Dynamics

Robot Dynamics is a natural extension of Robot Kinematics, which incorporates additional effects arising from the exchange of matter, energy and information with the environment. At mechanical level, information exchange is performed in terms of forces and momenta acting at joints and/or torques. To fix ideas, we shall concentrate our attention in energy exchange between components or in regard to the environment. Some meaningful extensions w.r.t. kinematics are related to activation-inhibition phenomena (w.r.t. critical thresholds for information arising from sensors), anticipatory/compensatory movements (for locomotion and/or manipulation tasks, e.g.) or drift and friction effects with the environment (non-holonomic constraints, e.g.).

Classical Mechanics has developed two approaches which are labeled as differential (Hamilton-Jacobi structural equations) and integral (reformulating Euler-Newton in terms of variational principles), which are sketched in the two first paragraphs. In the classical case, Symplectic Geometry on a smooth manifold M provides a framework where differential and integral formulation for Classical Mechanics are equivalent between them. The introduction of a moment map linked to the symplectic structure on the (co(tangent space provides a more general context for a unified treatment connecting algebraic and analytic aspects in an equivariant framework linked to orbits of the momentum map.

Next, we remark the role played by different kinds of (algebraic, infinitesimal, dynamical) symmetries, including some aspects related with total or partial integrability of Lagrangians (or their associated differential forms in the Cartan's sense), and other linked to breaking symmetries in presence of singularities. Finally, some general remarks about control and optimization are introduced in terms of an equivariant description of Mechanics having in account phase transitions as it occurs in complex movements of robots.

Finally, to improve the dynamical balance of the whole mechanism along complex tasks (locomotion or bimanual grasping, e.g.) it is necessary to coordinate several kinematic chains which combines anticipation and compensation effects. This coordination introduces additional troubles which are managed classically along phase transitions by means switching procedures between different control modes. Some simple algebraic considerations relative to different kinds of symmetries provide a more robust approach to control and optimization issues. From a mathematical viewpoint, the duality between both of them allows to re-interpret control problems in geometric terms, as it is well known from the eighties. A less developed approach consists of a reformulation of reachability and controllability issues in the Geometric Algebra framework. Roughly speaking,

- Reachability involves to feasible motions, from exploration of the working space according to constraints of configurations space, motion planning and simulation for each kinematic chain and/or platform.
- Controllability involves to linear independence of columns of a matrix which represents iterative application of an operator.

Following the classical approach, geometric, kinematic and dynamic singularities must be avoided. Their description can be easily obtained from controllability issues. These topics are sketched at last paragraph of this subsection.

3.3.1. A differential approach

Original formulation of kinematics was given by Lagrange for rigid solids extending the original Newtonian approach. A compact formulation which is compatible with variational principles (Euler) is obtained by introducing an operator (now called *Lagrangian* of a mechanical system) given by the *difference*

$$\mathcal{L} = \mathcal{T} - \mathcal{U}$$

between the kinetic energy $\mathcal{T}$ and the potential energy $\mathcal{U}$, difference to be minimized in order to perform a transfer between both energies (according to the energy conservation principle).

The relation with the the newtonian approach is given by taking the total energy of a system (given by Newton as the sum of both energies), as the Lagrangian to be preserved. In this case, the total energy can be understood as a functional on the tangent bundle of a support variety M, i.e. is an element of the cotangent bundle, which gives a strong connection with topological aspects (in terms of critical points or, more generally, the discriminant locus of the map).

In the symplectic framework, *Lagrange's equations* can be obtained by minimizing the first Lagrangian, and are given by

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = F_i \qquad 1 \leq i \leq m = dim(M)$$

where F_i represent the *generalized forces* associated to the generalized coordinates q_i for $1 \leq i \leq m = dim(M)$. This approach is extended in a natural way to articulated mechanisms by decoupling components and composing their effect along a kinematic chain or, more generally, a multibody system. If M is the configurations space for a robot, then $m = dim(M)$ denotes the number of joints; the first case to be studied corresponds to a kinematic chain. Their coupling is a little bit more complicated and it is developed along the fifth and sixth modules of these notes.

From a theoretical viewpoint, classical differential approach uses *Hamilton-Jacobi structural equations* as structural constraints for motion equations of an ideal system composed by N particles. This formulation provides an ideal framework which is linked to the preservation of a symplectic gradient. Let us remember that a conservative systems is characterized by the preservation of a function which is called a "potential"; in our case, the Lagrangian $\mathcal{L}$ is a functional defined on the tangent space $T\mathcal{C}$ of the configurations space (hence, it is represented by a 1-differential form); the symplectic structure existing on the cotangent space (Darboux) means that the natural gradient to characterize conservative systems is given by the symplectic gradient, now.

In practice, it is "very difficult" to have a completely integrable system on the cotangent space [65]. Usually, we have only a distribution $\mathcal{D}$ on vector fields defined on the configurations space which is represented by means of

$$\dot{y}^{\beta} + A^{\beta}_{\alpha}(x^{\alpha}, y^{\beta})\dot{x}^{\alpha} \;=\; 0 \qquad \beta = 1, \ldots, k \leq m$$

w.r.t. a decomposition of generalized coordinates $\underline{q} = (x^{\alpha}, y^{\beta})$. The "simplest" case corresponds to a linear system, i.e. one supposes that entries of the matrix A^{β}_{α} are linear forms. Under integrability conditions one finds k-dimensional integral subvarieties whose solutions are geometrically described in terms of foliations [66]; when $k = m$ one obtains the so-called Lagrangian subvarieties which can be re-interpreted as m-dimensional subvarieties of the $2m$-dimensional phase space $T\mathcal{C}^*$ [67]

Along the last modules, one develops an extension of this approach to the non-linear case for locomotion tasks of biped robots or animats; in the simplest cases, integrability conditions are theoretically analyzed in terms of the coupling of non-linear oscillators (inverted triple pendulum) for each leg. Solutions of this ideal approach display a very interesting quasi-cyclic behavior along walking tasks (see [Fin01]), which seemingly can be extended to other more complex locomotion operations (running or jumping, e.g.).

3.3.2. An integral approach

The integral approach is related to the minimization of an integral functional linked to the minimization of some "geometric quantity" (length,

[65] More precisely, integrable systems must fulfill a set of algebraic equations and, consequently, they are a high codimension variety w.r.t. the topology of coefficients

[66] See the module 4 of my notes on Differential Geometry for details and references

[67] Its analysis is performed in Algebraic Geometry in terms of the Grassmannian $Grass(m, 2m)$ which supports again natural symplectic structure (Arnold).

area) or a "kinematic quantity". *Euler-Newton equations* provide a first approach which was extended by Lagrange to more general action functionals which are called Lagrangian, currently. To start with, let us remember the simplest case corresponding to the *D'Alembert-Lagrange principle* associated to the minimization

$$\delta \int_a^b \mathcal{L}(\underline{q}(t), \dot{\underline{q}}(t))dt \; = \; 0$$

of a Lagrangian $\mathcal{L}(\underline{q}(t), \dot{\underline{q}}(t))$ w.r.t. to a distribution $\mathcal{D}$ of vector fields defined on the configurations space $\mathcal{C}$. In this case, variations $\delta\ \underline{q}$ of the solution curve $\underline{q}(t)$ of the distribution $\mathcal{D}$ are chosen in such way that $\delta\underline{q}(a) = \delta\underline{q}(b) = 0$ where $t \in [a, b]$. Hence, one has

$$-\delta\mathcal{L} \; = \; (\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} - \frac{\partial\mathcal{L}}{\partial q_i})\delta q_i \; = \; 0$$

for all variations fulfilling the integrability constraints linked to the distribution $\mathcal{D}$. Decoupling of of generalized coordinates $\underline{q}$ in terms of (x^{α}, y^{β}) provides a transformation A_{α}^{β} between maximal integral k-dimensional and $(m-k)$ subspaces in the phase space which can be reinterpreted as a linear map between $Grass(k, m)$ and $Grass(m-k, m)$, or alternately as an element of the tangent space to the grassmannian, which is easily re-interpreted in terms of Geometric Algebra, as natural support for dynamic issues.

The above interpretation is seemingly new, but it poses the problem of giving an extension for the (eventually singular) non-linear case. Here, the notion of stratification reappears again to provide a support given by a collection of nested subspaces associated to kinematics and dynamics, and along different phases of complex tasks (locomotion and handling, e.g.). Linearization of a good stratification (such to the corresponding EMAD) provides a behavior which is locally described by a dynamical flag whose elements are tangent spaces to the regular strata of such stratification. The development of this simple idea requires to work on flag bundles which are the natural extension of grassmannian bundles. This extension is not quite elementary, but contact constraints appearing in the EMAD are structurally linked to symplectic constraints linked to the cotangent space where Lagrangians are defined.

3.3.3. Locally symmetric spaces for dynamics

Symmetries are ubiquitous in geometric models and their extensions. Three major approaches to the use of symmetries in Robotics are related with the original work started by F.Klein (equivalence between classical groups and linear geometries), S.Lie (continuous groups G for ODE and infinitesimal treatment in terms of their algebras $\mathfrak{g}$) and E.Noether (symmetries linked to variational principles which preserve generalized Lagrangians) at the end of the 19th century. All of them are subsumed in a more general treatment performed by E.Cartan who unifies their treatment in terms of symmetric spaces.

Roughly speaking, each point $x \in X$ of a locally symmetric space X has a neighborhood which is geodetically complete, i.e. each point is reachable along a path which is a subset of a geodesic. Every Lie group G and any G-homogeneous space is a symmetric space, but the converse is not true. Intuitively, one can think of symmetric spaces as if they would be given by "pieces" of spaces which are homogeneous spaces, not necessarily of the same dimension. with "good" incidence relations in the boundary. Thus, the notion of symmetric space is much more general and flexible than the notion of homogeneous space. Furthermore, a homogeneous space is necessarily regular, but a locally symmetric space can display different kinds of singularities which are usually contained in the closure of G-orbits. These "pathologies" are the key to understand phase transitions and model them in terms of equivariant bifurcations linked to symmetries breaking.

All approaches (Klein, Lie, Noether) are related between them. For instance, Classical Groups (a la Klein) are more commonly described in terms of Lie groups, because in this way it is possible to apply the extension of differential and integral calculus to manifolds. Additionally, their infinitesimal version eases the estimation of algebraic transformations (it is much easier compute in a vector space, instead of a manifold) and allows to obtain additional infinitesimal symmetries linked to ODE (Lie, himself) or variational principles (E.Noether).

Furthermore, all of them provide invariants which are linked to conservation laws by following an increasing order of difficulty: (a) Classical Groups are characterized by the conservation of a quadratic (a distance, e.g.) or a bilinear form (symplectic or Poisson structure, e.g.). (b) Lie Groups and Algebras allow to characterize the conservation of physical amounts (such as the energy or the momentum, e.g.); (c) infinitesimal symmetries provide criteria for the conservation of variational principles (linked to a lagrangian, e.g.). Hence, it is natural to consider their extension to Mechanical Engineering and, in particular, to Robotics.

Modern formulations of Classical Mechanics are based on the conservation of a geometric or a kinematic "amount" such as the momentum or the energy (Hamilton). In more down-to-earth terms, the conservation of energy means that the kinetic energy is balanced by the power generated by forces and momenta applied at joints and torques, e.g. More generally, infinitesimal symmetries arising from variational principles provide first integrals which allow partial resolutions of motions equations. More recently, the above three approaches (algebraic, infinitesimal, variational or dynamics) have been included in a Differential Theory of Invariants which requires to develop the Geometry of Jets Space (see [Olv95][68] for an extensive treatment).

To fix ideas from the homogeneous viewpoint, structural motion equations (Hamilton-Jacobi) are preserved by the action of symplectic group which induces a equivariant structure linked to the moment map which is superimposed to each ambient space. On the other hand, the natural contact structure linked to the description of k-jets (prolongations of order k for the transference map τ, introduces additional structural constraints which are very useful for a right interpretation of captured data by sensors. Additionally, each constraint linked to an optimization procedure can be read in terms of a contact constraint for the problem to be solved.

The introduction of general conservation principles for articulated mechanisms is translated in *structural constraints* involving now to matrices which are linked to the ideal formulation of mechanics associated to each kinematic chain, and consequently the whole multibody for more complex mechanisms. Structural constraints are very important for estimation of dynamic parameters which are necessary for control and optimization issues (see next'paragraph). However, mechanical constraints involve to matrices representing transformations (in Lie groups or algebras) or differential operators (Poisson algebras) verifying structural constraints linked to their Lie-Poisson structures.

Obviously, this formulation has an ideal character and in practice, it is necessary to have in account additional terms (usually non-linear ones) linked to sliding or dissipation effects, e.g.. From the late years of the 19th century, these non-linear effects are modeled in terms of non-holonomic constraints, linked to the lack of exact integrability. Non-holonomic constraints can involve to differential formalism or variational principles (giving the

[68]P.J.Olver: *Equivalence, Invariants and Symmetry*, Cambridge Univ. Press, 1995

also called "vakonomic" constraints), which are no longer equivalent between them in this enlarged context w.r.t. the original symplectic context. Roughly speaking, differential non-holonomic constraints are more useful for kinematic analysis, whereas vakonomic constraints are more useful for control issues. A very complete treatment of these issues has appeared recently in Bloch (2003). [69]

3.3.4. Some control issues

From the *mathematical viewpoint*, the simplest control system on a variety X at a point $x \in X$ can be described in two ways:

- as $\dot{x} \in V_x \subseteq T_x X$ where V_x is a region contained in a subspace of the tangent space $T_x X$ at $x \in X$; this description is used to introduce linear constraints on velocities for systems propelled by motors, as rolling constraints, e.g.;
- in parametric terms, as a function $\ldots x = f(x, u)$ where for each $x \in X$ the map $u \mapsto f(x, u)$ has V_x as its image; typical examples are related to the use of actuators which modify the current state in the configurations space.

In both cases, one can consider linear combinations $\sum_{i=1}^{k} u_i \xi_i(x)$ of several vector fields $\xi_i(x)$ at $x \in X$ which are used for delimitation of regions where "accessibility" and "controllability" are feasible to stabilize solutions of dynamical systems. First case to be analyzed corresponds to linear time invariant systems whose discrete version for the state equation is given by

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

where $\mathbf{A}$ is a $n \times n$-matrix and $\mathbf{B}$ is a $n \times r$ involving the parameters of the system. By iterating the application of this transform, one proves that the system is *controllable* (i.e. reachable) if the $n \times nr$ matrix

$$\mathbf{C} = [\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \cdots\ \mathbf{A}^{n-1}\mathbf{B}]$$

has full row rank, i.e. $rk(\mathbf{C}) = n$. The rank constancy defines an open set, but degeneracy locus (corresponding to rank default) gives instabilities (ill-conditioned problem in computational terms) which can require additional

[69] [Blo03] A.M.Bloch: *Nonholonomic Mechanics and Control*, Springer-Verlag, 2003.

strategies based on output controllability or trajectories tracking involving kinematic aspects which are developed from the third module.

A *first example of a controllable system* corresponds to a kinematic chain composed by two links in a plane, where **A** and **B** are two column matrices; the controllability condition for **C** means that the rank of the matrix [**B AB**] must be equal to two, i.e. links are not aligned. Next case, shows that controllability is fulfilled if the resulting $(3 \times 3r)$-matrix has rank 3, and so on.

Beyond the trivial cases (corresponding to $r = 1$), it is clear that controllability conditions can be re-interpreted in terms of Grassmann manifolds $Grass(n, nr)$, with degeneration conditions parameterized by the manifold of complete flags corresponding to subspaces of dimensión $< n$. It seems natural to give a more compact description of these issues in terms of ODE on multivector algebra in the framework of Geometric Algebra, but seemingly it is not still performed. A theoretical approach to this problem with an application to control of biped gait was exposed in my talk of Agacse'01 (Cambridge, UK) by adapting some arguments of Riccati's equations to matrices representing Grassmannians. The basic idea consists of replacing optimal solutions by fixed points of maps $f : Grass(k, N) \to Grass(k, N)$ on a Grassmann manifold.

The existence of different dimensions for subspaces appearing even for each space of the PACW cycle, suggest the introduction of Flag manifolds for control issues involving "stratified spaces" according to previously specified mechatronic architecture. A flag of a N-dimensional space V^N is a finite collection of nested subspaces

$$L^{r_1+} \subset L^{r_1+r_2} \subset \ldots \subset L^{r_1+\ldots+r_k} = V^N$$

which are described by their successive quotients $Q^{r_i} = L^{r_1+\ldots+r_i}/L^{r_1+\ldots+r_{i-1}}$. In this way, one obtains flags of "nationality" $(r_1, \ldots, r_k)$ corresponding to all possible partitions of $N = dim(V)$ which are related between them by common refinements of two initial partitions. Two extreme cases corresponds to Grassmann manifolds corresponding to the partition $(k, N - k)$ and complete flag manifolds corresponding to $(1, \ldots, 1)$; both play a fundamental role in Geometry and our approach to Robotics. They are homogenous manifolds given initially as $GL(N)/[GL(r_1) \times GL(r_k)]$. Their tangent space at each point is given by $\oplus_{1 \leq i < j \leq k} Hom(Q^{r_i}, Q^{r_j})$ representing the "influence" or the interaction of the i-th module with the j-th module involving geometric, kinematic and dynamic aspects [70].

In this enlarged framework, the basic idea consist of replacing the Grassmannian (space of regular rectangular matrices) by a Flag Manifold $\mathbb{B}(r_1, \ldots, r_k)$ associated to the partition $(r_1, \ldots .r_k)$ of N. Its tangent space (support for Optimization) is represented by upper-block (resp. lower-block) triangular matrices corresponding to forward (resp. feedback) strategies. Multilinear control functions extending traditional approaches (a la Lyapunov, e.g.) are defined on the dual of these Flag Manifolds. This approach solves the problem at Kinematic level, but not to dynamic level. To avoid undesirable nonlinear effects appearing along the interaction, we incorporate "nilpotent operators" in charge dissipating non-linear vibrations, e.g. [71]

From the mechanical viewpoint, the most common control strategy involves to position and/or force coordinates acting on joints and torques; the resulting hybrid control is called *position-force* based control. A complementary strategy is the *Impedance-based control* which is implemented on manipulators for controlling the environmental interaction linked to inverse dynamics. It involves to a hybrid formulation of control law in the phase space in terms of non-linear (scalar, vector, tensor) fields, which allows to represent non-linear coupling effects linked to contact conditions; in realistic cases, such effects appear as sliding and friction effects which require the introduction of non-holonomic constraints for advanced dynamic modeling.

