A450 Stratifications. An introduction

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Índice

Previous remarks: These notes correspond to a short version of the Introduction corresponding to the module A_{45} (Stratifications) of the matter A_4 (Differential Topology). It is necessary to have previous knowledge of Basic Differential Topology A_{41} and some familiarity with Singularities of map-germs A_{44}. It is advisable to have some knowledge of GAGA A_{33} to understand examples and applications to Geometry.

For applications to scientific and technological issues (developed in the two last chapters) it is advisable to have some basic knowledge at the level of materials included in the matter A_1 (Differential Geometry). Additional issues concerning to Artificial Intelligence are included to a descriptive level; a more formal treatment will be developed in several modules of the part II.

As usual, materials are organized in four sections to be given along one month (one per week). They contain a list of exercises for self-verification of understanding of materials. Subsections or paragraphs marked with an asterisk ($*$) have a higher difficulty and can be skipped in a first lecture.

0.1. Introduction to the chapter A440

The notion of stratification is ubiquitous in Science and Engineering. Initially, a stratification is a decomposition of an object or the ambient space X in a disjoint union of “subsets” (topological subspaces X_i) displaying a “similar behaviour” w.r.t. some mathematical property. When such “subsets” (or their adherences) fulfill “good incidence conditions” between them, one labels them as “strate” (see below for more details).

Preliminary or naive versions of stratifications are implicit in regard to the introduction of *hierarchies* use for classification issues. These hierarchies can involve to morphological or functional properties, i.e. relative “shape” or the “behaviour”. Some antecedents for *morphological aspects* are linked to (ascendent vs descendent) nested collections of

- algebraic structures such as groups, vector spaces, rings, modules, fields extensions, finitely generated k-algebras e.g.;
- topological structures involving nested subsets, correspondences, embedded probability distributions, e.g.
- symbolic representations for discrete structures involving graphs, lattices, networks, e.g.

Similarly, *functional aspects* are linked to functions with the basic set-hierarchy corresponding to the strict inclusions $k[\underline{x}] \subset C^{\omega}(n,1) \subset C^{\infty}(n,1) \subset C^{0}(n,1)$, and their adaptation to their corresponding piecewise versions. These inclusiones are naturally extended to C^r-maps and morphisms (not everywhere well-defined) between suoperimposed structures involving to

- Homomorphisms between algebraic structures (groups, vector spaces, rings, modules, fiellds extensions, finitely generated k-algebras e.g.) and their factorizations including ;
- Operators and C^r-maps between topological structures involving nested base spaces B or regular morphisms between superimposed structures (bundles, fibrations, sheaves, e.g.)
- Functionals defined on symbolic representations for discrete structures (graphs, lattices, networks, e.g.) for Optimization and Control issues, e.g.

The introduction of graded complexes performed in the module A_{22} (Homology and Cohomology Theories), and its extension to Cell Complexes A_{23} provides a “natural” duality between morphological and functional aspects. Relations between components of different “degree” or (co)dimension are described in terms of generalized boundary ∂ and coboundary δ operators. They can be applied to superimposed structures on the base space B or the total space E, and similarly for linear operators defined on the corresponding graded structures.

The most structured case corresponds to “nested collections” of vector subspaces, which are called “flags”. In the finite dimensional case, for each partition $(r_1, \ldots, r_k)$ of $N = dim(V)$ one obtains flags of vector subspaces of “naitonality” $(r_1, \ldots, r_k)$. They are the elements of a G-homogeneous space $G(N)/[G(r_1) \times \ldots \times G(r_k)]$, where $G(h)$ is any Classical subgroup of $GL(N; \mathbb{K})$, e.g. This approach is extended to flags of modules, and to superimposed structures such as Flag Bundles in a similar way as for Grassmann Modules and Grassmann Bundles.

Similarity criteria involving the base space B or the superimposed structures E can involve to any kind of mechanical hieraarchy in terms of geometric, kinematic or dynamic criteria in any of usual C^r-categories. In particular, the *Ehresmann's theorem* says that a proper submersion between smooth manifolds is a locally trivial smooth fibration (extending the notion of vector bundle). Submersions are key to reeduce dimension, and allow the construction of pseudo-inverses.

(*) The initial notion of stratification is applied to the relative case of pairs (A, X) with $A \subset X$. This extension is formulated in terms of stratified maps (applying strata in unions of strata), and the corresponding maps $f_{ij} : S_i(X, A) \to S_j(Y, B)$. Here, the decomposition involves to strata in source and target spaces of C^r-maps $f : (X, A) \to (Y, B)$, and it is naturally extended to stratified morphisms $\varphi : E- \to F$ between total spaces of superimposed structures corresponding to “embedded systems”.

In addition of their applications to Natural and Health Sciences, stratifications are ubiquitous in Engineering, also. They appear with different names in Materials Science, Hydrodynamics, Meteorology, Geology, Social Sciences, Biomedicine, Physics and Mathematics. To fix ideas, along the first three sections of this chapter, we will restrict ourselves to the mathematical framework, where we sketch relations between different notions of stratifications. The fourth section is devoted to stratifications in Engineering.

Along the part II of these notes, Artificial Intelligence (AI) plays a central role to unify different aspects of Computational Mechanics B_1, Computer Vision B_2, Robotics B_3 and Computer Graphics B_4. A nexus with Stratifications can be illustrated by Self-Organizing Maps (SOM), displaying a strong parallelism with different aspects of Differential Topology [Koh97][1].

(*) A general strategy in SOM consists of “Learning Subspaces” which can be considered aas an Optimization problem in a Grassman Manifold. This technique is extended to Adaptive Subspace SOM (labelled as ASSOM). The aadaptation can be understoodd as a “projection” (in fact a submersion) on the tangent space to an “ideal” PS-manifold representing objects or tasks to be learned. The existence of universal properties for the Gauss mapping provides a theoreetical foundation to understand the “universal character” of SOM in AI at the end of nineties.