Multivector language (introduced at the end of the second section) provides a support for a self-dual representation involving to position-force based control (involving and "extended duality" between geometric and dynamic issues) and impedance-based control (involving kinematic issues, properly said which are formulated in a self-dual Grassmannian of lines). An additional contribution concerns to the introduction of "complete objects" involving trajectories whose support is geometrically represented by incidence and tangency conditions which are, not only projectively invariant, but dual between them in the projective compactification, also.

In this extended bivector framework the lack of controllability is related to a lack of total integrability; in the non-controllable case, one must compute (infinitesimal) symmetries linked to Lie brackets. Let us start with the "easy" case: The inclusion of *non-linear effects* for *dynamic controllability* is initially described in terms of a distribution $\mathcal{D}$ which is locally given on the tangent space by a control-affine form

[70] The reasoning si similar if one replaces the General Linear Group by any other Classical Group.

[71] more details in the module B_{34} (Robot Dynamics).

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{g}_i(\mathbf{x}) u_i$$

with similar controllability conditions which are expressed now in terms of iterated Lie brackets $[ad^k_{\nabla\mathbf{f}}\nabla\mathbf{g}]$. More generally, if ξ, η are two arbitrary vector fields (not necessarily gradient ones), one defines the *iterated Lie-Poisson bracket* as:

$$ad^0_\xi(\eta) = \eta \ , \ ad_\xi(\eta) = [\xi, \eta] \ , \ \cdots \ \ , ad^k_\xi(\eta) = [\xi, ad^{k-1}_\xi(\eta)]$$

which are the natural extension of ordinary adjoint representation $ad_X(Y) = [X, Y]$ by means of iterated Lie brackets for matrices:

$$ad^2_X(Y) = [X, ad_X(Y)] = [X, [X, Y]], \ \cdots \ , \ ad^k_X(Y) = [X, ad^{k-1}_X(Y)] \quad \forall X, Y \in \mathfrak{g}$$

In particular, for the above linear system $\dot{x} = Ax + Bu$ with the control

$$\dot{x} = f_0(x) + \sum_{i=1}^{r} u_i f_i(x) \quad \text{with} \quad f_0(x) = Ax \ , \ f_i(x) = B_i$$

being $B_i \in \mathbb{R}^{n+1}$ and $B = (B_1 \dots B_r)$, the application of the above criterion (following Kalman) for iterated Lie brackets gives

$$ad^k_{f_0} f_i \ = \ (-1)^k A^k B_i$$

and consequently, the controllability criterium written above corresponds to the computation of the rank of the subspace generated by $ad^k_{f_0} f_i$ at the initial point. The worse case corresponds to nilpotent operators. Let us remember that $\mathfrak{g}$ is a *nilpotent Lie algebra* iff

$$[X_1, [X_2, [\cdots [X_n, Y] \cdots]] = \mathrm{ad}_{X_1} \mathrm{ad}_{X_2} \cdots \mathrm{ad}_{X_n} Y \in \mathfrak{g}_n = 0$$

for all $X_1, X_2, \dots, X_n, Y \in \mathfrak{g}$. In particular, ad_X is a nilpotent endomorphism for any $X \in \mathfrak{g}$ when $\mathfrak{g}$ is a nilpotent Lie algebra, i.e., $ad^k_X Y = [X, ad^{k-1}_X Y] = 0$ for some $k > 0$. Thus, the lack of controllability is strongly linked with the geometry of the nilpotent variety. Intuitively, nilpotent fields are the responsible of rank default, which can be controlled in matricial terms.

From the infinitesimal viewpoint, a recent strategy for classifying finitely determined singularities (i.e. equivalent to its k-jet) consists of identifying infinitesimal deformations which transform them in its k-jet. One can prove that the responsible for k-determinacy is a nilpotent algebra which characterizes the equivalence class of the k-jet [72].

Along late nineties, we have characterized all "simple" singularities of polynomials (ADE classification) in terms of nilpotent Lie algebras; these cases cover all meaningful cases appearing in corank 1 singularities. Hence, to control the behavior at singularities, it suffices to construct generic nilpotent vector fields in the neighborhood of the singularity. Nilpotent operators are useful in Robotics because, between other things, they provide a complete list of algebraic models for dissipative models.

The specific choice of a control strategy depends on the task to be performed and the capability of measuring and interact with the kinematic variables. Thus, there is no a universal solution but an analysis linked to symmetries, analytical stratifications and multi-vectorial formulation provides complementary criteria to improve the design of control algorithms. The introduction of a hierarchy linked to the geometry of G-orbits and the transfer map τ (and their prolongations given by its k-jet $j^k\tau$) provide a general equivariant framework for a natural feedback between different kinds of control which are described in terms of the large diagram relating the stratified maps $j^k\tau$.

Our proposal consists of the following strategy: the lack of controllability can be corrected not only identifying local infinitesimal symmetries (classical approach which does not closure the iterations), but by introducing some the control through nilpotent elements which are the responsible to stabilize orbits linked to singularities. These topics are more advanced than usual and will be developed from a mathematical viewpoint at the end of the module 4, and applied to humanoid robots and animats along modules 5 and 6.

3.4. Expert Systems in Robotics

Expert systems try of imitating smart functionalities in charge of (a) information processing and analysis arising from sensors; (b) fusion, evaluation and tracking of changing information; and (c) support to decision making including adaptive interaction with evolving environments. Some of the most meaningful advances from the late nineties have been performed in the domain of distributed intelligence in advanced sensor networks, and their assistance to Decision Making.

[72] J. W. Bruce, A. A. du Plessis, C. T. C. Wall: "Determinacy and unipotency", *Inventiones Mathematicae*, 88 (3), 521–554, 1987

In the early years of the 21st century, this concept was restricted to static sensor networks based in RFID (Radiofrequency Identifiers) e.g. with different standards for communication protocols. More recently, the development of ubiquitous support for communications and facilities for integrating quite different kinds of information in a common suport has allowed the development of increasingly efficient sensor networks with a wide range of applications to mobile multiagents. One of the most interesting problems concerns to automatic driving which is discussed at the end of the module 2.

Some general problems concern to grouping around meaningful values, clustering around typical shapes (for signals and/or objects), adapting to expected trajectories by using pattern recognition for signals (accumulation of sigmoids or Gaussians in the simplest case, e.g.), and reuse of probabilistic reasoning (fuzzy systems) as support in presence of uncertainty. At the end of the §3,3,4 one can find a coarse description of ANNs (Artificial Neural Networks) and some the most frequent extensions given by GAs (Genetic Algorithms), EP (Evolutionary Programming) and SOMs (Self-Organized Maps).

Along this subsection, one gives some additional details which are meaningful to introduce an embedded intelligence for increasingly smart robots. Obviously, ANNs and their variants (GAs, EP and SOMs) do not cover all the approaches which can be performed to provide a modeling of embedded intelligence in Robotics. Other relevant approaches develop specific architectures (such as Cellular Automata or Support Vector Machines, e.g.), or put the emphasis on estimation (Principal versus Independent Component Analysis, e.g.) or uncertainty about reasoning (Fuzzy approaches, e.g.). However, all of them can be integrated in the current scheme as weighted linear or non-linear scalar bounded functions taking values in $\mathbb{R}$.

Beyond the initial description of ANN in terms of elementary linear algebra, there is a geometric interpretation which allows to think of as a (discrete) surface given explicitly by a system of weights w_{ij} representing the "weight" at each neuron n_{ij}. Several inconvenients are low convergence rate for learning procedures, sensitivity w.r.t. the initial values, lack of flexibility for adapting results, etc. These inconvenients were already known in the early nineties. To overcome them, several strategies have been designed which consist of introducing different kinds of algorithms which intend to accelerate the convergence, be tolerant with qualitative changes (GAs), include evolutive behaviior to improve the adaptability (EP) or to propagate locally the current optimal values to small neighborhoods of optimal solutions (SOMs) to reinforce the stability and persistence.

The *main novelty* of our approach to AI consists of all additional superimposed structures can be reinterpreted in terms of discrete versions of superimposed structures which are commonly used in Differential Geometry (fiber bundles, principal bundles, equivariant fibrations), and functionals defined on them.

This reinterpretation provides a natural feedback between Mechanics (formulated in terms of topological fibrations) and AI tools in charge of their management. In particular, the Perception-Action Cycle appears as a propioceptive extension of the mechanical feedback between configurations and working spaces. Obviously, this proposal requires the development of a discrete version of differential and integral tools which are commonly used in Differential Geometry such as vector fields, differential forms, and connections, between others.

This program is still far from being completed, Because of this, some topics are discussed in next section which is devoted to open research lines of Mathematics applied to Robotics. In other words, in this section we shall introduce some basic aspects (most of them are well-known) related to Expert Systems, but from a geometric viewpoint to ease the transition with topics which are presented in last section of this introduction.

3.4.1. Artificial Neural Networks

There is no a formal definition for an *Artificial Neural Network*(ANN in the successive). Roughly speaking, it is a model inspired in biological neural network for approximating and estimating any kind of shape or function, and consequently, imitating (through different kinds of learning) tasks which are described in terms of functions. Thus, it includes morphological and functional aspects. It is specified by an architecture (given by different layers with topological connections between them), a collection of activity rules (action-reaction or more involved patterns) and learning rules for the knowledge acquisition.

Algorithms based in ANN have a lot of applications including analytical issues (approximation of functions, e.g.), statistical problems (non-linear regression, e.g.), computer science (machine learning, e.g.), computer vision (pattern recognition, e.g.) or integral design (environment representations, e.g.) between others. They are useful to solve information processing, fusion and analysis arising arising from different sensors, to design and implement different kinds of data classification and learning, and to assist decision making to execute commands relative to tasks to be performed inside the Perception-Action Cycle. Thus, they appear in almost all areas of Robotics as high-level algorithms; they are developed in the module 2, but their refinements appear along all these notes.

From the architectural viewpoint, simplest representations consist of three layers which are labeled as input (corresponding to sensors, e.g.), hidden (for information processing and analysis, e.g.) and output layers (for the response which are connected with actuators, e.g.). More interesting ANNs have at least two hidden layers, but they are more difficult for task-oriented programming. Often, one supposes that all nodes of two consecutive layers are connected between them, from which a lot of proofs and simulations in order to identify the "right" weights for each neuron. Initially, the hidden layer acts as a black box, with (initially random) weights which are modified according to *unsupervised learning strategies* according to different optimality criteria, including the proximity of obtained results w.r.t. expected results, e.g..

Usual approach to convergence issues for ANNs is performed in terms of proximity criteria w.r.t. (pseudo-)euclidian structures superimposed to a discrete version (a regular mesh) of the (pseudo-)euclidian structure which is superimposed to a manifold or, more generally, a variety. Non-linear behaviors are modeled in functional terms locally described by a system of weights w_{ij} associated to a collection of nodes or neurons $n_{ij} = (x_i, y_j)$. The optimization of weights w_i assigned to a task can be geometrically interpreted as a surface $w_{ij} = f(x_i, y_j)$ which is as near as possible to the system of weights. An optimal approach to a task is nothing else than an optimal adjustment to a objective goal represented by such a function.

A more general approach is based on Geometric Algebra, where one considers multivector functionals instead of scalar functions. This approach seems more natural having in account the overlapping of different signals and commands which are translated in environmental representatons, small motions at joints in a 3D architecture, and execution of spatial movements. This approach has been initially developed by D.Hestenes and his school from the early nineties; see [Bay00] for an integrated approach [73] This idea is very suggestive; for higher applicability it is necessary to design and implement more efficient algorithms for control and optimization issues according to the last paragraph of the precedent subsection. Due to their more advan-

[73] [Bay00] E.Bayro-Corrochano: *Geometric Computing for Perception-Action Cycle*, Springer-Verlag, 2000.

ced character from the mathematical viewpoint, these issues are considered in last section of this introduction.

3.4.2. Some extensions of ANN for Robotics

To overcome the difficulties of original approach to ANN (slow convergence, sensititivity w.r.t. initial values, unability for mutations, evolution of self-adapting), it is necessary to superimpose different kinds of more flexible structures able of self-adapting to changing initial conditions or environmental constraints. Roughly speaking, there are three classical options for *superimposed Expert Systems* which are compatible with large systems. They are labeled as

- *Genetic Algorithms (GAs)* which involve to the capability of mutations (changes of shape, state or phase transitions, e.g.) in the dynamics of involved systems.
- *Evolutionary Programming (EP)* which involves, between others, to the capability of reconfiguration of modular systems as a result of multiple interactions at different scales.
- *Self-Organized Maps (SOM)* which are based on unsupervised competitive learning (instead of minimizing errors) by preserving the topology of the input space and represent results on a map.

The two first ones provide more robust solutions (but also more rigid ones), whereas the third option provides more adaptive solutions with eventually chaining hierarchies. Thus, the choice depends on the problem characteristics. In most treatises, there is no a connection between these approaches. Thus, seemingly it is not possible to transfer results between such approaches. However, all these procedures can be represented as different kinds of fields or functionals which can be defined on a generalization of the (co)tangent bundle (the old phase space of Poincare) given by successive jets of the transfer map $\tau : \mathcal{C} \to \mathcal{W}$ between configurations and working spaces of a robot. This reformulation provides a structural link between different mechanical aspects from the analytic viewpoint.

The open character for initial and boundary conditions, gives an eventually infinite-dimensional space of solutions as support for Optimization and Simulation issues, with a weak topological structure. Thus, usual hypotheses (convexity, e.g.) about the integration domain are not longer valid;

consequently, one can not expect an optimal solution, but a collection of suboptimal solutions which can be considered as possible milestones to be visited before achieving the final result. Very often, shortening of verification procedures incorporate a supervised heuristics-based approach.

It is necessary to advance in regard to the design of reconfigurable systems involving systems with a low number of DOF involving the most meaningful components (principal vs independent component analysis, e.g.). There are multiple examples arising from biological behaviors or chemical reactions which provide patterns for distributed networks. Their exploration and comparison would must provide more robust and adaptive behaviors of electromechanical devices.

The most relevant recent contributions of Machine Learning for Robotics arise from Deep Neural Networks (DNN). Their distributed and modular architecture incorporates a lot of layers which allow self-learning without waiting a final output. Basic algorithms are shared with traditional approaches including CNN and RNN (Convolutional and Recurrent Neural Networks) but with a much higher performance. They have been successfully applied to Speech and Image classification in the second decade of the 21st century. Some challenges are related with the automatic labeling and classification of video sequences for automatic navigation and 3D contents for interaction in complex environ

Some contributions of these notes concern to a more systematic use of symmetric principles for learning, the extension of learning subspaces method (appearing already in SOM along nineties) to nested subspaces (flags), the use of algebraic (injective vs projective) resolutions of modules (introduced in the chapter 4 of the module A_{23}), and the construction of "universal" models for linear approaches to learning problems in terms of maps between Graded Complexes.

3.4.3. Self-Organizing and Reconfigurable Systems

Self-Organizing Maps (SOM) were developed along the eighties and nineties with a lot of applications to almost knowledge areas. The most complete reference is [Koh97] [74] where one can find a much more detailed presentations and an impressive list of references involving different subjects. A commented list of mathematical developments performed along the nineties can be read in [Koh97], pp.280-284.

[74] T.Kohonen: *Self-Organizing Maps* (2nd ed), Springer-Verlag, 1997.

An additional *mathematical motivation* for SOM development (in regard to the applications to Robotics) arises from the need of "inverting" the correlation matrix involving to an input-output problem, where observed data are pruned to errors. In this context, SOM can be understood as a typical optimization problem where one tries of identifying optimal transfer operator which minimizes residual errors. This generality makes possible to apply SOM strategies for solving signals transduction in Electronics, clustering data in Computer Science, unsupervised learning in Expert Systems, or modeling under uncertainty from noisy data in any kind of scientific, engineering or economic knowledge field.

Kinematic extensions of the SOM-based approach require a probabilistic approach to vector fields compatible with SOM, which is given by Hidden Markov Models (HMM). Main problems to be solved concern to non-linear character, high dimensionality of the parameter space and control at boundary regions. All these problems are considered along the last chapters of the third module by using probabilistic extensions (in jets spaces) of classical vector bundles notions whose basic elements (deterministic approach) are introduced in the second module (Navigation).

From the viewpoint of a computational treatment of *Information Theory*, Data Clustering is focused towards Pattern Recognition, which can include (supervised or not) emergence and learning procedures. The whole design of algorithms allows to reconfigurate the involved systems according to stability criteria. Possible transitions between local and global aspects are a hard problem to be solved which requires additional research. From the mathematical viewpoint, there exists a hierarchy between topological, differential and infinitesimal stability criteria [75]. It is required a probabilistic and computable reformulation of this hierarchy involving different kinds of stability, compatible with the classical hierarchy (topological, differential, infinitesimal) for stability issues.

Reconfigurable Systems require a modular design able of identifying not only stable patterns, but transitions between them. In other words, one requires an adaptation of basic principles of Bifurcation Theory which, in our case, must be compatible with different kinds of group actions; Golubitsky, Schaeffer and Stewart have developed a quite general approach, but it is necessary to perform a computationally implementable adaptation compatible with a probabilistic approach. Contrarily to an extended opinion, the

[75] See my introductory notes to Differential Topology for additional details and references.

number of DOF is not a problem, because the topological classification of function germs is performed in terms of codimension, firstly, and corank of the Hessian matrix, next. The above remarks suggest additional developments for SOM which can be adapted to these problems which will be developed in the module four.

3.4.4. Evolutionary Robotics. Some remarks

Following [Nel09], the main goal of Evolutionary Robotics is the development of methods for automatically synthesis of intelligent autonomous robot systems [76]

Evolutionary Robotics is a knowledge domain where artificial neural networks, evolutionary programming, cognitive science, genetic algorithms and programming are overlapping to create some kind of artificial life. The corresponding behaviors go from the simplest stimulus-response or reflex motions till the most complex decision making and reflexive-consciousness situations.