However, hierarchies are not unique. Often, they depend on different kinds

[1] T.Kohonen: *Self-Organizing maps* (2nd ed), Springer, 1997.

of functionals to learn more complex objects or behaviours. This simple remark suggests the development of Learning Flags of subspaces; their space-time evolution, the need of Flag Bundles. S.Watanabe (2007) proved that spaces for Learning Manifolds are singular varieties. Instead of desingularizing them, we develop a strategy based on stratifications to propagate and complete the informations. They allow an interpretaton and management of possible degenerations to pass "through them" by introducing "control conditions" on adherence of strata.

The next paragraphs are devoted to introduce some basic notions in an intuitive way as possible, by following an increasing complexity. The coarsest approach corresponds to the notion of "topological stratification" extending the notion of C^r-fibration (Ehresman, 1946). Pieces are given by topological subspaces X_i making part of a partition of X which are called "strata". From a local viewpoint, they fulfill "good incidence conditions".

There is no a unique criterium for such partition: they depend on the category (for the absolute case) and the choice of a map $f : X \to Y$ (for the relative case). So, for the C^0-topology, incidence conditions involve to the adherence $\overline{X_i} \subset X_j$ of adjacent strata; in the PL- or the PS-category they involve to non-empty intersections between linear subspaces or between Ps-manifoilds. Their linearization gives again flag bundles.

The specification of "good matching conditions" depends on the C^r-framework and applications to be performed, and involve to "basic objects" X, Y and C^r-maps $f : X \to Y$ between them. So, they can be formulated in quite different frameworks, from set-theoretical ones till the most sophisticated interactions in dynamical environments. The last ones give "dynamical stratifications" on a generalized "tangent bundle" τ_P of the Phase space $P = TM$. Their local version is locally expressed in terms of matrix equations for dynamics, e.g..

To avoid pathologies of geometric-based approaches based on C^r-equivalences on intersections $U_i \cap U_j$ of domains for coordinate charts, one uses "attaching maps". In the last case, "good conditions" are formulated in terms of regular maps (submersions and immersions in the smooth case) for matching together eventually singular "objects". Their extension to non-regular maps to the Phase space P requires to identify generic singularities of Lagrangian and Legendrian manifolds [2]

The approach performed in terms of "attaching maps" is more flexible than usual geometric approaches, it does not require birregularity conditions nor a "complete" previous information about objects or processes to be identified. Furthermore, it can be adapted to discrete or statistical frameworks. In the last case, one has a Riemannian structure for parametric distributions w.r.t. the Cramer-Fisher-Rao metric on multivariate normal distributions, in the Geometric Information Theory (GIT) context [Ama16]. Constraints on this space give eventually singular subvarieties to be stratified [3]

[2] See the chapter 4 of the precedent module A_{44} (Singular map-germs).

[3] The adaptaation of GITto Topological, Kienmatic and Dynamical contexts is developed

Applications to Engineering start with discrete data which provide the support for superimposed structures by following different hierarchies. Clustering strategies of sparse data can be described in terms of "weak stratifications" giving initially an infinite number of strata for each dimension. Furthermore, the topological space X can have infinite dimension, as it occurs with spaces of maps or more general fields appearing in Geometry and Analysis. Dimensionality reduction is a first task to be accomplished.ç

Most effective methods for stratifications are not valid in purely topological structures, nor even on PL-superimposed structures on discrete spaces, e.g. This justifies our choice which is initially based on the PS (Piecewise Smooth) framework, and it is progressively enlarged to other more realistic, flexible and/or computable frameworks.

In the PS-framework, the first central result for stratifications is the Ehresmann's theorem giving a C^r-structure for proper submersions. Some weakening of these hypothesis in regard with applications to other scientific and technological areas are developed in the last two chapters of this module. The PS-framework can be understood as a "regularizatiion" (from "brute force" to Tikhonov mthods, e.g.) of PL-structures. Polynomials provide a more computationally manageable approach.

Furthermore Ehresmann's theorem, from a historical viewpoint the most important initial contributions in the PS-framework are due to H.Whitney and R.Thom along 1940s and 1950s. Algebraic an topological extensions were developed along 1960s and 1970s (Mather, Lojasiewicz, Hironaka, Teissier, Verdier), including explicit relations between different characteristics from the middle of 1970s (Teissier and Le).

(*) A topological reformulation in terms of Stratified Morse Theory (SMT) was performed in the 1980 (Goresky and MacPherson), where one makes a systematic use of attaching procedures mentioned above. An advantage of SMT consists of the incorporation of singurlaities in terms of Tangential and Normal Morse Data (TMD vs NMD), which are the key to relate propagatin models along the smooth part, with different possible behaviours at singularities in terms of local products.

From a topological viewpoint, the classical approach consists of reducing the study of global aspects to a local analysis by using a trivializing covering for some C^r-structure. The open sets $U_i \in \mathcal{U}$ are equipped with a C^r-equivalence $\phi_i : U_i \to \mathbb{R}^n$ for some fixed $n \in \mathbb{N}$ fulfilling compatibility conditions on $U_i \cap U_j$ for any $i, j \in I$.

In stratified theories, the main innovation consists of replacing the atlas $\mathcal{A} = (\mathcal{U}, \Phi) = (U_i, phi_i)_{i \in I}$ by a locally finite set of attaching maps which can match together eventually singular "objects" and maps" of different dimensions. In this way, semi-analytic and semi-algebraic subsets are incorporated from the scratch as "natural objects", by avoiding their treatment as "exceptional" cases of classical approaches.

in the corresponding modules of B_1 (computational Mechanics of Continuous Media).

0.1.1. Different frameworks for stratifications

Relations between C^r objects and maps in Mathematics and/or other Applied Sciences are a source of inspiration for notions and results relating the corresponding areas. The most common for the first half of this module are obtained for $r = \infty$ (smooth category) and $r = \omega$ (analytic category). They provide two general frameworks for usual top-down approaches. In both cases, stratifications are linked to local or formal properties of Differential Calculus.

Alternately, if we adopt a bottom-up methodology, irregularities in data distributions (sparse vs dense, e.g.), and the initial lack of structure, suggests a PL-approach (PL: Piecewise Linear) superimposed to clustered ata, as a first-order approach to PS-models. Polynomial approaches allow meaningful reductions of information to be managed, and introduce natural hierarchies in terms of graded complexes.