From the computational viewpoint, a basic related notion is the Evolutionary Computing or, more specifically Evolutionary Algorithms. Their goal is to provide a representation of the scene $\mathcal{S}$ (as a local representation of the environment $\mathcal{S}$) and the possible smart eventually mobile objects $\mathcal{O}$ which are embedded in $\mathcal{S}$. This representation must include not only the shape and observed trajectories, but an estimation of relative trajectories and possible behaviors depending on these observations.

This goal is too ambitious, currently, and first advances have been focused towards try of understanding some biological processes, where constraints relative to shape and function can be "more easily" incorporated. Some extensions to artificial models for colonies of small social insects (ants, bees) have been developed; in this way, it is possible identify behavior patterns, with simple rules which are individually learned by imitating basic gestures. To achieve these goals, several adaptations of the above superimposed structures (Generic Algorithms, Self-Organized Maps) to ANN have been developed from the late nineties. [77]

[76] [Nel09] A.L. Nelson, G.J. Barlow, L.Doitsidis: "Fitness functions in evolutionary robotics: A survey and analysis", *Robotics and Autonomous Systems* 57, 345-370, 2009

[77] More details on these issues are provided along last chapters of the modules B_{21} (Anchored Robots) and B_{22} (Navigation).

3.4.5. Elements of Deep Learning in Robotics

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Deep Learning strategies for ANN (DLANN in the successive) incorporate almost all strategies developed in multilayer ANN from the sixties. Thus, they are becoming the "new frontier" for the most powerful tools for all variants of mathematical modeling (differentiable vs discrete), massive architecture (parallel vs distributed) or learning procedures (supervised vs unsupervised). Depending on the framework choice (electronic, reasoning patterns, statistical, fuzzy logic), there are several architectures for DLANN. To fix ideas and by analogy with the approach performed in the first paragraphs of this subsection, we shall restrict ourselves to those based in reasoning patterns which are supported by CNN (Convolutional Neural Networks) [78]

Nevertheless the high complexity of the subject, we shall give some brushtrokes to understand the reaching of DLANN in regard to a theoretical approach from the functional viewpoint. A basic feature is the use use of ANN with an almost arbitrary number of layers (old ANN have at most five layers, i.e. at most three hidden layers furthermore input and output layers). Batch processing is translated to the internal structure, in such way that the processing can be performed by using "blocks" having in account the local information for "near paths"; traditional processing in ordinary ANN was only performed along individual paths connecting neurons corresponding to successive layers.

For learning procedures, DLANN can incorporate not only local backpropagation methods (discrete version of the gradient descent for the differentiable case) in a much more efficient way, but feedforward (acyclic) mechanisms labeled as FANN and recurrent (cyclic) mechanisms labeled as RANN. They involve to a a sequential or a cyclic treatment of packs of threads. From the mathematical viewpoint, this artificial scheme reproduces basic behaviors of solutions, but involving to packs of information. Hence, it allows create and process information corresponding to distributed packs of inputs, which furthermore being a very interesting contribution, it is a little bit worrying.

From a theoretical viewpoint, this approach requires an apparently new evolving spatial representation, which is reformulated in our notes in terms of a discrete version of integrable systems of degree k differential forms.

[78] See last chapter (Expert Systems) of the module B_{21} (Anchored Robots) for more details and references.

In this way, it is possible to benefit of the spatial representation by using convergence criteria not only towards critical values of functions (as in the old learning procedures linked to traditional ANNs), but towards integral varieties as solutions of the integrable differential systems. Along the next section, one sketches some more basic related elements.

4. Mathematical challenges for Mobile Robotics

In above sections, we have tried of proving the utility of Mathematics for solving some problems in Robotics; hence it intends to be useful mainly for Mechanical Engineers and experts in Computer Science and their applications to Computer Vision, Artificial Intelligence and Computer Graphics. A common leit-motiv is the interplay between local and global aspects which use advanced models of the interplay between Analysis and Geometry in the domain of Global Analysis; obviously, this interplay involves also to the estimation strategies (feedback between probabilistic vs statistical approaches) and propagation models (ODE- or PDE-based vs Differential Systems or Jets spaces).

Roughly speaking, all of them can be embedded in different variants of fiber bundles including vector bundles (for propagation models), principal bundles (for systems with symmetries), or more general fibrations (to include reaction capability of artificial systems in regard to interaction in complex environments). Their locally trivial structure allows to patch together local models, and propagate their behavior not only for issues involving the internal structure (principal bundles are commonly used in Robotics from the nineties), but the interaction with the environment through a reformulation of Expert Systems. The latter ones is possible thanks to a reformulation of the basic elements (detectors, descriptors, classifiers) for Artificial Recognition in terms of basic elements (functions, sections and isomorphism classes, respectively) defined on fibrations[79].

The above remarks justify our choice of the *Global Analysis* (simultaneous extension of Analysis, Algebra, Geometry and Topology) as the theoretical framework for mathematical issues in Robotics. Obviously, this framework must be completed with estimation methodologies (including probabilistic and statistical aspects), and a reformulation of Optimization and Control issues for their effective application to Robotics. The former ones are specified along each module, whereas the latter ones are introduced in the module B_{24} (Robot Dynamics), where one develops a dual presentation for Optimization and Control issues.

Along this introductory section, to lower the general formalism of Global Analysis, we adopt a more down-to-earth viewpoint by presenting a collection of problems arising from Robotics of great interest for a more systematic development in terms of advanced mathematical-physical mo-

[79] See the last chapter of the module B_{11} (Anchored robots) for a first presentation.

dels and the corresponding development of software tools. In the same way as in precedent sections, the selection is not neutral, and there is a clear predominance of geometrical and topological methods in detriment of other analytical or statistical frameworks.

Main mathematical problems for *anchored robots* concern to high operational accuracy (in euclidian environments), optimization for hyperredundant robots, and robust control in regard to inspection and manipulation tasks for robotic arms. Simplest cases correspond to planar robots or non-redundant spatial robots whose applications to industrial environments are well known and explained along the first chapters of the first module of these notes. Anyway, they provide the initial framework to integrated models and tools arising from different areas The most difficult case correspond to applications of hyperredundant manipulators (to assisted chirurgy), and to Stewart platforms (useful for flight simulators, e.g.). Both of them display several open problems which are studied in terms of multicriteria optimization and configurations in flag manifolds, respectively.

Mobile Robotics include a very large amount of situations including mobile platforms (Stewart, wheeled robots, wheelchairs, cars and trucks, e.g.) and multilegged robots (including humanoids and animats, e.g.). Other interesting examples which are not considered along in this section include terrestrial animals (including snakes-based models for spatial research or for assisted chirurgy, e.g.), maritime devices (for bathimetry in turbulent zones or operations in depth sea, e.g.) and aerial devices (fleets of unmanned vehicles, e.g.). At the end of the module B_{22} we introduce some remarks about fleets of terrestrial vehicles in monitored environments. They display a complex interaction with other agents and the environment in regard to typical applications concerning to mobile surveying tasks or assistance to automatic driving; along this course we don't consider the almost obvious military applications. To fix ideas, most applications are restricted initially to kinematic aspects of mobile platforms (module B_{23}) and multilegged robots (module B_{26}.

Computational geometry (module B_{11}) and *Computational topology* (modules B_{12} and B_{13})) are focused towards proximity queries and related rigid or deformable structures, e.g.. They provide a starting point for computational kinematics and dynamics which can be understood as successive steps of a larger Computational Mechanics. The discrete approach to motion equations is performed in terms of successive differences of motion snapshots can be applied for motion and manipulation planning, motion generation and execution. The capture and display of data involving changing environments (visual, haptic and auditory devices) provides crucial materials to improve the simulation of dynamical systems linked to motions, and ease the feedback inside increasingly realistic situations.

The very high complexity of problems requires to introduce some organizer principles (symmetries, stratifications, evolving geometric representations) to manage all this information in an efficient way by means expert systems and their corresponding algorithms. Three transversal axes for further developments are organized around (a) symmetries acting at different levels (infinitesimal, local, global); (b) hierarchies for complex structures to ease stability, optimization, and control issues; and (c) interaction capability with an evolving environment with changing constraints, including rank default conditions linked to each layer of Mechanics. All of them provide *structural criteria* with their corresponding *exceptions*. So,

- The *lack of complete integrability* (non-holonomic constraints) for subsystems must prevent us about the possibility of finding a complete system of first integrals.
- The presence of *qualitative changes* (described initially as phase transitions) arising from accumulation near to critical levels can generate jumps which are not foreseen in the most general hierarchical models.
- Finally, multivector calculus can describe generic phenomena at "human scale which are linked to extended matricial representations [80], but they are not able of describing all *pathologies* (local singularities, e.g.) *linked to propagation models* and different kinds of C^r-equivalences used for their classification.

In other words, we are not claiming that materials presented along these notes give a quite general approach to the application of geometrical and/or topological methods in Robotics. They provide some guidelines which are useful as organizing principles, and motivate the development of a more profound collaboration between experts in Mathematics, Physics and (Mechanical, Industrial, Communications, Informatics) Engineers with common interests in Robotics.

According to the above remarks, the comments are organized around three subsections (containing more advanced developments related to the above structural axes), and a fourth subsection with a hybrid character (some kind of jumble), where we try of re-combining the precedent developments.

[80]Thom-Boardman singularities are naturally associated to default-rank conditions for matricial representations, but they do not cover all possible singularities linked to maps between spaces

First subsection is devoted to discover symmetries for Mechanical Design, involving geometric, kinematics and dynamic aspects. The initial framework for simplest models is given by Discrete and Computational geometry which are organized around models, data structures and algorithms. This basic scheme is extended to computational topology , kinematics and dynamics. Along the first subsection, we sketch some problems in Non-integrable Systems for Robotics where geometrical techniques are specially useful from a local viewpoint.

Non-integrability means that some kinematic (resp. dynamic) constraints can not be described in terms of geometric (resp. kinematic) constraints; in more geometric terms, their support is contained in the kernel of the projection map going from the space representing kinematics (resp. dynamics) onto the space representing the geometry (resp. kinematics); this phenomenon is well known in Algebraic and Analytic Geometry. The problem is how to solve controllability issues which require to solve how to lift distributions of vector fields in presence of uncertainty. Our proposal consists of reintroducing symmetries to replicate the behavior on the kernel of the differential vertical map.

Second subsection is devoted to a generalization of flag manifolds as universal models for static stratified models to "flag bundles" for "dynamic stratifications". A *flag bundle* has a flag (collection of nested subspaces) as fiber at each point; the main advantage is the existence of splitting principles which allow to perform decoupling for mechanisms and tasks. A *set-theoretical stratification* is a decomposition of a topological variety in a disjoint union of open subsets which are given as a union of cells of the same dimension; singularities appear at boundaries.

The execution of tasks generates phase changes giving evolving subvarieties, which evolve (appear and disappear) following quasi-cyclic patterns with their corresponding control modes; a typical examples concerns to locomotion or re-grasping tasks, which can display an alternance between open- and closed-loop systems for control. In these cases, *dynamic stratifications* change accordingly to dynamic systems linked to the interaction with the environment. Hence, the general models for static stratified varieties given by flag manifolds are not enough on the Dynamics space, and require an evolving version which is described in terms of flag bundles.

Third subsection is focused towards a better understanding of phase transitions and qualitative changes from the Geometric Algebra viewpoint. The initial idea is very simple: such "pathological" phenomena do correspond to default rank matrices which are linked to vectors representing links, their kinematics or the lack of integrability of equations giving the dynamics. Thom-Boardman singularities provide a general framework to treat any kind of singularities arising from rank default conditions.

Unfortunately, the problem is a little bit more complicated, because the successive inclusions between Geometry, Kinematics and Dynamics, and the obstructions to lift and descent distributions of vector fields (or their dual in terms of differential forms) require some additional developments of flag bundles and their tangent "spaces". The static case is well-known (see third subsection, below), but the dynamics case deserves still some issues to be solved which impose additional constraints for Mechanical Design in regard to embedded constraints. Our strategy use "complete objets" in jets spaces as support for dynamics involving extended configurations and working spaces.

Seemingly, the above above descriptions quite new and because of this, all of them remain as a theoretical purpose which has not been experimentally verified. To finish, we shall comment some additional aspects of Robotics which are still waiting for more developments from the mathematical viewpoint.

4.1. Discovering symmetries for Design

Nature is not necessarily symmetric, and contrarily to the original Lagrange viewpoint, the whole Mechanics can not be explained in terms of universal Kinematic laws extending the Geometry. There are (a) statistical aspects involving complex dynamics, (b) matter granularity which are contrary to the existence of universal laws up to scale, and (c) information exchange which can be only explained in terms of topological (non-metric) or even discontinuous models. Some typical examples of Robotics which do not have a geometric analogous correspond to (a) probabilistic or fuzzy networks, (b) lack of scalability for motion of insects and mammals, and (c) shape changes for microscopic organisms or differences of potential for activation/inhibition phenomena.

Nevertheless, the ubiquitous presence of symmetries, the emergence of self-organized hierarchies and the geometric representation of mechanical interactions with the environment, simplify the treatment of a lot of situations at different scales which can be developed in a common framework, even being aware of troubles to find universal laws.

4.1.1. Symmetries, conservation and motion laws

At the most abstract level corresponding to symbolic representations, one can find symmetrical or anti-symmetrical behaviors involving representations of tasks or functionals defined on base spaces linked to kinematics or dynamics. This remark is almost obvious, because in the linear framework any square matrix can be decomposed in a sum of a symmetric and a anti-symmetric matrix; however, this principle is extended to more general operators defined on spaces of functions or discrete representations (graphs). To detect them, it is necessary to design and implement proximity and similarity queries involving input signals and output commands.

The extremely large diversity of patterns for grouping and changing behavior of clusters motivate a simultaneous development of simplified representations (paths, simplicial structures) and superimposed structures (homotopy, co-homology theories) able of self-adapting to evolving shapes and dynamical deformations from a discrete viewpoint. A mathematical treatment of these issues (and more detailed references) can be read in my notes on Algebraic and/or Differential Topology. A computational approach can be read in my notes on Computational Mechanics (both in Spanish language in my web page or the MoBiVAP web site).

One can not expect that patterns or structures appearing at a scale be present at different scales. Each LoD (level of detail) has its characteristic features for composition (granularity) and behaviors. So, neural plasticity, inter-connectivity or capability for generating patterns by firing, can not be found at the scale of organs. Hence, it is necessary to work at different resolutions, with different logical patterns and different laws for each LoD. Furthermore, transitions between discrete and continuous models are very tiny, and thresholds for different patterns and behaviors are not easy to identify.

From an historical approach, Geometry appears as a limit case of Topology, where the group of (local) homeomorphisms fixing a point is replaced by its linearization, giving the general linear group as a general framework for all classical geometries. Linear subspaces provide an approach to non-linear subvarieties, with the same rules for boolean operations between basic primitives. More generally, the topology of Dynamical Systems provide a natural extension of static topological features by allowing a zoo of different phenomena relative to deformations, re- or de-composition processes.

Similarly, the discretization of metric properties for a riemannian manifold (M, ds^2), are replaced by a collection of isolated control points which re-organize the ambient space in terms of distance maps (Voronoi diagrams) with their corresponding combinatorial structures. This coarse scheme is extended to arrangements of simple geometric primitivas (lines, planes, spheres) or to a collection of "sites" acting as attractors as self-organizers of dynamics. This reasoning scheme can be applied to any kind of sites (not only 0D), allows to manage any kind of constraints (geometrical and dynamical ones), and it involves to quite different aspects going from learning patterns to motion planning in presence of mobile objects.

Furthermore, even in presence of hundreds of thousand of sites, this strategy allows to combine deterministic patterns with behaviors under uncertainty in the proximity of saddle points or repulsors linked to Voronoi diagrams. Hence, this reasoning scheme provides an assistance to navigation (module 2) from a finite collection of "sites" in different environments (indoor vs outdoor scenes, coupling with GIS, internal navigation for assisted chirurgy, e.g.). Additionally, constraints can be incorporated to different scenarios in different ways involving geometric, kinematic or dynamic restrictions given by algebraic or differential inequalities. Hence, queries about proximity and similarity involve not only to geometric eventually deformable shapes, but to kinematic trajectories and dynamic behaviors (including corresponding optimization and control issues). Some typical examples concern to eye-hand coordination, haptic interfaces for dynamic simulation, and integration of VR/AR for simulation and learning.

4.1.2. Smart environments and Self-Organizing Robots

A smart interaction with monitored environments requires a careful design for placement of sensors (to avoid dark zones, e.g.), a balance of power for emitter-receptors, and a coarse representation of the scene to find optimal solutions along motions execution. All these topics make part of models and algorithms for motion planning [81]. Usual approaches are based on symbolic representations given by graphs (including trees, as a very interesting case) or, more generally, forests. The comparison between graphs provides a first hard kernel of NP-problems, where heuristic and/or probabilistic searches can be very useful.

In presence of increasingly complex scenes, it is necessary to develop optimization strategies on graphs, which avoid the examination of all possi-

[81] An application to the Maritime Museum (Barcelona, Spain) has been designed and implemented by the MoBiVAP for smart wheelchairs along 2011 in the framework of the Patrac Project

bilities, with an inassumable cost. Thus, embedding techniques, isomorphic subgraphs, automatically balanced graphs, and elimination of redundancy become central problems for the execution of complex tasks by a robot. A typical example is given by the generation of an environmental map from an autonomous vehicle equipped with an omnidirectional camera, e.g. The availability of an approximated location of structural elements (obtained from plans or sketches, e.g.) provides qualitative criteria for navigation which will be examined in the module 2.

At the early nineties, an underlying hypothesis was the unicity of a robot interacting with the environment. Currently, this hypothesis can not be maintained, because there can be be different autonomous mechanical devices able of sharing a space, communications and actions. Thus, it is necessary to study different kinds of "collective intelligence" which are already present in populations of insects, invertebrates or small animals from several millions of years ago. To achieve efficient scalable solutions it is necessary to perform a right distribution of simple tasks which involves the whole system architecture. These biologically inspired topics are included in last module of these notes (animats). More involved issues concern to robots with a changing shape (as in Transformers) to optimize mechanical complexity (in regard to the scene or the task to be performed) which will not be considered here.

4.1.3. Performing tasks in unknown environments

The above paragraph supposes that a representation of the scene is available. This hypothesis is not necessarily true and sometimes, one finds situations where robot must operate under uncertainty or with incomplete information provided by sensors. Typical examples can concern to navigation (including locomotion) or manipulation tasks involving hazardous missions (inspection in nuclear installations or military operations, e.g.) or tasks in unaccessible environments (deep sea or planetary research, e.g.).

In these cases, it is necessary to complete the available information relative to the scene (VR/AR tools) and the task to be performed (dynamic simulation from the available kinematic and dynamic models of similar situations). The development of models and software tools for managing these situations is a hard challenge. Our proposal uses a combination of dynamical systems and probabilistic models associated to the different kinds of (scalar, vector, tensor) fields under uncertainty conditions. Strictly speaking, this approach is not a novelty because the formulation of dynamics in terms of such fields is of common use in Advanced Visualization, and Markov fields is a well known paradigm for probabilistic formulation of fields.

A specific contribution is linked to the specification of mechanical hierarchies (linked to geometric, kinematic and dynamical aspects), and the use of statistical tools based on Sampling Consensus (sac) which avoid the blind character of Stochastic Processes. There are a lot of variants for SaC which include Ransac (Ran for Random), Impsac (linked to importance functions), MLESac (Maximum Likelihood Estimation), etc. All of them are inspired in the use of these tools in Computer Vision from the late nineties; variants of Ransac are explained with more detail in the module 4 of the CEViC. The most meaningful are introduced and comment in the module 5 on Humanoid Robots, where we are trying of imitating human behavior for humanoid robots.

The introduction of mechanical hierarchies intends to avoid the gap between higher level programming for tasks (typical in Advanced Control Theory), by means the introduction of intermediate agents in charge of managing geometric aspects (relative to allowed shapes and position-force control, e.g.), kinematic issues (relative to velocities and accelerations at joints, and impedance-based control for stable trajectories) and dynamical issues (relative to dynamical balance feedback between anticipation and compensation effects by means force-based control, e.g.). A crucial issue for unification is to perform a feedback in terms of different kinds of symmetries.

4.1.4. Motion Planning and Kinematics

A typical issue related to motion planning is the traveling salesman problem which involves to identify the path with minimal length to be performed by an agent which must visit a collection of "sites" which can be interpreted as the nodes of a planar graph. [82]. From a computational viewpoint the TSP is a NP-hard problem, but it can be solved by means a combination of Minimum Spanning Tree (a subtree of the Delaunay triangulation linked to the set of sites) and a collection of "certificates" which allow to incorporate the heuristic.

Motion Planning in Robotics is more complex because it involves to other functionals furthermore the distance (the energy expenditure, e.g.),

[82] [Law85] E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy Kan, and D. B. Shmoys (eds): *The traveling salesman problem.* J.Wiley, New York, NY, 1985. is the classical "Bible" for this subject

environmental conditions (dangerous or unaccessible environments for humans, e.g.) or feasibility of tasks in 3D. This problem has been approached from different perspectives which allow to incorporate logical, metric, topological, geometric, statistical or heuristic constraints [83].

For once a robot, a complete solution of the problem is guaranteed by using topological arguments which are based on cellular decompositions of the configurations $\mathcal{W}$ and/or the working space $\mathcal{W}$. The corresponding brute-force algorithms display an exponential complexity, Hence, this solution is not useful in practice, and it is convenient to incorporate differential methods (based on attractors vs repulsors) and an appropriate heuristics (based on certificates, and similar to the TSP described above). To warrant the feasibility of task to be performed it is necessary to maintain bounded related functionals and assert the connectivity of the graph representing paths to be followed according to the scene characteristics and robot constraints.

The problem becomes much more sophisticated in presence of several mobile platforms which can display different behaviors. Accordingly to ecosystem modeling, a basic distinction is related to competitive versus cooperative behaviors which can be clustered; even so, the problem is not elementary, because there are different strategies for each team of players. A more detailed analysis of related issues is developed in the Module 2 (Navigation).

4.2. A geometric approach to Distributed Robotics

In the report "The Interplay between Mathematics and Robotics. Summary of a Workshop", the author outlines three most important challenges in distributed robotics, where mathematics is likely to make significant advances are:

- Developing formal models that allow principled comparisons between distributed robot algorithms and give performance guarantees;
- Developing dynamic control for non-smooth systems;
- Developing methods to characterize the power of modular robot systems and of matching structure to task.

The proposal developed along these notes uses different types of hierarchies which are represented by flags (i.e. superimposed nested linear subs-

[83]The most complete reference is still [Lat91] J.C.Latombe; *Robot Motion Planning*, Kluwer, 1991

paces). A typical case which appears in a recurrent way is linked to strata of stratified maps such as the transference map $\tau : \mathcal{C} \to \mathcal{W}$, their extensions (given by k-jets) and their local inverses (for inverse Kinematics and Dynamics). This idea is inspired by the systematic use of Grassmannians for the local case as geometric objects which are naturally linked to integrable distributions on any of such spaces. Matching of these local data is performed in terms of Grassmann bundles, or more generally by Flag Bundles which provide a natural framework for distributed robotics.

Linear reduction of non-linear information to subspaces is a common strategy in any Engineering problem, including all stages of the PAC (Perception-Action Cycle), and any kind of mathematical framework including manifolds (where appear as co-tangent spaces at each point), signals (spectral decomposition and data compression in frequency domain for 1D/2D/3D signals, e.g.) or statistical treatment (Principal or more advanced Independent Analysis), between others. From nineties there are a lot of increasingly efficient algorithms for a computer treatment of all these issues, and their extension to non-linear cases, including their approach by algebraic varieties. In all cases, Grassmannian varieties provide a classical framework (in fact a universal space) for the treatment of information relative to subspaces of *fixed* (co)dimension of a vector space.

Along nineties, it was necessary a very careful selection of parameters to ease clustering, concentration measures (regression, correlation, covariance), projection methods onto appropriate subspaces (SVM initially, auto-correlation) and stabilization of eventually mobile data (distances between distributions, Kalman filtering, e.g.). Recent developments of self-organization and self-learning from advanced Recognition models provide more powerful tools. However, the huge amount of data requires the introduction of multiple hierarchies to improve data management. Our proposal consists of introducing evolving nested subspaces for a simultaneous management of subspaces.

4.2.1. Grassmann bundles for Robotics

Elements of a Grassmannian of $(k+1)$-subspaces L of a $(n+1)$-dimensional vector space V are given by points p_L. Thus, the introduction of Grassmannians allows to consider a collection of a finite (initially constant) amount of data which evolve in an uncorrelated way. Superimposed structures (different vector bundles, typically) allow to manage differential systems which involve to a simultaneous evolution of linearly independent data represented by

$k+1$-dimensional subspaces. To fix ideas, we remember their set-theoretical description:

- The *Grassmann manifold* $Grass(k+1, n+1)$ is set-theoretically described as the set of $(k+1)$-dimensional subspaces L^{k+1} of a $(n+1)$-dimensional vector space V. This construction can be projectivized in a natural way. The simplest non-trivial case is given by the set $Grass(2,4)$ of projective lines in a 3-dimensional projective space $\mathbb{P}^3 = \mathbb{P}(V^4)$ which is a non-degenerate quadric in $\mathbb{P}^5$ called the Klein quadric. Grassmannian of lines are commonly used in Ray Tracing/Casting (inside Computer Graphics), 3D Reconstruction (inside Computer Vision) or to represent motions of articulated mechanisms in Robotics (kinematic chains, e.g.).

- A *geometric flag* is a finite collection of nested subspaces L^{d_i} of a n-dimensional vector space V for $1 \leq i \leq r+1$ with quotient spaces $Q_i = L^{d_i}/L^{d_{i-1}}$ being $L^{d_0} = (0)$ and $L^{d_{r+1}} = V$; the r-tuple $(d_1, \ldots, d_r)$ is called the "nationality" of the flag. The direct sum of quotient spaces Q_i gives a decomposition of V in subspaces of dimensions $q_i = d_i - d_{i-1}$ with $q_1 + \ldots + q_{i+1} = n$, i.e., the $(k+1)$-tuple $(q_1, \ldots, q_{i+1})$ is a partition of n giving the flag manifold $\mathcal{F}(q_1, \ldots, q_{i+1})$. In particular the partition of n linked to the Grassmannian $Gass(k+1, n+1)$ is equal to $(k+1, n-k)$. The set of partitions is a poset (partially ordered set) which induces a hierarchy inside the set of different types of flag manifolds. Any flag manifold linked to a partition of n can be embedded in the complete flag manifold $\mathcal{F}(1, \ldots, 1)$ associated to the partition $(1, \ldots, 1)$ (the most symmetrical one). Flag manifolds are used at the end of the first module to describe parallel manipulators (such as Stewart platforms and their generalizations). In fact, the mechanics of any articulated mechanism can be described in terms of flag manifolds or more generally flag bundles; even the inclusion of Geometry in Kinematics and finally in Dynamics can be represented as flags with variable "nationality" in the second extension $J^2\mathcal{X}$ where $\mathcal{X}$ denotes the configurations $\mathcal{C}$ or the working space $\mathcal{W}$.

The Grassmannian is a "universal manifold" in the following sense: The assignation which to each point $x \in M \subset \mathbb{R}^N$ assigns the tangent space T_xM induces the so-called Gauss map $M \to Grass(m, N)$, which allows to recover any linear structure (given by a vector bundle, e.g.) on M up to deformation. More generally, if we consider a vector bundle with total space E on a variety X (not necessarily a manifold), the *Grassmann bundle*

$G_k(E)$ matches the Grassmanniana of k-planes on each fiber E_x for each base point $x \in X$. Typical meaningful examples concern to k-dimensional distributions on the base variety which can be integrable or not (as it occurs for non-holonomic constraints, e.g.).

4.2.2. Flag bundles for Robotics

Similarly, for any stratified variety X (configurations and working spaces, e.g.) with good incidence conditions for strata (this condition is only generic), one has a *generalized Gauss map* going which assigns to each base point $x \in X$ a linear description of geometric, kinematic and dynamic aspects. All of them can be considered as nested linear subspaces which can be analytically described (in terms of second order jets) or geometrically described (in terms of "packs" of multivectors). In the same way as above, these partial flags can be embedded in the complete flag manifold with a very high dimensionality, which allows to consider (kinematic or dynamic) singularities and "degenerations" in terms of lower-dimension subspaces of the flag.

In the same way as above, one can construct the *flag bundle*$\mathcal{F}\ell(E)$ linked to any vector bundle E with total space E a variety X, by matching together the flags on each vector space E_x by means the transition function of the fiber bundle. Roughly speaking, morphological aspects of any robot can be described on an universal flag bundle, where inclusion relations involve to the architecture or, more generally, to the natural stratification between geometric, kinematic and dynamical aspects. Similarly, functional aspects are described in terms of scalar, vector and tensor fields, which display a similar hierarchical scheme. Furthermore, the flag bundle structure on the mechanical architecture induces a similar hierarchy for all the above aspects, which provides a support for distribution, control and optimization of tasks.

Seemingly, the use of flag bundles is new in Robotics. It can be motivated by parallel robots (such as Stewart platforms, e.g.), by the natural hierarchy in Mechanics (linked to geometric, kinematic and dynamic aspects) or from the basic notions of distributed Robotics (where nested subspaces represent the hierarchy between components or between functionals to be applied, e.g.). All of them are developed along these notes. A distinctive feature is that it involves not only to the mechanical architecture, but the whole Perception-Action Cycle (PAC) such it appears in the following paragraph.

4.2.3. Formal Models for the PAC

The *leit-motiv* for this paragraph consists of the framework provided by flag bundles to integrate mechanical issues in a common model can be adapted to the whole Perception-Action Cycle. Fusion of information arising from different sensors can be thought in an abstract way in terms of successive extensions of 1D/2D/3D signals and their spatial representations:

1. Fourier analysis is originally developed for *one-dimensional signals*. They are incorporated as acoustic, infrared or laser signals, e.g. They can be treated in terms of traditional Fourier analysis or in terms of wavelets. A compact algebraic treatment of the linked Harmonic Analysis can be developed in terms of representations of $\mathfrak{s}\ell(2)$.

2. *Two-dimensional signals* appear linked to the analysis of isolated images or video sequences [84] Typical related processes involve to 3D Reconstruction (module 2 of the CEViC), motion analysis (module 3 of the CEViC), and recognition (module 4 of the CEViC).

3. *Three-dimensional signals* can be relative to the dynamic analysis of video sequences (including kinematic characteristics of motion) or the propioceptive analysis of the whole evolving configuration of the robot. In both cases, we use a quaternionic representation arising from the problem formulation in the Geometric Algebra framework.

The common development of Fourier tools for all of them provides a hierarchical framework which allows to connect estereoceptive and propioceptive perception before performing any action, and to obtain a natural feedback in terms of evolving configurations or shapes. Navigation issues can be understood as a conversion of a flow image in a flow scene, and making decisions involves their integration which is ideally performed in terms of representations of $SL(2)$ and $SL(3)$.

In our case, one has a stereoceptive flow (in terms of information fusion relative to the environment) and a propioceptive flow (in terms of the self-consciousness of evolving shapes in regard to the environment), which are in feedback through interaction. The management of flows is performed in terms of fields for which we provide ideal structural relations between $\mathfrak{s}\ell(2)$- and $\mathfrak{s}\ell(3)$-representations. Unfortunately, in this case one has infinite-dimensional representations (in consonance with tools of Harmonic Analysis and their solutions) which are a little bit more complicated to manage.

[84] See the chapter 6 of module 1 of the CEViC for more details and references.

A typical "example" is given by the harmonic functions as solutions of the Laplace operator which is naturally extended to the harmonic forms as solutions of the generalized Laplacian operator $\Box = dd^* + d^*d$. Hodge Theory provides a general framework for these issues which have been extended to the singular case along eighties [85]. Recent developments of Computational Conformal Geometry open the door to obtain on-line solutions (at least in low dimensional case) which will be introduced in the module 6 in regard to advanced tools for simulation of gestures.

4.2.4. Reconfigurable robots

The main motivation of reconfigurable robots is to create more versatile robots by using reconfiguration of systems which have been introduced at the end of the precedent section (see §3,4,3). This idea is biologically inspired and intends to develope models where hundreds of small modules will autonomously organize and reorganize according to hierarchies linked to evolving geometric or topological structures. Hierarchies are not unique, and can change according to different phases of a task or according to evolving environmental conditions. Thus, a modular design is quite necessary.

Modular character involves mainly to the capability of recombining and self-organizing *functionalities*, not to the reuse of mechatronic components which is also allowed, but in the design framework (see next subsection). Functional modularity follows a scheme which is reminiscent of SOA (Services Oriented Architecture) and their recombination in POA (Processes Oriented Architecture) which appears at the end of nineties. In our case, tasks to be performed by the robot is a composition of basic units (which extend the notion of services) as they would be a process.

The main problem to be solved for autonomous robots is the capability of self-adapting to evolving environmental conditions or to changing shapes, even in presence of incomplete information. A typical and seemingly "innocent" issue is the automatic resolution of a puzzle by a robot, with additional troubles arising from lacking or excedentary pieces. This problem can be considered as an extension of assembly tasks which are introduced at the end of the first chapter of the first module. Automated detection of symmetries relative to object and the task simplify the design and implementation of corresponding algorithms.

[85] See Chapter 0 of P.Griffiths and J.Harris: *Principles of Algebraic Geometry*, J.Wiley, 1978 for a geometric approach.

4.3. Mechanical Design at different LoD

Issues in the design of robotic systems as mechatronic devices, the kinematics and dynamics of their operation, the physics of their interactions with the surrounding environment, and decision making procedures have preoccupied robotics researchers and practitioners since the earliest days of robotics research. Basic problems can be considered as solved from the electro-mechanical viewpoint.

Nevertheless the impressive advances from the eighties, our ignorance is still very high about how seemingly more simple organisms are much more efficient for all kinds of interaction of isolated elements (from invertebrates till superior mammals) with the environment. Furthermore, embedded intelligence in large populations of microrganisms or social insects display patterns for distributed architectures which we are not still able of mimifying. The question to be solved is not to introduce more advanced models (from the physical, mathematical or electronic viewpoint), but to try of understanding interrelations between components which can be recomposed in reconfigurable architectures which can be applied from nanostructures to very large systems (involving intelligent transportation systems at large scale, e.g.).

From a biological viewpoint, most live beings display a mixture of centralized and distributed architectures, which allow to combine different functionalities and reconfigure partial solutions which can be supervised or not by a global planner. It is necessary to combine different kinds of logical rules for knowledge, and their spatial and (bio)mechanical representation to different LoD (Levels of Detail) to assist making decision procedures.

Their full integration involves the most advanced topics of Perception-Action Cycle, and in view of the very large amount of partial solutions, it is priority the identification of organizer principles and dynamical systems which can operate at different LoD. Along this subsection, the attention is paid to basic principles underlying to mechatronic devices, whereas along next subsection, we shall concentrate our attention on organizer principles for the embedded intelligence.

Basic principles involving architecture and function are modularity and interoperability. To give shape to these principles it is necessary to develop abstract structures whose specification at each LoD allows to recover already known “instances” relative to evolving shapes and functionalities. General mathematical ideas such as symmetries (algebraic, infinitesimal, dynamical), semi-analytical stratifications (for spaces, maps, functionals) and multivector calculus (for scalar, vector, tensor fields) provide ubiquitous tools which act as organizer principles for the dynamics of complex mechanisms. Some illustrations of the above ideas as organizer principles are the following ones:

- *Symmetries* can be understood as mathematical tools, but they allow to generate mechanical behaviors (following action-reaction patterns), identify invariants or replicate basic patterns on symbolic representations (graphs) extending the currently available configurations. They provide growing patterns preserving original shapes or modifying them by breaking symmetries, e.g.

- *Semianalytic stratifications* allow to visualize feasible configurations according to mechanical limitations, represent any kind of mechanical constraints involving mechanisms or interaction with the environment, provide restrictions for optimization and control issues, and allow to identify possible behaviors or other agents by replication of self-restrictions in order to improve the interaction in multiagent environments.