In presence of group actions, hierarchies between geometries and their structural groups provide the framework to relate stratifications in different Classical Geometries, their extensions to G-structures and their applications. An "absolute approach" concerns to the space X, whereas a "relative approach" involves to stratified maps $f : X \to Y$ between spaces. One says that f is a stratified maps if the image of each stratum in X is a finite union of strata in Y.

Fields-based approach are key to "complete information" by using propagation models. A non-trivial problem consists of extending Different kinds of (scalar, co-vector or, more generally, tensor) fields defined on strata, to their adherence. Remark that their adherence can display differenc "branches" for the support, and even discontinuities for higher derivatives.

The existene of discontinuities for scalar functions motivates the use of stepped functions(commonly used in Statistics). Similarly, one must introduce vector fields with corners (dgenerating in PL-vectors), and the corresponding PL-versions for covectors. The notion of vector fields with corners is due to R.Thom (1969), who introduces the notion of controlled submersion for the management of possible discontinuities for derivatives [4]

The introduction of analytic ringed spaces $\mathcal{X} = (X, \mathcal{O}_X)$ and morphisms $\mathcal{X} - \to \mathcal{Y}$ between them allows a simultaneous management of morphological and functional properties in terms of the support X, and the behaviour as $\mathcal{O}_X$-modules. In some advanced applications, behaviours in the relative case are formally described in terms of maps between tensor algebras or k-jets spaces and bundles.

This approach is compatible with the usual C^r-categories for $r = alg$, $r = \omega$ and $r = \infty$. Piecewise Linear (PL) approaches provide the nexus between discrete and continuous models. In both (absolute and relative) cases, one can follow two alternative and complementary strategies:

- If we follow a *top-down methodology* one can discretize local coordinate

[4] Their exteension to the disrete case is performed in the module B_{13} (computational Differential Topology).

systems and/or local fields defined on them, to understand internal relations or complex interactions between hierarchical structures.

- If we follow a *bottom-up methodology* one can develop clustering methods at different levels going from unstructured data to hierarchical models.

The feedback between top-down and bottom-up approaches for spaces and maps between spaces is the key for some recent applications to IST (Information Society Technologies) which are developed in Chapter 9 of this module A_{45}.

On the other hand, in (Differential, Algebraic, Analytic) Geometric frameworks a basic strategy consists of developing local procedures which are based in matching fibred structures (vector bundles, principal, bundles, sheaves, e.g.) by using local triviality conditions on the overlapping $U_i \cap U_j$ of open sets of a covering $\mathcal{U}$ of the base space X. This strategy is the key for the common support provided by (Commutative and Homological) Algebra. In Stratified spaces one replaces compatibility conditions on intersections by "attaching maps".

The resulting topology based on attaching maps is more adaptive, and it can be applied to semi-algebraic and semi-analytic spaces (with the corresponding superimposed structures). Furthermore, it does not need a previous knowledge of the "ambient space" as a whole, which makes possible a better behavior to conditions in absence of complete information or in presence of uncertainty (statistical manifolds, e.g.). These ideas can be traced to *Esquisses* of A.Grothendieck, where he introduces the notion of tame topologies (moderate behavior) and "devissage"(unscrewing) for the management of stratified structures.

0.1.2. The interplay between smooth and analytical approaches

As always, the simplest situation corresponds to the smooth cases, i.e. for $r = \infty$ which is well understood from Differential Geometry A_1. Algebraic Topology (first three modules of A_2) provides a PL-approach, which can be adapted to geometric objects and morphisms in the Geometric Topology framework (last three modules of A_2). If we look at polynomial approaches (instead of PL ones), one must use arguments of Algebraic Geometry A_3. The incorporation of advanced PS-techniques is performed in terms of k-jets spaces and the corresponding maps in A_4 (Differential Topology). Local Algebra provides a common language for a lot of issues involving all of them. But, it has a strictly local character.

To recover global aspects one must reintroduce global topological methods. In a lot of applications one has not an idea of global characteristics of ambient spaces and their transformations. Thus, they must be constructed on the way in terms of attaching maps, able of incorporating "events" as singularities of some kind of (scalar, vector, covector) fields involving properties, trajectories and constraints to understand complex behaviors. This program requires a reformulation of classical results involving even to the initial Ehresmann's fibration theorem, continuing with Whitney conditions for "good stratifications" till arriving to properties or maps between superalgebras which provide an integration of all above issues.

To start with, let us remember that the differential df of a map $f : N \to P$ between two smooth manifolds gives the linearization $d_x f$ of f at each point $x \in N$ which is locally represented by the Jacobian matrix after fixing coordinate systems. The linearization allows to analyze the singular locus of the local representation $\mathbb{K}^n \to \mathbb{K}^p$.

The classification of singularities of map germs $f \in C^r(n, p)$ has been performed according to different $\mathcal{B}$-equivalence relations along the module A_{44} for $\mathbb{K}$ the real or the complex field. Unfortunately, in most cases relations between $\mathcal{B}$-orbits have not a "nice" (locally trivial) structure.

To be more precise, and according to the Ehresmann's fibration theorem if a proper smooth surjective submersion $f : N \to P$ between smooth manifolds is a locally trivial fibration. However, if the map is not proper or a submersion, then the Ehresman's theorem is no longer true. A first strategy consists of decomposing the source space in strata (by using the geometry of the Discriminant Locus) such that the restriction to each stratum fulfils the hypotheses.

Unfortunately, superimposed (tangential and normal) structures on adjacent strata do not match between them necessarily in the singular case. A classical solution consists of introducing the Whitney's conditions relative to the limits of tangent spaces, which are "controlled" by secant spaces; alternately, one can use "controlled submersions" [Tho69]. More modern approaches uses Stratified Morse Theory [Gor87].

The two main approaches to stratifications are due to R.Thom and H.Whitney. They are related with the precedent smooth and analytical approaches:

- The *Thom's approach* can be considered as an extension of the rank stratification for spaces of matrices (representing the Jacobian matrix, e.g.). The deep relations with cellular decompositions of the Grassmannian $Grass(k, N)$ of k-dimensional linear subspaces L^k of a N-dimensional space V^N appear already in [Tho69]. The notion of controlled submersion (very useful for a lot of applications in Engineering) is crucial for a local maintenance of "good conditions" for fibrations (locally trivial structures) and appears in [Tho69]. Cell complexes provide a large amount of "examples" where the Thom's approach is applied successfully.