- *Multivector calculus* allow to manage any kind of geometric quantities (scalar, vector, multivector), and their evolution along time in terms of fields and their corresponding flows, incorporates an intrinsic (non-coordinate) representation of the current state of the robot and its environment, reduces the number of parameters connecting discrete and continuous representations, identifies redundancies or pathologies (instabilities, singularities) in evolving mechanisms.

The introduction of hierarchies linked to the above organizer principles eases the distributions of processes, and in particular the development of parallel architectures in regard to complex constrained tasks and possible responses of the environment or other intelligent agents. Thus, (algebraic, semi-analytic and multivector) hierarchies reappear along all modules of these notes under different aspects.

To fix ideas, we develop some of these ideas along four paragraphs involving some aspects relative to design, kinematics, dynamics and their integration in Optimization and Simulation modules. We are not intending to give an overview of possible developments in this area, but to shed some light on some aspects which are not enough developed from the mathematical viewpoint. The consecution of this goal requires additional contributions and a reformulation of some concepts which is illustrated at each paragraph.

4.3.1. Towards a Modular Design Methodology

Nevertheless the quite unrealistic approach developed in the movies of "Transformers" saga, it poses a very suggestive issue. Is it possible to design robots which can be deployed and re-composed by changing the original morphology to perform different tasks?. In some cases, the answer is affirmative. It suffices to look at spatial stations which are orbiting around the Earth, with a lot of "deployable" components. Deployability is understood as relative to the "capability of folding"; this notion is developed in the second module in regard to the navigation of spatial platforms. An underlying principle is linked to PL-structures which can be recombined in different ways to recompose different shapes, by extending the old tangram constructions to mechatronic constructions.

From an opposite side, other micro-organic examples linked to Pharmacological Design involve to the re-composition of ligand molecules w.r.t. a receptor molecule which break down (at least) a link in each molecule, to generate a coupling between both types of molecules. The idea arising from molecular design is very suggestive, because a good pharmacological design must self-adapt to the illness evolution (as it occurs in malaria, e.g.). This adaptation follows stable arrangements of spheres whose energy levels are not optimal (in the equilibrium there is no activity), but suboptimal [86].

The goal of a *modular design of robots* is to ease the assembly of mechatronic pieces to achieve the desired movement. If one adopts a bioinspired approach, this goal is highly non-trivial because each motion involves to a extremely complex combination of components of muscle-skeletal architecture. In the same way as for pharmacological design, arrangements of links and joints are not necessarily always at equilibrium, and they can adopt suboptimal configurations to allow motions in a stable way, as it occurs with quadruped or biped gait in regard to locomotion tasks, e.g.. Reconfiguration involves must involve to mechatronic components and their functionalities (which are considered in the following paragraph in regard to kinematic issues).

"Brute-force" approach introduces a "design space" involving all possible choices for system parameters relative to materials and functions to be integrated in an optimal way. This approach is quite usefulness because one can not intend to have artificial devices able of performing any kind of terrestrial, marine or aerials displacement, e.g. Modularity in robotics con-

[86]Private communication of Dr.Patarroyo along a Cellular Biology Congress held in Alicante (Spain, 2002)

cerns to reduce the number of variants for mechatronic devices, till smart embedded systems in charge of their management.

Modularity of mechatronic devices involves to automation of design methods including VLSI system design (where Computational Geometry provides some keys to minimize distances on printed circuits which use Manhattan distance, e.g.), their integration in MEMS (MicroElectroMechanical Systems), capability of reprogramming progressively smart sensors (by using CMOS devices, e.g.) and microactuators along directions which combine robustness and adaptability to evolving environmental conditions.

Possible extensions of this approach to multi-functional architectures requires additional developments linked to a modular approach to tasks to be accomplished by robots. Some aspects linked to cooperation between nanorobots are presented in the last module in regard to cooperative models for animats.

4.3.2. Kinematic Synthesis

Kinematic synthesis is an important enabling technology for robotic systems. There are obvious opportunities to apply methods from (differential vs semi-analytic) geometry and (algebraic vs differential) topology to the kinematic synthesis problem in regard to evolving phenomena represented by open vs closed chains occurring along the interaction with the environment. A key feature consists of connections between different components, involving mechatronic architecture or the embedded intelligence (see next subsection). Connections are performed at critical points for functionals or fields; thus, one needs a differential framework to reformulate these principles which allows an exchange of "mechanical amounts".

Along these notes, kinematics of robots is understood in terms of successive k-jets extensions of a transfer map $\tau : \mathcal{C} \to \mathcal{W}$ or their extensions which include additional *maps* (multipaths and multiconstraints) and their local inverses, extensions of original piecewise smooth structures given by *manifolds* to semi-analytic varieties; and finally development of *superimposed structures* (fiber bundles, principal bundles, equivariant fibrations, e.g.) and invariant data (connections, e.g.) over them. Hence, our reformulation of kinematics involves to all of them.

An abstraction of the mechanical strategy described in the above paragraph can be extended to functionalities represented by different kind of mathematical objects such as

- *Maps* which are matched by means OF continuation (or also called, homotopy) methods extending usual interpolation. This strategy can be applied to other vector or tensor fields and it can be illustrated with multipaths linked to different trajectories or control points, with the corresponding control strategies. Tensor fields are crucial for elasticity, plasticity and adaptability in regard to contact, friction or sliding effects.

- *Manifolds* which can be matched using "chirurgy" techniques in the PL-framework (by using cellular structures, e.g.) or in the PS-framework (by using cobordism techniques, e.g.). The introduction of metric criteria (natural for pseudo-riemannian manifolds) allows to compare on a common geometric representation different systems that solve the same task using different sensors. Some typical examples concern to motion planning and other complex tasks such as locomotion or grasping and handling operations modeled on Lie groups

- *Superimposed structures* such as fiber bundles or, more generally, fibrations by means formal operations (tensor product) or by enlarging distributions or systems of differential forms, e.g.

All of them can be recomposed between them to support reconfigurable functionalities to be applied to sensors and actuators providing the support for the PAC (Perception Action Cycle). Furthermore, they can be reformulated at different scales involving from minuscule configurations to very large mechanisms. Additionally, they are compatible with continuous vs discrete models, deterministic vs random approaches, and support the information related with any kind of (scalar, vector, tensor) fields and their associated flows.

4.3.3. Contact and interaction Modeling

Contact modeling continues to be the main problem to be solved for Advanced Robotics in regard to interaction in partially structured environments. A classical pipeline is based in

1. the right positioning in regard to the task (grasping, locomotion, e.g.) to be developed;

2. the evaluation of contact along friction cones (between objects or with the ground, e.g.) whose axis is ideally given by the the unit normal to the surface at each contact point;

3. a modeling and simulation of tasks (grasping, locomotion) to be developed including collision, contact and friction effects;
4. a kinematic control of velocities at joints which display sudden jumps at contact or collision points;
5. the application of anticipatory forces to re-balance the architecture along the planned motion;
6. the motions executions (manipulation of grasped objects, e.g.) in a safe and stable way;
7. the tracking and evaluation of dynamic effects along monitored trajectories, specially at phase transitions;
8. the application of compensatory forces and moments to adjust the current to the expected trajectory.

The most difficult parts concern to the control of velocities at phase transitions, and the evaluation and correction of dynamical effects arising from the interaction between objects or with the environment. It is necessary to combine different devices to avoid vibrations linked to jumping velocities (including switching procedures to maintain almost-uniform accelerations), and use different kinds of control (robust vs adaptive) in correspondence with the alternance between closed-loop and open-loop transfer functions. Furthermore, it is absolutely necessary to validate results from an experimental viewpoint, under different environmental conditions.

4.3.4. Optimization and Simulation:

System design is tightly linked to optimization whose constraints can involve to the shape, the function (task to be developed) or a combination of both of them.

- *Optimization issues relative to the shape* are easier and they can be solved in terms of FEM (Finite Element Method) involving to metric criteria on products of Lie groups associated to configurations and working spaces [87].

[87] Invariant metrics on classical Lie groups are constructed from Maurer-Cartan forms, but we need a PL- or PQ-approach (PL: Piecewise Linear; PQ: Piecewise Quadratic) to ease their computational implementation

- *Optimization issues relative to the task* are more difficult to solve, because they involve to trajectories to be followed by each component, which are controlled in terms of a finite number of control points. Unfortunately, the space of trajectories is not finite-dimensional and one must introduce effective criteria for solving the corresponding variational principles linked to minimal length, energy, work, power or any other kind of kinematic quantity (described as Lagrangian actions, typically).

Classical approach in Mechanical Engineering consists of introducing a "space of states" which tries of identifying transitions in terms of snapshots linked to the space-temporal evolution of the whole mechanism. So, by applying a brute-force approach, there appears spaces of states with thousands or even millions of parameters with their corresponding constraints with are presumably solved with computers more and more powerful. In this context, numerical techniques for the corresponding optimization problems would provide the requested solutions for each problem.

Nevertheless the increasing power of computers for increasingly complex systems, the "brute-force" approach has no sense, because it ignores the topology of the space of solutions. Most systems are non-integrable in the ambient space, even if approximate solutions can be computed by using numerical analysis. However, if a generic small perturbation of a solution is near to unstable phenomena, solutions corresponding to stable regions are not longer valid in the vicinity of bifurcation regions.

Thus, previously to this blind machinery, it is necessary to perform a qualitative (topological) analysis in order to identify stable vs unstable regions in terms of bifurcation diagrams linked to the dynamical systems (given locally by ODE or the PDE) governing the dynamics. A key feature consists of bifurcations diagrams (concerning kinematics and/or dynamics) inheritate symmetries arising from the highly symmetric architecture of artificial mechanisms, and elementary motions of articulated mechanisms given as a composition of planar or spatial rotations and translations. Thus, ordinary and infinitesimal symmetries reappear again as an organizer principle for kinematics and dynamics.

The *simulation* of processes is a crucial step before any fabrication of mechanical devices. Simulation involves mainly to kinematic and dynamic aspects which are linked to the structural equations for true movements in the working space. In the same way as above there is an "force-brute" approach (based on the systems specification involving to huge spaces of states) or a more intelligent topological analysis of expected trajectories. In both cases, one can use numerical methods (such as interval-based methods, interval arithmetic, and automatic differentiation tools), but the search must be guided by some kind of "learning" which allows to identify ideal trajectories in terms of PL- or PQ-approaches linked to the realizable movements to be performed by the robot. Advanced Visualization techniques provide a structural framerwork for simulation which is based on flows associated to (scalar, vector, tensor) fields [88]

Usual approaches for learning issues are linked to *expert systems*. There is a very large amount of expert systems going from cellular automata, till ANN (Artificial Neural Networks) or Fuzzy Systems, depending on the type of logic used for programming; the above description requires a lowering precision form parameters linked to the tasks to be performed. A strong preccsion is an appropriate constraint for industrial environments in bounded and safe zones. However, the relatively high variability w.r.t. initial conditions and environmental constraints makes useless the ANN-based approach in less controlled environments. The introduction of fuzzy logic provides a more amenable behavior to the descriptive logic which is used by humans and superior mammals which are used in their daily activities.

4.4. Towards an integration of embedded intelligence

Embedded intelligence for an individual robot involves to a stereoceptive and propioceptive representation of external world and the whole mechatronic architecture. Its goal is to improve the interaction by means a progressively well adapted and smart behavior in regard to the tasks to be performed. External world is modeled in geometric terms which are extended to a reformulation of Mechanics. Internal perception is learned from tasks which are represented as (multi)paths under multiple constraints, which are modeled in topological terms (algebraic, differential, dynamical). Interaction between them is performed through learning procedures which are based on manifolds or, more generally, varieties with superimposed structures.

The integration of embedded intelligence is performed at different levels accordingly to a distributed systems which includes from reflex motions till the most advanced artificial capability of reasoning. Thus, it is strongly related with different types of logic (classes, propositional, descriptive), different types of expert systems (ANN, self-evolutionary, fuzzy systems) and a balance between centralized and distributed architectures for control and

[88]See module 6 of my notes on Computational Mechanics for details and references.

optimization issues.

It is necessary to introduce hierarchies for clustering actions performed by devices going from nanoscale to macroscales, able of exchanging information at critical points or zones according to different kinds of constraints. A qualitative version of maps of distances w.r.t. different shapes or functionals provides a general strategy which can be adapted to quite different situations appearing at different scales. Following this scheme, coordination is performed at critical point, making possible phase transitions, re- or de-coupling between components, and consequently the integration of phenomena appearing at micro- and macro scales.

Replication phenomena are controlled in terms of symmetries, including breaking or grouping symmetries at boundary of feasible regions for robots mechanics. Bounded variation introduces not only criteria for operability under constrained functionals, but a structural framework (given by Semi-analytic Geometry) for different kinds of stability (topological, differential, infinitesimal) relative to tasks to be performed.

Finally, Clifford calculus is not only a trick for representing evolving multivector quantities, but a context to develop a global intrinsic analysis. Furthermore, it allows to integrate optimization and control issues for multibodies in a natural way, by treating entities linked to multivectors as "points" in a "superspace" with higher symmetries than those which are present in ordinary space. Thus, far from being an irrelevant extension it is the kernel for an overall integration of all aspects concerning to Mechanics.

The above remarks involve mainly to isolated robots or the interaction of a particular robot with its environment However, it remains the challenge of developing a global framework for a multitask robot or, even more difficult to different robots which can interact between them in increasingly open environments. It is not necessary thinking of science-fiction films; it suffices to consider several (terrestrial or aerial) vehicles with semi-automatic navigation embedded devices in intelligent environments. From Computer Science viewpoint, this issue involves to the development of multiagent systems which is a well known topic in Artificial Intelligence. However, the transition between local and global aspects it is not well understood, still.

A variant of the compact-open topology introduced at the beginning of the subsection §2,4 of this introduction provides the natural framework for multiagent systems, but its formulation requires to specify the transition between local and global issues. For topological objects (any kind of manifolds, e.g.) this transition is performed by means atlas of coordinate charts

(U, φ); for functions it is a little more involved, because its compatibility requires (1) the comparison at each base point $b \in B$ by taking the "inverse limit" (germ of a function, in more formal terms) w.r.t. open sets U containing the base point; (2) the matching of superimposed data given by functions on successive intersections $U_{i_0,\ldots,i_k} := U_{i_0} \cap U_{i_1} \cap \ldots U_{i_k}$ of open sects. The same argument is valid for (distributions of) vector fields and (systems of) differential forms or, more generally, any kind of tensors.

The Čech cohomology allows to know if the resulting system (of any kind of tensors) is solvable in terms of the vanishing of cohomology groups on a manifold or variety X. In particular, non-vanishing of some cohomology group implies that there exists some "obstruction" to the resolution of system of equations relative to "sections" of some superimposed structure onto X corresponding to functions, distributions of vector fields, systems of forms or, more generally, tensors.

At first sight, this topological approach is seemingly too abstract and useless for Robotic issues, but it is not true. To fix ideas, the open sets U_i involve to subsets in a space-temporal representation of the configurations or the working space for the i-th robot $\mathcal{R}_i$; any command can be understood a s a function defined on an open set U_i linked to the i-th robot which is "traveling" with $\mathcal{R}_i$ supporting task under structural constraints which live in a compact subset $K_i \subset U_i$ (compact-open topology). Geometric interactions between different robots $\mathcal{R}_i$ and $\mathcal{R}_j$ (semi-automatic vehicles, e.g.) are described in terms of (classes of) functions defined on non-empty subsets U_{ij} where the interaction is possible, including possible "holes" or (eventually mobile) obstacles B_α in the scene which can give obstructions to solvability of equations linked to commands [89].

Similarly, Kinematic and Dynamical issues can be formulated in terms of distributions of vector fields or systems of differential forms, or in a more compact way as sections of k-th order jet bundles for low values of k. Anyway, solvability of resulting systems is discussed in terms of vanishing of Čech cohomology groups $\check{\mathrm{H}}^k(X, E)$ where E represents the "sheaf" linked to the superimposed structure given by distributions $\mathcal{D}_i$ of vector fields, systems $\mathcal{S}_i$ of differential forms or, more generally, any kind of tensors $\mathcal{T}_i$ for the i-th robot $\mathcal{R}_i$. This approach is well known in G.A.G.A. (Algebraic Geometry-Analytic Geometry in French language) from the sixties, but seemingly it has not been used in Robotics. We shall illustrate the utility of this approach for navigation issues of several vehicles at the end of the second module

[89] Let us remark that obstacles are not necessarily physical ones; they can involve to opaque zones for communications or to limitations of sensors

(Navigation of floats), and some more advanced applications along the fourth module (simultaneous Dynamic Control of several semiautomatic vehicles).

Obviously, the consecution of this program requires a reformulation of different approaches to ease the information transference between different approaches. Along this last subsection, one displays some key ideas to re-organize these materials.

4.4.1. Beyond the Differential Geometry

The approach performed along these notes can be considered as geometrically based, where Geometry is understood in a broad sense according to:

- *Geometry following Klein*, i.e. as the set of invariant properties for the action of a classical Lie Group, with rotations and translations which are coded in the euclidian group, e.g.
- *Geometry following Lie*, i.e. in terms of "continuous" groups of analytical transformations, and their infinitesimal versions (Lie algebras), to ease the estimation of the most common transformations and formal integration of motion's equations.
- *Geometry following Darboux* i.e. linked to the preservation of a bilinear 2-form which represents the Hamilton-Jacobi structural equations for an ideal description of the dynamics of rigid or articulated bodies.
- *Geometry following Noether* i.e. in terms of infinitesimal symmetries linked to the first integrals of an action functional, such as the total energy of a system, e.g.

Differential Geometry provides an ideal initial framework for a regular dynamics on smooth manifolds M; local homogeneity (given by continuous or discrete symmetries) are described in terms of Lie groups for algebraic aspects or Lie algebras for infinitesimal aspects. Global properties can be described in terms of fiber bundles or, in presence of symmetries, in terms of principal bundles. Displacements of geometric quantities given by (scalar, vector or tensor) fields are described in terms of (metric, affine Ehresmann) connections.

In practice, the support is not smooth because there are (geometric, kinematic and dynamic) singularities. Groups are not a priori fixed and actions can have a regular orbit, but several sub-regular orbits with singularities at boundaries. The (dis)apparition of symmetries for solutions of ODE linked to motion structural equations is represented in terms of equivariant bifurcations which allow to recover phase transitions in complex tasks (re-grasping, locomotion). These remarks motivate the introduction of the *Semi-Analytic Geometry* as a natural extension of the Differential Geometry, whose objects can be approaches by PL-varieties by making easier a computational approach [90].

From a kinematic viewpoint the apparition of non-holonomic constraints (i.e. relative to velocities which can not be deduced from position-orientation coordinates) extend the classical lagrangian approach. They provide an explanation of rolling or sliding contacts which can not be deduced from differential (Hamilton-Jacobi) nor integral (Newton-Euler) equations corresponding to variational principles. Minimization of distance functionals in the space of trajectories or, more generally, Lagrangian action functionals in the space of connections requires the introduction of some topological tools linked to Morse Theory and their extensions to locally symmetric spaces.

Variability of shapes (relative to morphological aspects linked to robot architectures) and fields (relative to functional aspects linked to tasks) require a topological approach which displays a more flexible and adaptive behavior than geometrical approach. In abstract terms, *Topology* can be understood as a natural extension of the Geometry which is obtained by replacing a finite-dimensional group by an infinite-dimensional group of C^r-equivalences which is appropriate for optimal design of tasks. Topological approach is also hierarchies by following a progressively complex scheme

- *Set-theoretical topology* with homeomorphisms as typical actions which are locally obtained by integrating vector fields linked to any kind of motions (proper or external).
- *Algebraic Topology* which provides a PL-approach (PL: Piecewise Linear) to set-theoretical topology, with affine maps as the basic tool; it provides tools for homotopy methods, superposition of meshes and evaluation of functionals on PL-structures giving invariants for pattern recognition, e.g.
- *Differential Topology* as a natural extension of Differential Geometry with diffeomorphisms as actions in regard to the ideal preservation of "quantities" or functionals linked to actions performed by robots.

[90] See my notes on Computational Mechanics for additional details and references

- *Topology of dynamical systems* which provides a quite general connection between geometry (support where dynamical systems are defined) and topology (of spaces of solutions). It allows multiple interpretations in terms of deformations and/or flow, providing key invariants linked to flows in configurations and working spaces. Finally, it provides some structural connections between local (ODE, PDE) and global formulations (fiber bundles) of spatial phenomena which are evolving along time.

Transference of dynamical effects along critical zones is formulated in terms of *Lagrangian Actions* which is topologically reinterpreted as an enlarged version of the old Morse Theory [91]. The variational reformulation for action functionals allows to connect with the old integral version (Euler-Newton-Lagrange) of Mechanics, identify optimal solutions (extending the usual geodesics), associate invariants linked to infinitesimal symmetries, evaluate topological characteristics of flows in regard to dissipation of some "quantity" (as symplectic diffeomorphisms in regard to generalized Moment Map, e.g.) and control the ideal evolution in terms of preserved quantities or phase transitions in terms of controlled submersions, e.g.

4.4.2. Non-standard Optimization and Control

Classical dynamics is deduced from Hamilton's principle which is based on the momentum preservation (in the algebraic framework) or the energy conservation (in the variational framework). Both of them can be described in terms of algebraic or infinitesimal symmetries in the Lie framework which is extended to Principal Bundles in the global framework.

However, the apparition of non-holonomic constraints breaks down the above scheme. In particular,

- Energy is preserved, but not the moment; hence, classical moment map must be replaced by other kinds of symmetries.
- Infinitesimal symmetries can be described in terms of Poisson first integrals, but the bracket does not satisfy the Jacobi identity.
- Furthermore, the volume is not preserved and, consequently, usual formulation in terms of any subgroup of the special linear group is not longer valid.

[91] This idea is reminiscent of the standard model for unification between different interactions which appears in Theoretical Physics

It is necessary to identify Poisson symmetries (which do not generate a Lie algebra, necessarily), to estimate the lack of holonomy linked to such symmetries and to analyze stability properties away from the integrable regions. The notion of differential closure for related fields must be explored in regard to controllability issues.

The distinction between integrable E and non-integrable Q subbundles poses the problem of analyzing complementary behaviors which can assist control and optimization issues in stable regions of the phase space $\Omega^1_{\mathcal{C}}$ (cotangent bundle of the configurations space, initially). If $r = rk(E)$ and $q = rk(Q) = n - r$ are the ranks of such subbundles, then the Grassmannian bundle $Grass(k; \Omega^1_{\mathcal{C}})$ provides a support for the design of control and optimization issues. An obvious differential first-order relation between both subbundles can be read in terms of $Hom(E, Q)$ which is nothing else than the tangent bundle to the grassmannian bundle. This argument is immediately extended to flag bundles; a coarse description can be read in [Fin99] [92]

From the local viewpoint, both of them require to solve dynamical systems on the total space of the grassmannian which are locally given by matricial ODE defined on spaces of $k \times (n-k)$-matrices. Riccati's equations provide an enough general type for control issues. This idea can be easily reformulated in terms of k-multivectors which appear in the Geometric Algebra framework. However, we have not still a compact solution for this kind of problems. From the locally homogeneous viewpoint, one can not expect a complete system of G-invariant solutions, but one can look at G-stable regions where the control must be performed. Our proposal develops some basic facts arising from SOM by their adaptability to formulate local issues.

4.4.3. Computer Graphics and Computer Vision

Computer Graphics has been extensively used in Robotics from the beginning. CAD/CAM tools provide the support for Geometric Design (module 1). Three-dimensional meshes allow to represent (eventually deformable) objects and elasticity effects relative to components or interaction of the robot with the external world. Simulation tools allow to represent evolving robots or very complex tasks related to tele-operation or assisted surgery interventions, including VR/AR elements. They are used in the second half of these notes for simulating dynamical effects (module 3), represent in-

[92] J.Finat and M.Gonzalo-Tasis: "Lie Techniques in Neurocontrol for Intelligent Hierarchised Systems" (CAIP'99, Florida), 1999

teractions of humanoid robots with the environment (module 5) and for entertainment industry (module 6).

Computer vision has been applied from the early eighties to all areas of Robotics including fabrication, quality control, visual feedback, reconstruction of external world, objects recognition, evaluation of flow scene from flow image, interactive navigation, learning by imitation, assistance to medical robotics, teleoperation, augmented reality, image/video based rendering, between others.

Visual information can be captured from monocameral, bicameral or multicameral devices; furthermore, omnidirectional cameras provide additional devices for spherical representations of the whole scene. Typical visible spectrum devices are complemented by other devices which provide range information (based on structured light, infrared -IR- or laser devices, e.g.), which are very useful for interactive navigation of autonomous vehicles.

Some of the most important areas of Computer Vision are

1. *Image Processing* (focused towards information extraction from segmentation) *and Analysis* (focused to grouping in geometric or topological primitives).

2. *Three-Dimensional Reconstruction* from stereo devices or mobile cameras.

3. *Motion Analysis* for an eventually mobile camera with several eventually mobile agents in the scene; and

4. *Recognition* of objects and tasks with description, detection and classification as the main issues to be solved.

Each one of the above topics is developed in the modules 1-4 of the CEViC. As always, one needs robust mathematical models, efficient management of data and real-time algorithms able of providing a response to improve the interaction in the above mentioned domains.

Almost all advanced robotic systems incorporate some modules of Computer Vision. Originally, it was oriented towards Active Vision, i.e. the search of a precise object or a well-defined task to be tracked and controlled by robot. Automatic navigation poses additional challenges which were essentially solved in structured scenes along the nineties; this approach has been extended to monitored roads, but a much more hard challenge is its extension to arbitrarily environments for any kind of vehicles (including subsea navigation). Perhaps the most difficult challenge concerns to hand-eye coordination and learning by imitation, which are commented in last paragraph of this subsection.

4.4.4. Statistical learning theory

Learning theory is linked to different kinds of logic, which can be labeled as logic of classes, propositional and descriptive logic, by following an increasingly order of difficulty. The most elementary one (logic of classes) needs to fulfill if parameters describing objects or tasks are identical or not; hence, it involves only to a metric accuracy which is applied to meaningful control points. Propositional logic implies some kind of reasoning which can be modeled in terms of Automated Proof of Theorems [93]. The most interesting case corresponds to descriptive logic, where one must introduce clustering criteria, establish similarity criteria, and develop different kinds of reasoning (fuzzy logic, e.g.) for supervised or unsupervised learning.

Data flow relative to propioceptive sensing requires efficient algorithms for clustering, analysis and interpretation. Initial techniques for linear clustering are not enough for complex dynamics. From Expert Systems, Self-Organized Maps (SOM) provide a general framework for treatment of local issues which are compatible with the continuity of environmental spaces and the space of tasks (understood as paths or multivectors linked to multipaths, e.g.); in addition, they are able of adjusting by themselves to a changing dynamics. However, they are not appropriated for learning issues, where vector quantization can become much more effective.

Statistical approaches to robotics provides an initial framework to represent uncertainty in Robotics linked to PAC, because it allows to incorporate the inherent uncertainty linked to advanced learning. As robots move away from factory floors into environments populated with people, the need to cope with uncertainty is enormous. Active Vision is not enough to give a response in complex open environments; it is necessary to implement more advanced Recognition modules.

Along nineties there has been joint developments which combine metric, differential and statistical criteria to solve mobile robot localization and mapping. They provide inputs for structured environments, but in presence of large volume of data one requires a more flexible approach able of self-adapting. Probabilistic approaches provide criteria which can be oriented by structural models (as it occurs in stochastic processes) or by heuristic

[93] See the module 1 of my notes on Computational Mechanics for details and references

models (as it occurs with variants of Ransac models) in evolving environments. Due to an easier connection with perception modeling, we privilege the development of methods based on Sample Consensus (Ransac, Impsac, Mlesac, etc)

Additional research (models and algorithms) is needed involving the feedback between Recognition and Learning. Representation of uncertainty in robotics domains is modeled in terms of variants of Ransac, but they require additional hierarchies able of organizing open models for knowledge, with fast algorithms for reasoning under uncertainty. Statistical learning on manifolds has been introduced at the early years of 21st century.

Our proposal is based on a small extension of this approach which develops a probabilistic version of fiber bundles $\xi = (E, \pi, B)$ on base spaces which initially correspond to manifolds (or more generally, varieties). Intuitively, descriptors and detectors are represented by elements $b \in B$ of the base space B and (sections of) the fiber $F_b = \pi^{-1}(b) \subset E$.

The introduction of group actions on geometric primitives of B and E provide classes, and consequently, classification criteria for objects to be recognized and, consequently, learned. As always, one has finite groups (given by symmetric or alternate groups, e.g.), finite-dimensional groups (given by a subgroup of the general linear group, e.g.) or infinite-dimensional groups (given by a subgroup of the homeomorphism group, e.g.). The former ones are a discretisation of the second ones, which are a linearization of last ones; hence, one has a natural hierarchy for classification criteria, also [94]

[94]Additional details for this approach are developed in the module 4 of the CEViC.

5. Scheme of the Course

This Course includes 4 modules with six chapters per module. Each chapter is explained along two weeks which makes a total of 3 months per module, according to the usual time distribution. Last month of each period is focused towards the development of a practical work to be performed by at most two students. Relative to contents, these notes are organized in two parts which can be given in two Courses according to the usual organization in Mechanical Engineering, Applied Mathematics and/or Computer Science:

1. The first part is composed by two modules with a generalist character, nearer to Mechanical Engineering than Computer Science; theoretical kernel is given by mathematical formulation of mechanics following Lagrangian and Newton-Euler paradigms. Problems and challenges to be solved are nearer to basic research, rather than applications.

2. The second part is more specialized towards artificial models of multibodies with human and animats as main paradigms; theoretical kernel is given by Biomechanical Engineering for multibodies models and Computer Vision and Graphics for capture, simulation and advanced visualization of motions linked to animats (insects, birds, mammals, humanoids). Problems and challenges to be solved are nearer to applications in Prosthetics/Orthopaedical surgery or assistance to multimedia production, rather than basic research.

Along this subsection, we display some comments about the contents of each module, by including the main topics to be developed by chapter. Contents of each chapter is devoted to present theoretical foundations and information processing; more details about foundations and more detailed proofs can be found at the given references.

Each chapter is organized in sections and subsections; at the end of each paragraph, one includes a very reduced number of exercises which are written to self-validation of contents understanding. It is not necessary to solve with detail these exercises; it suffices to sketch a solution, but if the student has no idea about how to solve them, then he/she would must read again and/or consult the tutor. Furthermore, at the end of each chapter there is a collection of problems, which provide some guides to orient the practice to be developed for each module, according to the tutor.

5.1. Anchored Robots

The first module has a basic character and provides a support for the other modules. Along this module, each robot is considered ideally as a kinematic chain which is connected to a fixed body which is anchored at a fixed platform. Most of industrial robots operating in industrial environments are anchored robots. Robots for assisted surgery, and parallel robots for for flight simulators (training pilots) or for entertainment industry are anchored robots, also. These "examples" justify our terminology.

5.1.1. A description of the module 1

The first module B_{31} (Anchored Robots) contains the following chapters:

1. **Geometric Design of Robots**. Basic principles. Degrees of freedom. Links. Types of joints. Platforms. Control points. Planar and volumetric robots. Low DOF chains for generating forces. High DOF chains for generating moments at torques. Redundant and hyperredundant robots. Microbots. Nanorobots.

2. **The planar case**: Simplest robots are planar, i.e., they are composed by a finite collection of components (joints and segments) which are represented by a planar polygonal contained in a bounded region of $\mathbb{R}^2$. Motions of their components and its end-effector are constrained to rigid displacements which are contained in a fixed plane π. Even so, this architecture allows to introduce basic concepts, robotics architectures involving components, solve reachability issues.

3. **Spatial kinematic chain** are composed by a collection of mechatronic components which can operate in a bounded region of the space. Their geometry is composed by a collection of consecutive segments which are connected between them by spherical or Cardano joints. Following a similar approach to the performed in Chapter 1, we shall represent each configuration as a spatial polygonal contained in $\mathbb{R}^3$. Motion of end effector can be represented by a paths which are controlled by composing different actions of Lie groups. In a very similar vein to the precedent chapter, we are specially interested about forward and inverse kinematics which are expressed in terms of Lie Groups linked to internal movements at gears, and to external motions performed by links which are translated to a final motion of the end-effector.

4. **Simulation of robotic operations**. Before performing any task, it is necessary to simulate not only the robot, but the interaction with the environment where the tasks will be developed. In this chapter one introduces some basic tools to simulate tasks which are strongly related with the software tools for mechanical design which have been presented in the first chapter. Additionally, one simulates some basic aspects of kinematics for robotic arms composed by one a kinematic chain. An advanced example concerns to snake-like robots.

5. **Industrial anchored robots**. In this chapter we consider planar and spatial robots acting on conventional industrial environments. Main operations to be modeled and implemented are Grasping and handling, inspection, cutting, welding, assembly and painting. In all cases, a high accuracy is required. Usually, robots are isolated in isolated zones by safety and security reasons. Thus, we don't consider possible interactions with other robots or human in mixed environments.

6. **Assistance to chirurgy** Versatily and redundancy. Metric calibration for extreme accuracy. Information fusion from different biomedical images. Navigation in deformable environments. Cutting and assembly operations. Precission and reliability. Laparoscopy. Neuronavigators for assistance in brain chirurgy.

7. **Hybrid architectures** Some of the most interesting anchored robots are given by *Stewart Platforms* and their generalizations. These mechanisms are composed by two polygonal plats and a collection of bars connecting vertices of both plats. They are used for different purposes going from flight simulators, hip simulation or prosthetics of artificial hands, e.g. We follow a geometric approach which is based on *flag manifolds* to interlink representations in working space. Flight simulators.

8. **Expert Systems for Robotics**. Cellular automata. Basic elements of Artificial Neural Networks (ANN). Associative memory. Dynamical aspects. Statistical Pattern Recognition. Learning. Some extensions. Neural computations.

5.1.2. Some references for the module 1

Two good references for Classical Mechanics of rigid bodies can be found in [Abr78] or [Arn89] including a large sample of differential and topological methods which are commonly used in Mechanics. Unfortunately, neither of them considers problems related with the Mechanics of articulated mechanisms or multibodies. A more modern approach in the Differential Geometry framework can be found in [Mur94], which includes the first systematic approach to Robotics in terms of Lie groups.

From a computational viewpoint, it is necessary to extend the original contributions of Computational Geometry to Mechanics including aspects relative to Kinematics and Dynamics. We adopt the same scheme as in Computational Geometry, which is organized around models, data and algorithms. This extension is currently labeled as Computational Mechanics, which includes additional aspects relative to Computational Topology. Unfortunately, there is no a general reference covering all these topics [95]. The following list includes some of the most relevant textbooks. I apologize by omissions of meaningful references-

[Abr78] R.Abraham, and J. E. Marsden: *Foundations of Mechanics*, Addison-Wesley, 1978.

[Arn89] V.I.Arnold: *Mathematical Methods of Classical Mechanics* (2nd ed), Springer-Verlag; GTM 60, Springer-Verlag, 1989.

[Boi88] J.D.Boisonnat and J.P.Laumond (eds): *Geometry and Robotics*, LNCS 391, Springer-Verlag, 1988.

[Cal04] G.A. Calvert, C.Spence and B.E. Stein (eds): *The Handbook of Multisensory Processes*, The MIT Press, 2004.

[Cor13] P.Corke: *Robotics, Vision and Control: Fundamental Algorithms in MATLAB*, Springer Tracts in Advanced Robotics (1st ed. 2011, reprt 2013), Springer-Verlag, 2013.

[Gol89] D.E.Goldberg: *Genetic Algorithms in Search, Optimization and Machine Learning*, Addison-Wesley, 1989.

[Kob89] A.A.Kobriski and A.E.Kobrinski: *Bras manipulateurs des robots*, Ed. Mir, 1989.

[Koh97] T.Kohonen: *Self-Organizing Maps* (2nd ed), Springer-Verlag, 1997.

[Kul88] V.S.Kuleshov and N.A.Lakota (eds): *Remotely controlled Robots and Manipulators*, Ed. Mir, 1988.

[Mur94] R.M. Murray, Z. Li and S. S.Sastry: *A Mathematical Introduction to Robotic Manipulation*, CRC Press, 1994.

[Nuc09] A.Nüchter: *3D Robotic Mapping. The Simultaneous Localization and Mapping Problem with Six Degrees of Freedom*, Springer-Verlag, 2009.