- The *Whitney's approach* uses geometric conditions about limits of tangent lines $\ell = t_x X$ and tangent vector spaces $T_x X$ at regular points $x \in X^0 = X \backslash Sing(X)$, as control conditions to avoid "bad behavior" at singularities in order to recover some kind of fibred local structure. These conditions could be inspired in usual Lipschitz conditions for ODEs, and are labeled as Whitney's conditions. Quasi-projective varieties provide a large amount of "examples" where the Whitney's approach is applied successfully.

From the late 1960s there appear different extensions of both approaches by following topological and algebraic approaches. Limitations linked to the study of semi-analytic (Lojasiewicz)and semialgebraic varities (Hironaka) motivate some first extensions where there appears the need of reformulating topological and geometric aspects.

Some more sophisticated motivations arise from limitations for the applicability of the above approaches to more general topological spaces (CW-complexes, lens spaces, e.g.) or to Deligne-Mumford compactifications in moduli spaces. The development of Polar Varieties and the relations with the Discriminant Loci of (families of) maps is developed by B.Teissier and Le-Dung Trang along the 1970s and early 1980s.

All of them, suggest the development of "controlling the behavior at boundary" or "attaching maps" instead of using local coordinate charts to integrate features which were considered as "pathologies" in precedent approaches, specially in regard to the possibility of singular attachments. The need of controlling the behavior at these situations motivates the introduction of "tame" topologies by Grothendieck [5].

The attachment of "singular cells" occupies a central role in the Stratified Morse Theory, where the attachment is performed in terms of Tangential and Normal Morse Data performed by Goresky and Macpherson along the 1980s, which are integrated in the Homology Intersection Theory. The more recent introduction of Persistent Homology can be considered as an extension of this approach to the Discrete Topology, with the corresponding applications to Engineering (see the last section of this introductory chapter for more details).

0.1.3. From Geometric to Dynamic stratifications

As always, a first distinction involves to "absolute vs relative" approaches involving the C^r-spaces X vs the C^r-maps $f : X \to Y$ between spaces. The simultaneous consideration between both of them is formulated in terms of maps $\mathcal{X} = (X, \mathcal{O}_X) \to \mathcal{Y} = (Y, \mathcal{O}_Y)$ between analytic spaces, where $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) denotes the "sheaf" of regular functions on X (resp. Y). Formal properties and some applications to GAGA have been developed along the module A_{33} (Sheaves, Cohomology, Schemes) of the matter A_3 (Algebraic Geometry). This approach has an essentially static character.

In presence of variations for evolving objects X_t and evolving morphism $f_t : X_t \to Y_t$, it is necessary to enlarge the initial framework. Furthermore, these variations can involve different underlying topologies including possible changes of state or phase transitions linked to singular behavior of deformations. A lot of "examples" have been described along the modules A_{43} (Singular functions) and A_{44} (Singular maps). Now, we are incorporating possible "degenerations" or "regenerations" of the whole superimposed C^r-structure, whose simplest types (vector bundles vs principal bundles) have been described in the module A_{42}.

[5] Some descriptions appear in his "Esquisses", where they are called "moderate" topologies

Furthermore, one must introduce elements able of "controlling" possible evolutions of living systems. Along the part II of these notes there appear a lot of illustrations involving different regimes (laminar, turbulent, chaotic) in Computational Mechanics of Continuous Media B_1, all kinds of events in video sequences in Computer Vision B_2, auotmatic navigation of autonomous vehicles in Robotics B_3, and interactions between avatars in Computer Graphics B_4, e.g. All of them can be considered as "examples" of dynamic stratifications.

To fix ideas, we borrow some terminology arising from Robotics which we adapt to more general frameworks:

- Let $\mathcal{P} = (P, \mathcal{O}_P)$ be the Perception space, where regular functions $s \in \mathcal{O}_P$ are the signals corresponding to sensors.The are packaged or decoupled in terms of ascending and descending sequences of ideals.
- Let $\mathcal{C} = (C, \mathcal{O}_C)$ be the Configurations space (involving selecteed clouds of data (initially points with attributes, e.g.), whose local regular functions $\mathcal{O}_C$ are given by any kind of scalar functions (including Dirac delta functions) on elements of C to improve the clustering;
- Let $\mathcal{W} = (W, \mathcal{O}_W)$ the Working space, where local regular functions $\mathcal{O}_W$ are given by the "joint contribution" of each clusters with similar characteristics; they provide the key to visualize the space-time evolution of the systems. corresponding to actuators (devices for controlling the system).
- Let $\mathcal{A} = (A, \mathcal{O}_A)$ be the Action space, whose regular functions $a \in \mathcal{O}_A$ are the actions (also called commands) corresponding to actuators (devices for controlling the system).

In the chapter 8, we construct the "knowledge map" $\kappa : \mathcal{P} \to \mathcal{A}$ which is locally an analytical fibration (extending the Ehresmann fibrations), giving the Perception-Action Cycle (PAC): $\mathcal{P} \to \mathcal{A} \to \mathcal{P} \ldots$. Basic models correspond to MIMO systems used in transmission/reception patterns linking input signals $\mathbf{x}$ and output signals $\mathbf{y}$, where the simplest relations are given by

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

being $\mathbf{H}$ the channel matrix and $\mathbf{n}$ the noise vector. In a first regular extension, all these vector data can correspond to points on a manifol, and the lienar map $\mathbf{H}$ is initially replaced by the differential of a regular map (a submersion or a immersion for forward and inverse patterns, e.g.). In stratified models points belong to stratified varieties, and the map is a stratified map, also.

The PAC morphism factors through a "mechanical transfer morphism" $\mu : \mathcal{C} \to \mathcal{W}$ linking the Configurations and Working space through a locally analytic fibration: the map μ provides the framework to understand any kind of motions for complex configurations of points [6].