[95] My notes of Differential Geometry and several modules of Computational Mechanics are in Spanish language, only.

[Pas95] A.Pasqual del Pobil, M.A.Serna: *Spatial Representation and Motion Planning*, LNCS, 1014.

[Per92] P.Peretto: *An Introduction ot the Modeling of Neural Networks*, Cambridge Univ. Press, 1992.

[Sel00] J.M.Selig (ed): *Geometrical Foundations of Robotics*, World Scientific, 2000.

5.2. Automatic Navigation of Autonomous Vehicles

The main problem to be solved in the module B_{32} (ANAV) is the *assistance to real-time navigation* by a mobile platform. We shall consider only geometric and topological aspects which are meaningful for smart behavior based on expert systems. We shall give an introduction to propioceptive systems which are useful for an interactive navigation, including a description of sensor-based systems which are integrated in the navigation systems, with a special regard to vision-based systems, by their capability to generate three-dimensional representations of the environment. We avoid more complex kinematic and dynamic issues which are developed in the third and fourth modules, respectively.

Initial examples are given by wheeled vehicles with a circular or rectangular basis. The former ones are commonly used in Robotic Labs corresponding to already known structured environments. A more advanced case does corresponds to wheeled chairs for disabled persons, with a supervisor module to avoid collisions with fixed or mobile obstacles (including persons); in this case, the environment is only partially structured. A third and more sophisticated example is linked to traffic scenes; first applications in real outdoor environments were performed at the late nineties in the USA. The incorporation of semi-automatic devices for driver assistance in smart vehicles is more recent and deserves relevant challenges which are being currently developed.

To start with, each mobile platform is considered initially as a material point which represents the c.o.g. of mobile platform. The main purpose is to provide an eventually changing geometric representation of the environment able of integrating the information arising from sensors, an unified treatment of signals, their interpretation and transformation in commands.

The *main goal* along the second module is focused towards an integrated representation of all the processed information in a 3D representation of the scene. Thus, we prior the visual information which combines 3D reconstruction and motion issues. This goal involves to different aspects which are organized by following a typical pipeline:

1. The real-time processing and analysis of mobile data provided by sensors

2. Updating of the 3D representation for the geometric environment.

3. Self-localization based SLAM (Simultaneous Localization and Mapping) methods.

4. Combination and selection of sparse and dense data at different scales.
5. Symbolic representations for grouped data (quickly updatable perspective models, e.g.).
6. Interpretation by dedicated expert systems
7. Decision making from the evaluation of the precedent information
8. Control devices to execute commands in charge of semi-automatic navigation.

The combination of the above aspects intends to provide a driver assistance in the most sophisticated models corresponding to open environments. This is a far-reached goal which is being progressively incorporated to some vehicles.

All the above aspects are integrated in a mathematical framework which is provided by the Differential Geometry (as an extension of the classical Projective Geometry) in a broad sense. Differential Geometry allows to integrate the flow image in a flow scene which provides an integrated framework for navigation issues. To fix ideas, we shall restrict ourselves to the regular case [96]

Our choice of a geometric context is initially justified by the conversion of information arising from different kinds of signals in the spatial domain. Furthermore, the geometric framework admits a natural extension (in the context of Differential Geometry, again) to Kinematics and Dynamics which appear as superimposed layers to be developed in modules 3 and 4, respectively. [97]

To achieve these goals, it is necessary to develop a flexible approach to the Differential Geometry going beyond usual Curves and Surfaces. Indeed, the high dimensionality of the configuration space requires a more abstract approach which is performed in terms of manifolds and their superimposed structures: fiber bundles (initially given by tangent and cotangent bundles or phase spaces), connections (in their metric and affine versions) and principal bundles (to include homogeneous arising from the action of Lie groups). Some mathematical foundations for these aspects are developed in the second chapter.

[96] The singular case is introduced in the module 3 in the context of Differential Calculus on Semianalytic Geometry which is a natural extension of the regular case.

[97] A spanish version of a computational approach for Mechanics as an extension of Computational Geometry appears in my web site.

Next, we consider some local aspects which concern to a feedback between qualitative properties of (image and scene) flows and actions to be undertaken for Navigation.

5.2.1. A description of the module 2

The *main problem to be solved* is the (semi-)automatic navigation of mobile platforms avoiding collisions with obstacles. This problem has a lot of approaches (see [Lat91] for a classical and excellent presentation). Roughly speaking it involves to a changing representation of the environment and the (absolute and relative) localization of the mobile platform. This description keeps away a lot of very interesting topics related to navigation such as maritime, aerial or satelitar navigation with more than relevant problems to be solved related to the noise, turbulence and stabilization, e.g.

The geometry of mobile platform imposes strong conditions about allowable motions. So, the problem for circular-based platforms in indoor known closed scenes is easy, but the problem becomes increasingly more complex when we consider open environments, multilegged robots or possible interactions with other intelligent agents. To simplify, we shall restrict ourselves to the simplest cases corresponding to circular platforms or semi-automated wheelchairs in known scenes, initially. Later, we relax these hypotheses depending on additional developments.

A 3D reconstruction of the environment requires information arising from additional sensors, including infrared, acoustic, laser and cameras of visible spectrum, mainly. All information is referred to a 3D representation of the scene; thus, visual information arising from video cameras is the most relevant one to integrate all information [98] Usually, one adopts a coarse-to-fine approach going from relative localization (based on an affine reconstruction, e.g.) to more accurate representations (corresponding to a metric reconstruction, e.g.). The main contribution is the simultaneous management of RT generated perspective and semantic maps by using topological tricks (perspective representations as "retraction" of semantic maps, e.g.) and underlying cellular structures.

Along the preliminary states, it is very important to detect main elements which are linked to perspective representations including lines, planes and volumetric representations which can be globally managed by means some

[98] Nevertheless, for a reactive navigation, infrared, acoustic and laser can become more meaningful by their capability for RT responses.

basic elements of multivectors. In more advanced stages and depending on more complex tasks to be performed, it is necessary to consider finer information linked to curved shapes, e.g. The problem becomes much more complicated in open environments and in presence of multiple intelligent agents, which require additional developments arising from expert systems (different variants of ANNs and their extensions). This kind of applications is very useful for intelligent vehicles which have been initially developed along the late nineties.

According to the above remarks, this module contains the following chapters:

1. **Sensors and Perception. A geometric approach**: Deduced reckoning. Scanning the environment. Image and range information. Space representations. Motion planning. Navigation strategies: Self-Location, features recognition, simulation. Instruments for navigation: Compasses and Gyroscopes.

2. **Linear Geometries and Manifolds in Robotics**: From the cartesian space $\mathbb{R}^n$ one considers the main Linear Geometries (Euclidean, Affine, Projective) which are meaningful as support for kinematics and dynamics. The matching of these structures provides models for Smooth and Algebraic Varieties. An important case of manifolds and their tangent bundles is given by structural Lie groups of the above Geometries and their corresponding Lie Algebras.

3. **Local Differential Analysis for Mobile platforms**: Local comparison between structures and/or configurations varying along time require some elements linked to several descriptions of tangent spaces. In particular, flow images and flow scenes can be understood in terms of distributions of vector fields; their evaluation is performed in terms of systems of differential forms. The relation with navigation systems is read in terms of the differential map of the transference map linked to multisensor functions which is locally given by the Jacobian matrix. Its introduction allows to identify the Kinematic Singularities of mechanisms which plays an important role along the Course.

4. **Motion Planning** for one or several vehicles in indoor structured scenes or outdoor unstructured scenes. A first problema concerns to a representation of the environment from the information fusion arising from different sensors; from this representation one computes ideal paths which must be approached by PL- or PQ-trajectories performed

by some control point of the robot (belonging to the platform or represented by the end-effector of kinematic chains). Next, it is necessary to solve motion equations, under constraints relative to the scene or the robot architecture. Usually, such constraints are represented by means of algebraic and/or differential (in)equalities. Unfortunately, most mobile systems are non-holonomic even for daily tasks such as manoeuvres for car parking. Thus, it is necessary to develop Algebraic and Differential Topology methods for solving these problems. Some meaningful examples examples are included to illustrate our approach.

5. **Autonomous vehicles and SLAM** (Simultaneous Localization and Mapping): SLAM is one of the most common strategies for generating a map of the environment from an interactive navigation. Geometric sparse and topological dense variants provide two complementary approaches which are commonly used in semi-structured (indoor or urban) scenes and arbitrary outdoor scenes. Resulting models combine visual and range information (from metric sensors, e.g.) in a geometric 3D reconstruction of the scene which is updated from egomotion. Typical software tools for these representations are given by automatic generation of perspective maps and estimation of fundamental/essential matrix, respectively. These topics are developed in chapter 5 of the second module of the CEViC.

6. **Optimization and Control for Navigation** . Manifolds and Lie Groups provide a structural framework for both of them. Modern descriptions of Configurations Space $\mathcal{C}$ and Working Space $\mathcal{C}$ for Robotics is given in terms of Lie groups. They are groups G with an additional structure of manifold or smooth variety. This description allows to apply usual tools of differential and integral calculus on manifolds on the Lie algebra $\mathfrak{g}$ of G. This description avoids the cumbersome use of coordinate systems for displacements at joints, and provides a more natural support for interpolation procedures. Optimization issues are solved on the Lie algebra $\mathfrak{g} := T_eG$ of each group. A general approach to control issues is obtained by dualizing this approach.

   Autonomous Vehicles. This chapter includes some elements for semi-automatic navigation in partially or non-structured. The integration of information arising from different sensors (visual feedback, mainly) is crucial to improve security and adapt the current navigation to incidences which can appear in non-structured environments. A special attention is paid to round platforms, wheelchairs for indoor scenes and cars for outdoor scenarios which provide two cases of use for the two first sections. An extension to free-collision navigation in aerial and maritime environments are sketched by using some variants of mobile Voronoi diagrams. We don't intend to replace the role of human operators, but to provide an assistance for semi-automatic navigation.

7. **Expert Systems for Navigation**. Mobile platforms require very efficient algorithsm for updating and tracking cluestered data and generate patterns from video sequences. In absence of general models for recognition, it is necessary to implement strategies able of self-adapting to changing environments. Modeling changes requires refinements of ANN-based algorithms. One provides a small introduction to the main approaches in terms of Genetic Algorithms, Evolutionary Programming and Self-Organizing Maps with some applications to the generation of space-temporal representations by following increasingly complex models. In absence of structural patterns, Fuzzy models provide a very general strategy which takes advantage of statistical tools for reasoning.

8. **Multiagent Systems for Automatic Navigation**. The presence of different mobile objects (robots, humans, vehicles) can be modeled in terms of multiagent systems. Along this chapter we adapt some well known results of this computational topic. We apply these results going from the local management of traffic scenes till very large intelligent transportation or logistic systems. In this case, one adapts typical methodologies arising from cooperative systems in contrast with competitive systems which will appear in the module 6. All of them can be thought as variants of ecosystems with their corresponding strategies for distributed memory between different agents in presence of uncertainty about the behavior of other agents. Thus, it is necessary to develop some elements of Expert Systems and fuzzy reasoning on scenes flows changing along time.

5.2.2. Some references for the module 2

Previous remark: Most references of the module B_{32} (Automatic Navigation) are only general textbooks. More specific and detailed references can be found in [Gir96], [Hal97], [Sha97]. I apologize by meaningful omissions.

[Aar96] E. Aarts and J. K. Lenstra (eds): *Local Search in Combinatorial Optimization*. Wiley, 1996.

[Ber97] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf: *Computational Geometry: Algorithms and Applications*. Springer-Verlag, 1997.

[Bue09] M.Buehler,K.Iagnemma, ans S.Singh (eds.): *The DARPA Urban Challenge. Autonomous Vehicles in City Traffic*, Springer-Verlag, 2009.

[Cho05] H.Choset, K.M. Lynch, S.Hutchinson, G.A. Kantor, W.Burgard, L.E. Kavraki, S.Thrun: *Principles of Robot Motion: Theory, Algorithms, and Implementations*, Bradford, 2005.

[Cue05] F.Cuesta, and A.Ollero: *Intelligent Mobile Robot Navigation*, Springer-Verlag, 2005.

[Eve95] H.R.Everett: *Sensors for Mobile Robots*, A.K.Peters, 1995.

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[Gri90] W.E.L.Grimson: *Object Recognition by computer. The role of Geometric Constraints*. MIT Prtess, Cambridge MA, 1990.

[Hal97] D.Halperin, L.Kavraki and J.C.Latombe: "Robotics" Chapter 41, in J.E.Goodman and J.O'Rourke: *Handbook of Discrete and Computational Geometry*, CRC Press, 1997.

[Iag04] K.Iagnemma, and S.Dubowsky: *Mobile Robots in Rough Terrain. Estimation, Motion Planning, and Control with Application to Planetary Rovers*, Springer-Verlag, 2004.

[Koh97] T.Kohonen: *Self-Organizing Maps* (2nd ed), Springer-Verlag, 1997.

[Lat91] J.C.Latombe: *Robot Motion Planning*, Kluwer, 1991.

[Li93] Z.Li, and J. Canny: *Nonholonomic Motion Planning*, Kluwer, 1993.

[Mar92] J.E.Marsden: *Lectures on Mechanics*, Math. Society LNS 17, Cambridge Univ. Press, 1992.

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[Mul93] K.Mulmuley: *Computational Geometry. An Introduction through Randomized Algorithms*, Prentice Hall, 1993.

[Oka92] A.Okabe, B.Boots, and K.Sugihara: *Spatial tessellations: Concepts and applications of Voronoi diagrams*. John Wiley & Sons, Chichester, England, 1992.

[Rou94] J. O'Rourke: *Computational geometry in C* (1st ed). Cambridge University Press, 1994.

[Sci00] L.Sciavicco and B.Siciliano: *Modeling and Control of Robot Manipulators* (2nd ed), Springer-Verlag, 2000.

[Sha97] M.Sharir: *Algorithmic Motion Planning*, Chapter 40, in J.E.Goodman and J.O'Rourke: *Handbook of Discrete and Computational Geometry*, CRC Press, 1997.

[Sie11] R.Siegwart, I:R.Nourbakhsh, and D.Scaramuzza: *Introduction to Autonomous Mobile Robots* (2nd ed), The MIT Press, 2011.

[Sta09]C.Stachniss: *Robotic Mapping and Exploration. SLAM for autonomous robots*, Springer-Verlag, 2009.

5.3. Robot Kinematics. A Hierarchised approach

The initial geometric framework for Kinematics is given by the tangent bundle τ_M of a manifold or, more generally, the cotangent bundle Ω^1_V (dual of the tangent bundle in the classical case) for an eventually singular variety V. Singularities appear in a very natural way in regard to geometric, kinematic and dynamic aspects. Roughly speaking they can be interpreted as a *default rank* of matrices representing the transference map $\tau : \mathcal{C} \to \mathcal{W}$ or linked to manipulation (dexterity, e.g.) or locomotion (locomotivity) tasks, e.g. The rank default conditions are intrinsic, i.e., they do not depend on the inicial chosen reference for localization and they can be extended to kinematic and dynamic frameworks; hence they can be formulated in terms of Lie groups, and more specifically in terms of the Geometry of the Momentum Map [Gui90]

Ordinary differential equations linked to structural descriptions for kinematics or dynamics can be interpreted as sections of a fiber bundle or, more generally, as Ehresmann connections on the support of fiber bundles. Last interpretation allows to re-interpret minimal solutions as geodesics, by solving in the way linked optimization issues. To understand the above concepts, it is necessary to develop some rudiments of Riemannian Geometry which are introduced in the central chapters of this module.

The most relevant contributions are linked to the systematic use of Lie groups for discrete, finite-dimensional and infinite-dimensional symmetries. They involve to signals persistence, preservations of geometric configurations, motion analysis of trajectories performed by control points (including end-effectors), data alignment and decision mechanisms in automatic learning procedures for tasks-oriented approaches, between others. Their Lie algebras provide the basic models for their estimation, and to evaluate their transformations along the motion, according to structural equations.

Hierarchies between different symmetry groups are described in terms of G-equivariant decompositions (including capture and breaking procedures) and superimposed structures, going from Principal Bundles (initially developed by J.Burdick) till G-fibrations (introduced in the modules and A_{24} in Algebraic Topology). The superposition of available information about inner mechanism and the outdoor environment is performed in terms of cell complexes A_{23} giving a Cellular Equivariant Stratification (CES) as a common framework for morphological and functional issues involving low-order extensions (Jets spaces) of the PACW cycle.

5.3.1. A description of the module 3

The module B_{33} (Robot Kinematics) the following chapters:

1. **Differential Geometry in Robotics:** Manifolds. Vector fields and distributions. Differential forms and systems. Fiber Bundles. Integrability of systems.

2. **Forward Kinematics:**. Differential of a map. Jacobian matrices. Regularity. Direct images of distributions. Kinematic singularities. Lifting vector fields.

3. **Inverse Kinematics:** Pseudoinverses. Local duality. Feedback between forward and inverse kinematics. Path following methods. Kinematic analysis of trajectories.

4. **A hierarchical approach** introduce a general hierarchy between geometric and kinematic which can be extended to dynamic aspects. Jets spaces and prolongations of maps.

5. **Lie formalism:** Lie groups in Robotics. Homogeneous Spaces. Symplectic geometry. Infinitesimal version. Locally symmetric spaces. Principal bundles in Robotics. Evolving symmetries in Robotics.

6. **Elements of Riemannian Geometry** Riemannian metrics. Metric and affine connections on manifolds. Connections on Fiber Bundles. Ehresmann connections. Extension to jets spaces.

7. **Optimization in Kinematics:** Convex optimization. Non-convex optimization. Contact geometry. Internal constraints. External constraints. Integrability issues. Reachability.

8. **Kinematic control:** Lyapunov control. Robust control. Adaptive control. Interaction with the environment. Impedance-based control

9. **Geometric Algebra for Kinematics** The introduction of vector calculus provides a unifying language for some aspects described above.

5.3.2. Some references for the module 3

Only textbooks are included. I apologize by meaningful omissions.

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[Kel13] A.Kelly: *Mobile Robotics: Mathematics, Models, and Methods*, Cambridge Univ. Press, 2013.

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[Lau07] C.Laugier,and R.Chatila (eds): *Autonomous Navigation in Dynamic Environments*, Springer-Verlag, 2007.