[6] In Robotics, the configurations space $\mathcal{C}$ is labelled as the Joints or Articular space; we use the configurations terminology to avoid confusion with the use of J (Joints space, Jacobian, maps, structural tensor in symplectic geometry, jets spaces) and A (action space, articular space, right-left equivalence)

In our approach the knowledge $\kappa : \mathcal{P} \to \mathcal{A}$ giving the PAC, factors out through the mechanical transfer $\mu : \mathcal{C} \to \mathcal{W}$ giving a *basic PeCWA pipeline*:

$$P \to C \to W \to A \to P \to \dots$$

between the support of the corresponding analytic spaces $\mathcal{X} = (X, \mathcal{O}_X)$, where the last map is construcetd in terms of the composition of pseudo-inverses for the precedent locla submersions. This composition is naturally translated to maps between regular functions locally defined on them, i.e.

$$\mathcal{O}_P \leftarrow \mathcal{O}_C \leftarrow \mathcal{O}_W \leftarrow \mathcal{O}_A \leftarrow \mathcal{O}_P \leftarrow \dots$$

given by composition of maps which can support differential and integral operators, also. Modules of vector and covector fields on the local rings of regular functions are the key to evaluate how first-order variations can be extended to this framework with their corresponding stratifications (locally given by Jacobian maps). Direct and inverse images follow the usual formalism developed for sheaves in A_{33}.

Similar arguments are applied to local systems of equations E_X defined on the total space X of the PeCWA pipeline, which are initially modelled as $\mathcal{O}_X$.modules. Higher order variations are expressed as always in terms of their k-Jets bundles. In particular for direct images, we would have

$$J^k E_P \to J^k E_C \to J^k E_W \to J^k E_A \to J^k E_P \to \dots$$

Their evolution along time is evaluated in terms of multi-dimensional time-series, where extensions of ARMA (auto-Regressive Mean Average) methods play a central role. Their stratification in subset with "similar behaviour" for low-order variatoin rates is crucial for segmentation issues (decomposition of data in sub-populations with a "similar behaviour") corresponding to strata. After segmentation, one performs tracking for evolving data packages in Kinematics, and predicition for possible interactions in dynamics.

Optimization along m multi-paths are given by maps $[0,1]^m \to \mathcal{X}$, whereas constraints corresponding to ℓ functions (scalar fields) are given by maps $g_j : Y \to \mathbb{R}$ for $1 \leq j \leq \ell$ in the Basic Analytic Diagram; their k where X or Y are any of the above spaces P, A, C, W (written as PACW shortly). Functorial properties allow their extension to Kinematic and dynamic aspects by taking the 4K4-th extension of the PACW.

0.1.4. A G-equivariant approach

Propagation models on locally homogeneous or isotropic media provide a G-equivariant decomposition of the ambient G-spaces for source X and target spaces Y of any G-equivariant map $f : X \to Y$. The resulting decomposition of X and Y as union of G-orbits provides a decomposition for G-equivariant stratified maps $f : X \to Y$. Typical "examples" correspond to Symplectic and Contact structures in Mechanics.

The G-equivariant description for stratified maps eases the study of kinematic and dynamic aspects which are superimposed to f in geometric terms (tangent τ_M and cotangent bundle τ_M^* in the simplest cases) or in analytic terms (jets spaces for ODEs or variational problems , e.g.).

The extension to the evolving singularities is formulated in terms of monodromy groups exchanging solutions given as “confluent” branches at singularities. Monodromy groups have been used for Milnor's fibration in regard to function germs (hypersurfaces) in the chapter 3 of the module A_{43}. They have been extended to more general map germs given by local complete intersections (lci) in the chapter 4 of the module A_{44}. In both cases, one has easily computabloe relations between different kinds of invariants.

Unfortunately, most singularities are not lci, and one must use a more general description is given by Ramification and Discriminant Loci in the GAGA framework A_{33}. To simplify, in applications to Engineering, most expressions to relate multivariate data are locally given by matrix expressions. The Cohen-Macaulay character for the corresponding rings and modules is the key to preserve a “good behaviour” under maps (including change of base point) for incidence conditions between adjacent strata.

For more complex situations it is necessary to use algebraic properties linked to the finite-determinacy of singular map-germs. The equivalence between the existence of a unipotent action and the finite-determinacy for k-jets [Bru86] is the key to describe ‘the behaviour around “bad singularities” in terms of deformations (versal foldings).

This remark motivates the need of developing a stratification theory for finitely-determined map germs. This approach must take in account the geometry of the unipotent variety. Let us remember that unipotent actions (or their infinitesimal version given by nilpotent actions) play a fundamental role [Bru86]. To understand how it does works, it is convenient to remember some basic features of ventually degenerate linear maps, before extending them to “deformations” given by local diffeomorphisms or analytical transformations for each regular phase of motion.

Along the module A_{44} (Singularities of map-germs) we have developed a similar reasoning scheme by replacing the above actions by the action of diffeomorphisms groups acting on the source and target spaces of a differentiable map-germ $f : (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, \underline{0})$. To understand the relation between both approaches one must have in account the Lie group structure of the diffeomorphisms $Diff_0(\mathbb{R}^n)$ is the space of vector fields fixing the origin $\underline{0} \in \mathbb{R}^n$, whose linear part is given by $\mathfrak{g}\ell(n) := T_I GL(n)$,

(*) Thus, the ambient space for the linear part of the $\mathcal{A}$-orbit $\mathcal{A}f$ is given by the product $\mathfrak{g}\ell(n) \times \mathfrak{g}\ell(p)$ corresponding to endomorphisms of the source and target spaces for each C^r-map $X \to Y$. Endomorphisms have not necessarily maximal rank. By this reason, it is necessary to develop methods for simultaneous compactifications or completions. An illustrativee “example” for

3D reconstruction issues in Computer Vision is developed in [Fin20] [7], but the method is quite general

The above descriptions can be adapted to other Classical Groups, preserving some "geometric quantity" or even to more general G-structures. So, one obtains the corresponding G-eqjuivariant stratifications for the base space B or superimposed structures (such as principal bundles, e.g.). Some of the most important regular cases are the following ones:

- the Special Linear Group $SL(N;\mathbb{R})$ for the preservation of oriented volumes in Fluid Mechanics, e.g.:
- the Orthogonal group $O(N;\mathbb{R})$ to preserve orthogonality conditions between frames;
- the Special Orthogonal Group $SO(n;\mathbb{R})$ for metric preservation;
- the conformal group $C(n;\mathbb{R})$ for the angles preservation;
- the Symplectic group $Sp(2n;\mathbb{R})$ for the preservation of the motion's equations in the Phase space under ideal conditions;
- the Contact group $C(2n+1;\mathbb{R})$ for the preservation of contact conditions w.r..t a hypersuprface in the Phase space $P = TM$

In a conservative framework, one can impose the preservation of the above quantities for any of he $\mathcal{B}$-actions (corresponding to the smooth or the analytic cases) where $\mathcal{B}$ is equal to $\mathcal{R}$ (right action on te source space), $\mathcal{L}$ (left action on the target space), $\mathcal{A}$ (decoupled right-left action on source and target paces), or $\mathcal{K}$ (contact action on the graph Γ_f of a map as support for some kind of coupling). The differential of the action at the neutral element e corresponding to each geometrical restriction gives each one of the above classical groups on the corresponding ambient space for Kinematics.

The above approach for the smooth cases has been extended to infinitesimal $\mathcal{B}$-actions on spaces of map germs $f \in C^r(n,p)$ for $r = \infty$ (smooth case) and $r = \omega$ (analytic case). The most interesting cases for $\mathcal{B}$-action correspond to the right-left or $\mathcal{A} = \mathcal{R} \times \mathcal{L}$ and the contact or $\mathcal{K}$-action, and have been extensively studied in the module A_{44} (Map-germs). All of them can be considered as an extension of actions on the tangent space to the $\mathcal{B}$-orbit, but the action is defined now on jets spaces.

An important particular case corresponds to *simple singularities*, where the introduction of boundary conditions on A_k and D_k series gives the B_k and C_k series, with the corresponding "dictionary" for classical groups. Unfortunately, the resulting classification in terms of infinitesimal actions are not stratifications in the Thom's sense (there are infinitely may orbits), nor in the Whitney's

[7] J.Finat and F.J.Delgado-delHoyo: "Complete Endomorphisms", arXiv, February 2020

sense (homological equations for computing neighborhoods is much more general than Whitney's conditions). It is necessary to enlarge the usual notions of stratification to include them.

Coming back to stratification criteria for the support X (instead of looking at functions or map-germs), the notion of G-equivariant stratification can be extended to objects or maps in the semi-analytic and the semi-algebraic frameworks. It suffices to consider the corresponding transformations in a stratified framework, where local descriptions on intersections $U_i \cap U_j$ of coordinate domains are replaced by attached maps along eventually singular elements. In different applications we will consider an interplay between

- *Discrete groups* which are described in terms of representations of symmetric and alternating groups (appearing for polyhedral symmetries, e.g.) acting on configurations of isolated elements (points or segments, e.g.);
- *Continuous finite-dimensional groups* such as the Classical Groups "preserving quantities" (in fact tensors), to ease locally homogeneous or isotropic structures appearing in the Phase space, e.g.; simplest stratifications in the semi-algebraic cases appear associated to "complete objects" including envelopes by linear elements[8]. The most interesting cases for semi-algebraic stratifications are linked to the action of the Unipotent Group which are associated to finite-determinacy conditions for k-jets.
- *Topological actions* on spaces of map-germs with usual smooth vs analytical frameworks, which are crucial to understand changes of state, phase transitions and the corresponding dynamical models in a lot of applications to Physics, Mechanics, Biological Sciences, Chemical-Physics and Engineering. They are strongly related with symmetries of ODEs and PDEs (giving the local expressions for sections of jets spaces), which are developed in the next module A_{46} (Dynamical Systems).

Some contributions of our approach consist of developing some tools for relating the above types of actions, their applications to several aspects of Mechanics, and the possibility of incorporating "dissipative phenomena" in a natural way at different (discrete vs continuous) levels. The idea is very simple again: Use (upper vs lower) triangular matrices whose diagonal elements are null. They can be visualized in a very easy way, because they induce "collapsing phenomena" (corresponding to volume contractions, e.g.) which can be reinterpreted in terms of vanishing nested modules.

In our apporach, *dissipative patterns* are described in terms of nilpotent operators acting on the (co)tangent fields. This approach is compatible with singular map germs, because the finite-determinacy condition for finitely determined map-germs is characterized by by nilpotent operators on the tangent space $T_f(\mathcal{B}f)$ of the $\mathcal{B}$-orbit of $f \in C^r(n,p)$ for any of the usual $\mathcal{B}$-equivalence

[8] In the Part I of my thesis there appear explicit analytic, algebraic and symbolic descriptions for the variety of Complete Quadriccs, e.g.

relations, i.e. $\mathcal{B} = \mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$ or $\mathcal{K}$. One must have in account that the resulting decompositions in $\mathcal{B}$-orbit spaces are not stratifications in the Thom's or the Whitney's sense because the number of cells is not finite, and one has not a locally Cartesian structure to describe limits of tangent or secant spaces.

Propagation and more general diffusion-reaction models are reformulated on the basic PeCWA pipeline and their k-jets extensions. Regular cases for forward composition is read in terms of successive submersions $P \to C \to W \to A$ reducing the amount of available information. Their linearization can be represented in terms of large block triangular matrices which are generically regular (i.e. its differential has maximal rank at each regular point).

(*) The problem becomes more complex at mechanical singularities corresponding to changes of state at (X, x), phase transitions $\Theta_{X,x}$ or higher order singularities. In all these cases, the rank is not maximal, and one must introduce additional control actions to prevent possible degenerations. Some of the most complex cases appear in regard to locomotion tasks in Animats B_{36} involving to transitions for gait phases in multilegged robots (quadrupeds, mainly). An efficient modelling in terms of a stratified approach to locomotion dynamics is the main challenge to be solved.

(*) From the AI viewpoint, semi-automatic learning of complex motion tasks (such as those appearing in mammals locmotion) requires not only the learning of "optimal subspaces" for each space appearing in the PeCWA pipeline (by using ASSOM, e.g.), but the learning of relatins between them. The linearization of this problem is reformulated in terms of an optimal trajectory in the space of evolving flags. The corresponding stratified aproach is formulated as a problem of learning flags, where transitions at each mechanical level are represented by geometric, kinematic or dynamic singularities..