[Mur12] S.Murata, and H.Kurokawa: *Self-Organizing Robots*, Springer-Verlag, 2012.

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[Oll07] A.Ollero, and I.Maza, Iván (eds): *Multiple Heterogeneous Unmanned Aerial Vehicles*, Springer-Verlag, 2007.

[Sat86] D.H.Sattinger, and O.L. Weaver: *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*, Springer-Verlag, 1986

5.4. Dynamics and control in Robotics

Two important issues which involve specially to dynamics are related with non-integrable systems and discontinuities linked to contact mechanics, including friction phenomena, impact and/or collisions. All these issues are crucial in multilegged robots which display additional troubles linked to coordination tasks in multibodies.

From a geometric viewpoint, mechanical systems may be holonomic or non-holonomic. Nontrivial holonomy represents the lack of conservation along a "transport" of geometric quantities given by some (scalar, vector, tensor) field along a path. In Differential Geometry parallel transport is performed in terms of a "connection" on a fiber bundle extending the (co)tangent bundle presented along the precedent module. A parallel transport does not preserves exactly the "shape", but it transforms geometric objects (tensors) according to certain rules which are compatible with (covariant) differentiation. Thus, in Mathematics one says that a connection is a rule of covariant differentiation.

A first taxonomy give us two types of affine and metric connections. Both of them are useful in Robotics because they involve to (real or apparent) deformations and accuracy in regard to the information fusion or the expenditure energy necessary to maintain motions to be performed. The main invariant of a connection is the curvature. Roughly speaking, the curvature represents the second order variation rate of "geometric quantities". The global representation of curvature maps provides a visualization of dynamical effects linked to internal constraints (propioceptive evaluation) or external restrictions (changing environmental conditions). The construction of curvature flows allows to manage the above quantities.

In more abstract terms, a *holonomic constraint* is a wholly integrable sub-bundle E of the tangent bundle τ_M of a manifold M. The system outcome for a non-holonomic system is path-dependent. Non-holonomic systems have been studied in robotics. Examples include: car-like robots, tractor-trailers, bicycles, roller-blades, airplanes, submarines, satellites, and spherical fingertips rolling on a manipulandum.

In robotics, a non-holonomic system is usually defined by a series of non-integrable constraints of the form $G_i(\mathbf{p}, \mathbf{v}) = 0$ on the tangent bundle. For example, whereas holonomic kinematics can be expressed in terms of algebraic equations which constrain the internal, rotational coordinates of a robot to the absolute position/orientation of the body of interest, non-holonomic kinematics are expressible with differential relationships only.

This distinction has important implications for the implementation of a control system.

It is well known that many under-actuated manipulation systems are naturally modeled as non-holonomic systems. An important control problem is the analysis of isotropic non-holonomic manipulation system in which multiple robots manipulate objects by using non-prehensile grasps, or by using prehensile grasps enabled by tools such as ropes. Bimanual operation poses additional problems from the dynamical viewpoint which are relative to coordination, compensation, control of velocities before and after contact, stabilization, anticipation of efforts and compensation of effects linked to manipulation.

Complex interactions with the environment require an adaptive behavior of mechanisms. To soften effects of contact and/or impact it is necessary to incorporate elastic models for multibodies. Elasticity is an old topic in Mechanics, but it is usually applied to materials. Usual mathematical tools are related to a formulation of small deformations in terms of Tensor Calculus, which very often impose additional mathematical constraints linked to the preservation of energy [99].

It is necessary to develop models for multibodies in a dissipative framework, having in account the propagation of impact effects, evaluation of friction effects, minimization of non-linear vibrations, stabilization of mechanisms along stable trajectories, and compensation mechanisms for the whole robot architecture.

5.4.1. A description of the module 4

The module B_{34} (Robot Dynamics) has the following chapters

1. **Dynamical aspects** The application of forces at joints and the generation of moments at torques, provide a collection of examples for planar and/or volumetric robots. Basic principles of mechanics are reformulated in terms of Lagrangian formulation and Eulerian approach . Several methods for resolution of structural equations are discussed. Finally, we display some applications to simple kinematic chains which operate in the cartesian space, with a special regard to 6R robots

2. **Optimization in Robotics** Along this chapter we develop an approach which follows an increasing order of difficulty. A short revision

[99]For a basic approach see the module 3 of my notes on Differential Geometry, e.g.

of linear methods is presented, including some elements of neural networks based approach. The core of this chapter is devoted to present and explain some of the most usual non-linear methods for optimization issues and their applications to anchored robots. Optimal Control Theory. Pontriajine Maximum Principle. General variational principles. Local and infinitesimal symmetries. Elements of sub and semi-analytic theory.

3. **Control and Stability** From a theoretical viewpoint, control issues can be considered as dual of optimization problems. A basic distinction involves to robust vs adaptive control systems; the choice depends on the characteristic of problems to be solved in regard with industrial vs more flexible environments. Besides general considerations involving linear and non-linear control, a special attention is paid to the applications of differential and integral methods involving PID (Proportional Inverse Derivative) methods and control systems based in Expert Systems. Main applications concern to force-position control; the incorporation of kinematic effects for more advanced control issues is performed from the module 2.

4. **Global Differential Geometry in Robotics**. Vector bundles. Basic notions. Integrability conditions. Non-holonomic constraints and control theory. Elements of Subriemannian Geometry. A geometric reformulation of Optimal Control. Stabilization from Energy-Moment map.

5. **Geometric Algebra for control issues** An excursion through Algebraic Geometry. Extending matrix calculus. Controllability. Accessibility.

6. **Elastic multibodies**. Contact. Impact. Anticipation and Compensation models. Elastic models on multibodies. Topological and different stability.

7. **Control of Vibrations**. Controlled submersions. Infinitesimal stability. Controlled Lagrangian Methods. Feedback matching. Stabilizing Non-holonomic systems.

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5.5. Humanoid Robots for disabled persons

Following Wikipedia, “a *humanoid robot* is a robot with its body shape built to resemble that of the human body. A humanoid design might be for functional purposes, such as interacting with human tools and environments, for experimental purposes, such as the study of bipedal locomotion, or for other purposes. In general, humanoid robots have a torso, a head, two arms, and two legs, though some forms of humanoid robots may model only part of the body, for example, from the waist up. Some humanoid robots also have heads designed to replicate human facial features such as eyes and mouths. Androids are humanoid robots built to aesthetically resemble humans.”

Along the fifth module, each robot is considered ideally as a humanoid given by a four kinematic chains and a head which are connected to a fixed body which is called as the trunk. From the beginning there is a human inspiration which is modeled from human biomechanics with its traditional hierarchy in skeletal, muscular and nervous systems (chapter 1). In practice and to ease a real-time robust behavior, it is necessary to adopt some simplifications (Chapter 2). An efficient design is crucial to improve functionalities linked to complex tasks such as locomotivity for human gait (chapter 3) and dexterity/manipulability for grasping (chapter 4).

The above general principles are applied to robotics of assistance to disabled or elderly people, where some advances in Prosthetics and Orthopedical Chirurgy are reported to ease their lifestyle. Finally, some of the most interesting research domains concerns to human-machine communications, from RT-interaction based on human gestures and speech recognition; these topics are related with information processing arising from different sensors and their integration to provide a RT response is a permanent challenge from the late nineties (Chapter 6).

Other important topics related to biomedical applications which are not considered here concern to the simulation of surgery interventions. In this case, it would be necessary to develop additional models and software tools linked to internal organs (using deformations controlled by meaningful parameters), the behavior of soft tissues and their response w.r.t. interventions. The simulation involves to advanced mathematical models for the most used software tools for surgery, elasticity and plasticity for internal organs, and which are not considered here.

Models to be developed must include Advanced Visualization tools for modeling internal organs, eversion procedures for interventions, interactive navigation tools inside the body, dynamical systems for elastic and resilient deformable objets, and real-time incorporation (in terms of VR/AR tools) of information arising from microcameras linked to very sensitive tasks [100]

5.5.1. A description of the module 5

The module B_{35} (Humanoid Robots) contains the following chapters:

1. **Elements of Human Biomechanics**. Along this chapter we recall some of the most meaningful facts about the three (skeletal, muscular and nervous) systems in the human body for the support, execution and control of tasks. We develop morphological and functional approaches, and we sketch some basic feedback mechanisms for understanding right behavior under normal conditions to obtain typical locomotion patterns. (De)coupling between components for each subsystem is essential to provide a distributed management which eases the validation of partial models and their effects on the whole human architecture.

2. **Simplified models of muscle-skeletal system**. The complexity of most tasks performed by human body and their imitation by artificial mechanisms requires some simplifications to morphological and functional levels. These simplifications are performed by introducing a hierarchy which involves initially to morphological aspects, and which incorporates in a more advanced stage functional aspects for control and optimization issues.

3. **Grasping and Handling**. Efficiency in these tasks require a right design going from the whole arm, till the fingers and their articulation in an artificial prosthetics hand. To start with, we shall consider three-fingered artificial hands with the thumb in an opponent location w.r.t. the other fingers. Main problems to be solved concern to position-force based control able of warranting dexterity and manipulability of objects arising from a visual feedback. The final section considers some problems related to the coordination of both hands for manipulating volumetric objects. Stability in manipulating objects is a very difficult problem, in part by the modification of gravitational effects linked to dynamical effects arising from robots; in addition, objects to be manipulated can display a passive or, contrarily, an active behavior, including energy exchange which must be modeled (in terms

[100]An introduction can be read in the module 5 of my notes on Computer Graphics

of Lyapunov models, e.g.) and absorbed/stored in components of the robot.

4. **Human Locomotion** is one of the most difficult tasks for human modeling. Even if we restrict ourselves to human gait (by forgetting jumping and different types of courses), there is no a universal model for locomotion. Thus, we adopt some simplifications which are initially centered in the coupling of lower limbs along the different phases of human gait. Our purpose is to try of providing a basic structural model which can be adapted to different subjects depending on morphological parameters. The most difficult issues concern to stability for the whole structure and kinematic control. Stability is performed around expected trajectories (never exactly performed in the same way). Adaptive control to changing environments is performed in terms of kinematics impedance having in account a discontinuous contact of limbs with the environments, which generates an alternance between open and closed loops. In both cases, it is necessary to perform a coupling between anticipatory and compensatory movements performed by the whole body.

5. **Robotics of assistance to disabled persons**. Some of the most outstanding applications for Robotics is the assistance to disabled people, such as paraplegic or tetraplegic persons. From the early nineties several mechatronic devices have been developed for assistance which are based on Functional Electro-Stimulation (FES), prosthetics (reciprocators, e.g.), artificial muscles (pMA, e.g.) or other hybrid systems. More recently, some of these elements are being integrated in components which are coupled with human bodies for traumatic amputations. Exo-skeletal structures have been developed from the early nineties and make part of chapter core. Recent advances in mechatronic miniaturization, performance of new materials and advances in the integration of bionic components open the door for excitant progress which try of improving the quality of life for disabled persons. [101]

6. **Simulating virtual characters** Recognition of gestures involving the human body and human face provide some keys to improve the interactions between different (human or artificial) agents. A basic distinction in human activity analysis concerns to human gestures,

[101] The url http://www.vph-institute.org/ provides a permanent update of related materials and connections with RTD and clinical groups working in this area.

actions, interactions and group activities. In this chapter we develop an approach which is mainly based in video processing for human gestures and individual actions. Main goals are the recognition, synthesis and simulation to improve human-machine interaction, to simulate optimal configurations and postures (ergonomics), and to provide a support for the production of multimedia contents. [102]

7. **Sportive performance**. Sportive performance is a challenge which requires the information fusion arising from different sensors linked to actuators. The main goal of this chapter is to provide an assistance for Medical Rehabilitation. Biomechanical and physiological models provide a support for advanced visualization [103] of more advanced functional models (cardiovascular, pulmonary). Current research involving integration in a whole model is incorporated to progressively complex Virtual Physiological Humans from biomodels (including (biomedical) databases. Paralimpic activities for amputees is an extreme case to evaluate the sportive performance of advanced prosthetical devices in order to try of improving the quality of life of citizens.

5.5.2. Some references for the module 5

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5.6. Animats. Simulation and animation

In despite of the progress performed in Robotics from the eighties, even the most advanced robots are still very far from achieving the efficiency of superior mammals. Even, the performance of insects is often much higher than the achieved by robots, currently. Several millions of evolution years have produced architectures which are very specialized for complex tasks, which we are still far from imitating in an efficient way. Progress in aerial and marine navigation are still very far from being satisfactory from the viewpoint of robotic prototypes.

In view of the unsuccessful state, there are several options which are centered in quite different approaches related to some basic functionalities, the development of evolutionary strategies for robots at algorithmic level, a better biomechanical understanding of connections between different components, simulation of movements and/or the modeling of Central Nervous Systems. In the same way as the precedent module, the main inspiration arises from Biomechanical Analysis, but in this case applied to other live beings different from humans.

One of the most relevant applications for Biomechanics concern to performance in complex tasks such as locomotion and grasping-handling. The evolution of several millions of years has provided very efficient architectures which are very well adapted to different environmental conditions and to different tasks related to predator-prey behavior. The adaptation is performed at different scales going from the simplest ones (corresponding to insects) to the most advanced (terrestrial, marine, aerial) animals. Seemingly, there is no a systematic study of all these aspects, and because of this the notes presented here have a very fragmentary character with different degrees of development.

Our main motivations arise from two quite different topics which are related to Sportive Medicine (including an analysis of some of the most common lesions) and the analysis, understanding and simulation of basic principles for aerial, terrestrial and marine movements of live beings.

Along the sixth module, each robot is considered ideally as a body given by several kinematic chains which are connected to an articulated body. All the models are biologically inspired in insects or more complex mammals, with a high number of d.o.f. The very large diversity of natural mechanisms makes very difficult a unified approach to animats. Thus, our approach is necessarily very schematic and it is more focused to suggest possible research lines and applications rather than detailed solutions. Connections with the industry of multimedia contents give a more speculative character to most applications developed in this module.

A typical approach consists of combining structural models (arising from biomechanical studies, as for humanoids) with appearance-based models (arising from the analysis of video sequences which are edited and manipulated in a manual or semi-automatic way. These applications are strongly related to advanced contents of Computer Vision which are developed with more detail in the CEViC [104]

From a theoretical viewpoint the introduction of different kinds of symmetries and the use of Lagrange multipliers allows to reduce the initial complexity of the problem. Symmetries allow to decompose complex systems to be solved by means the introduction of natural hierarchies associated to inclusion relations (from the kinematic viewpoint) or breaking symmetries (from the dynamical viewpoint). Symmetries are applied not only to working and configurations spaces, but to the tasks to be performed, including kinematic and dynamical aspects. Control and optimization issues are developed in terms of (de-)coupling and (ir)redundant mechanisms.

5.6.1. A description of the module 6

The module B_{36} (Animats) contains the following chapters:

1. **A stratified approach** Along this chapter we adapt the hierarchical approach for mechanics (Geometry, Kinematics, Dynamics) associated to the successive prolongations (in jets spaces) of the transference map $\tau : \mathcal{C} \to \mathcal{W}$ between configurations and working spaces for like-insect and like-mammal robots.

2. **Motion equations** Contact structure of the jets space provides structural constraints for motion equations which are exploited to obtain first integrals of motions equations. Along this chapter we develop the two classical formulations based in Lagrangian and Newton-Euler formalism. Finally, we give some examples of motion equations for several paradigms involving efficient animats at different levels corresponding to insects (the incredible fast course of a cockroach, e.g.), mammals (the fall of a cat as the best solution to recover equilibrium, e.g.) or birds (variable geometry of a falcon for efficient hunt, e.g.)

[104] Curso de Especialista en Vision por Computador, an on-line course which is coordinated by the author

3. **The role of symmetries**. In mathematical formulations of Robotics, symmetries are ubiquitous from the geometric design (opponent character between components), to the reformulation of kinematics in terms of Lie groups and its infinitesimal version (in terms of Lie algebras), and the resolution of motion equations in the Lagrangian formulation by using first integrals linked to conservation of "quantities". Furthermore, we show how singularities can be used to pass through singularities, by avoiding indeterminacy issues linked to the lack of "enough independent sections" of fiber bundles where control is developed.

4. **Simulation and Visualization**. Several millions years of evolution have provided a lot of architectures which we are still far of understanding. It is necessary to improve our knowledge of these architectures and their functionalities to try of understanding and imitating them. In this chapter we provide modeling and simulation tools for increasingly complex animats inspired by some insects and mammals. To accomplish these goals we introduce several hierarchies which are inspired by biomechanical principles. After understanding them, the main goal is to try of simulating and visualize results. To achieve this goal we adopt a hybrid approach which combines biomechanical structural principles and appearances-based modeling (Computer Vision tools).

5. **Architectures like-insect**. Most robots are still far of achieving the performance of seemingly simple insects. With very low energy consumption, some of them are able of transporting weights (ants, e.g.), develop very fast displacements (cockroaches, e.g.) or develop abilities for flight to improve the performance of current UAV in cooperative tasks (such those appearing in bees flight, e.g.). Thus, the study of architecture and functionalities of insects provides a permanent source of inspiration to improve the current robots. The analysis performed in this chapter is completed with some examples focused towards simulation and visualization of other insects (butterfly, dragonfly, etc) which are useful for multimedia productions.

6. **Architectures like-mammals**. The most known architectures involve to domestic terrestrial mammals. Thus, we concentrate our attention on several cases relative to horses, dogs and cats. The multilayered structural approach must include layers corresponding to skeletal, muscular, and internal organs. The appearance-based model has two components which are labeled as video cartoonization (as a part of

video processing and analysis) and video-based rendering (as an advanced development of video edition). [105]

7. **Artificial modeling of ecosystems**. An initial motivation is given by two typical strategies given by cooperative and competitive systems. Both of them can be considered as particular cases of Synergetics applied to Robotics. From a more applied viewpoint, soccer-matches provide some very interesting models which combine both strategies on the same space. Their management requires to combine supervised and unsupervised strategies. We illustrate some basic principles with robotic behaviors for multiagents in speculative financial markets which drive the self-evolution of smart systems towards the self-destruction.

Some references for the module 6 Only textbooks are included. I apologize by meaningful omissions.

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