0.2. An outline of the chapter A440

Along this module A_{45}, one follows an increasingly complex description of stratifications which starts from initial Ehresman and Thom's approaches for PS-frameworks, and the Whitney's analytic tools. Limitations of both approaches in regard to finiteness conditions or the lack of structure for geometric approaches provide the motivation for different theoretical extensions.

In addition of an intensive use of generalized flag buindles and their functional extensions (in terms of filtered algebras), the most relevant contributions are linked to the applications to other scientific and technological areas. A novelty is linked to the introduction of nilpotent operators to incorporate dissipation effects involving higher codimension strata. Materials of this chapter are organized in the following sections:

1. *From topological to analytic stratifications* where we develop a "static approach" to Stratifications, by using only topological, geometric and analytic properties in a classical framework.

2. *From Geometry to Kinematics* where we suppose that data are evolving under some constraints. In the smooth case, tangent and normal bundles provide a support for Kinematics. In presence of singularities and incomplete information, one must enlarge the theoretical framework, and use "attaching maps" at kinematic level also.

3. *Dynamic stratifications* where interaction between multiagents provide a motivation for dynamic aspects of stratificaitons. One extends the Geometry of the Moment Map of Theoretical Physics to include bifurcations and degenerations.

4. Some *applications to Engineering* are developed in the fourth section, with a special regard to the subareeas developed in the part II of these notes: Computational Mechanics B_1, Computer Vision B_2, robotics B_3, and computer Graphics B_4.

5. In the last section *outline of the module* A_{45}, one displays some connections with other scientific and technological areas.

The application to other scientific areas was pioneered by R.Thom himself, but with restricted to the topology of unfoldings of simple singularities in terms of Catastrophe Theory [9]. Some applications to Civil and Mechanical Engineering were developed along 1970s and 1980s. Their dynamical interpretation in terms of wavefronts and their applications to Geometric Optics was developed by researchers of the Liverpool school (Bruce, Giblin) [10]. In the last chapters of the module A_{43} (Singular function germs), we have introduced several applications of Catastrophe Theory to Engineering, Economic Theory and Boimedical Sciences in terms of singular function germs $f \in C^r(n, 1)$.

Thus, we focus towards some applications of map-germs for $f \in C^r(n, p)$, which can be illustrated as an extension of Multi-Input Multi-Output (MIMO) systems in Control Theory, e.g. We paid a special attention to Engineering areas appearing in the part II of these notes such as Computational Mechanics of Continuous Media B_1, Computer Vision B_2, Robotics B_3 and Computer Graphics B_4, which are commented in the last section of this chapter. Most mathematicians ignore basic aspects for these areas and because of this, one gives only an eye's bird view of these applications. A more detailed presentation is developed along the corresponding modules of the part II.

0.2.1. Some methodological issues

Most notions and results of this module are developed by following a top-down approach, i.e. they are based on previosuly known topological and geometric models, with their corresponding extensions to analytic models and their

[9] See the module A_{43} (Singular function germs) for details

[10] Dynamical aspects for wavefronts in stratified media are developed in the next module A_{46} (Dynamical Systems).

algebraic reformulation. However, to connect with several applications in other scientific and technological areas where stratified models and maps are ubiquitous, it is necessary to develop bottom-up approaches. In the last case, stratifications must "emerge" as a result of clustering data arising from sensors, by following different morphological and functional criteria. In this pàrageraph, we sketch some basic methods of both approaches.

The first general meaningful result of Stratification Theory is the Ehreseman's fibration theorem; roughly speaking it says that a proper submersion between manifolds is a C^r-fibration. In a more down-to-earth terms it can be described by local sections, even the topology can change. Trivial examples are given by vector bundlles and principal bundles. Less trivial examples are linked to multi-parameter families of algebraic varieties fulfilling the hypotheses of the theorem such those appearing in Classical Algebraic Geometry A_{32} (Quasi-Projective Varieties). Other more sophisticated examples with irregular behavioir at fibers appear in regard to unfoldings of simple singularities of function fgerms A_{43}.

A C^r-map germ $f : (N, x) \to (P, y)$ with $y = f(x)$ is not in general a regular map at $x \in U \subset N$. To analyze the lack of regularity one considers singularities of the differential $d_x f : T_x N \to T_y P$. After fixing local coordinate systems at (x, y), the differential is represented by the Jacobian matrix at $x \in U \subset N$. The locus of points $x < inN$ where the map is not regular is called the Ramification locus $Ram(f)$ of f, , and its image in P is the Discriminant locus $Disc(f)$; they are the natural generalization of critical points and critical values for a function $f : N \to \mathbb{R}$ (see chapter 5 of A_{41} for details). Thus, a first approach (R.Thom, 1956) to stratification issues for f is given by

$$\Sigma^r := \{x \in U \ \mid \ corank(d_x f) = r\} \qquad r \geq 0$$

where for $r = 0$ one has the "regular strata". Having in account the corank is a lower semi-continuous map, one has a finite collection of inclusions

$$\overline{\Sigma^0} \supset \overline{\Sigma^1} \supset \dots \overline{\Sigma^m}$$

where $\overline{\Sigma^k}$ denotes the adherence of Σ^k for $k = 0, \dots, , m = min(n, p)$, and $\overline{\Sigma^k} \setminus \Sigma^k$ is the singular locus of $\overline{\Sigma^0}$. Obviously, the subsets Σ^r are not necessarily connected nor regular necessarily, which gives a recurrent decomposition for each one of them, which can be formulated in a more intrinsic way (J.Boardman, 1963). The resulting decomposition is called the Thom-Boardman stratification and will be developed along the first chapter of this module.

On the other hand, $(p \times n)$-matrices (corresponding to the evaluation of the differential $d_x f$ of $f : N \to P$ at each point, e.g.) up to scale, correspond to k-dimensional subspaces L^k of a N-dimensional vector space V^N being $k = min(n, p)$) and $N = max(n, p)$. Thus, there exists a local dictionary between the vector space of matrices with constant coefficients and the Grassmann manifold $Grass(k, N)$ of k-dimensional subspaces L^k of V^N. Thus, it is "natural"the existence of a dictionary between the (co)rank stratificaiton of $M(k \times N; \mathbb{K})$ and the cellular decomposition of $Grass(k,N)$ in terms of the Schubert cycles. This viewpoint is developed by R.Thom in his 1956 article [Tho56] for the smooth case. Automated learning is performe din the SOM framework [Koh87].

The extension of Grassmann manifolds to Grassmann modules (GAGA framework) is performed in [Gro73] [11]. Roughly speaking, this means that one can replace matrices with constant coefficients by variable coefficients, representing maps between modules on a ring, instead of vector spaces on a field. The (co)rank stratification of maps between modules is locally developed in terms of determinantal varieties, an old topic where Algebraic Geometry and Local Algebra overlap between them; a classical representation, not easy to read, is [Roo37] [12]; a more modern presentation is the chapter 20 of [Eis95] [13]

In the analytical approach (Whitney) the comparison between limits of tangent lines $\ell = t_x X$ or tangent k-subspaces $T_x X$ at regular points of each stratum, and the "control" of their evolution is performed by imposing like-Lipschitz conditions involving the evolution of secant spaces. These conditions are feasible because the Grassmann manifolds appearing in both cases (secant and tangent spaces) are compact spaces. However, incidence conditions along different paths in the corresponding Grassmann manifold vary according to a Schubert cycle (making part of the cellular decomposition) where one takes the path (this idea is implicit in [Tho56]). Their dual version is formulated in terms of characteristic classes (Stiefel-Whitney vs Chern) depending on the base field (real vs complex).

In local terms, it is necessary to make a control of the intersection dimension of subspaces appearing when one takes limits. A solution consists of considering "all possibilities" of intersections between k-dimensional subspaces. They can be described in terms of the monoidal transformation of the product $Grass(k,N) \times Grass(k,N)$ with center on the diagonal $\Delta_G \simeq Grass(k,N)$. As a result, one obtains the Secant Variety. Nevertheless its synthetic character, this approach disregards richer structure involves the internal structure involving properties depending on the dimensionality of intersections. To remediate it, one introduces complete flag manifolds playing a universal role w.r.t. intersections [14].

The main contribution of these notes is the use of Flag manifolds $\mathcal{B}(r_1,\ldots,r_k)$ of nationality $(r_1,\ldots,r_k)$ corresponding to a partition of N as "universal spaces" for strattified spaces and their applications to represent multiple hierarchies. The usual superimposed structures on $\mathcal{B}(r_1,\ldots,r_k)$ given by tangent, cotangent and tensor bundles provide general patterns to represent complex behaviors in stratified media.

A formal advantage of this approach consists of the minimal character of

[11] A.Grothendieck and J.Dieudonne: *Elements de Geometrie Algebrique*, GMW, Springer-Verlag, 1973.

[12] T.G.Room: *The Geometry of Detrminantal Loci*, Cambridge Univ. Press, 1937.

[13] D.Eisenbud: *Commutative Algebra with a View to Algebraic Geometry*, GTM 150, Springer-Verlag, 1995.

[14] In the part II of my Ph.D. (Valladolid, 1983, non-published) there appear more details involving linear subspaces, which can be extended in a relatively easy way to modules on a ring.

the complete flag manifold $\mathcal{B}(1,\dots,1)$ and the splitting result for its tangent bundle as sum of line bundles on $\mathcal{B}(1,\dots,1)$. This result makes possible the decoupling of linear systems on the complete flag manifold (see part III of [Bot82] for a proof). Unfortunately, the result is not true for "intermediate" partitions. Hence, a tupical strategy for solving a problem in a stratified spaces X consists of reformulating in terms of osculating flags, remap on the complete flag manifold to solve it (by using splitting methods), and pull-back to intermediate flag manifolds.

0.2.2. Morse Stratified Theory and beyond

Classical Morse Theory develops a constructive approach to the topology of any compact smooth manifold M by "attaching cells". A k-dimensional cell $(e_i^k, \partial e_o^k)$ is a pair homeomorphic to the pair $(\mathbb{D}^k, \mathbb{S}^{k-1})$ where the $(k-1)$-dimensional sphere $\mathbb{S}^{k-1}$ is the boundary $\partial\mathbb{D}^k$ of the k-dimensional disk $\mathbb{D}^k$. The dimension k of the cell to be attached depends on the signature of the quadratic form given by the Hessian $Hess(f)$ of a Morse function, i.e. a function $f: M \to \mathbb{R}$ whose only singularities are non-degenerate critical points (i.e. the Hessian has maximum rank equal to m). The tangent bundle τ_M is described in terms of the gradient vector field ∇f, and this description provide the nexus with the simplest applications to other scientific and technological areas.

The above reasoning scheme is of *conservative* type, i.e. one supposes there exists once a function defined on a smooth manifold M or the Phase space P whose first variation rates (given by its differential or the gradient field) explains the Kinematics or the Dynamics of a system. Typical examples are given by the height for objects or the depth for static scenes, the energy or any other Hamilton function $H: P \to \mathbb{R}$ on the Phase space $P = TM$, or any other Lagrangian (curvature functionals corresponding to second order variation rates, e.g.) to be minimized along possible trajectories according to "connections" on the underlying structure.

However, most systems are not conservative and non-integrable; very often one ignores the nature of the support X (it is a PS-manifold in the best case), one has not information about internal forces acting one the system, there appear singularities at each mechanical level (Geometry, Kinematics, Dynamics), and they can display a different behavior depending on the scale and possible singularities. Thus, it is necessary a more flexible approach than conservative systems able of integrating not only the diversity of maps germs A_{44}, but all the richness of interactions between different components, and the corresponding bifurcations for partially unknown dynamical systems A_{46}.

Along this module A_{45} one develops several strategies to include quasi-static approaches to an eventually singular support X and superimposed structures (beyond vector bundles or sheaves, e.g.). To be more effective, one supposes that singularities involving objects and maps are not "too bad", i.e. their differential has low corank, and one has information about generic types appearing in a small neighborhood of singularities giving information about propagation phenomena.

Stratified Morse Theory provides a first framework, fulfilling the above conditions. In despite of its name, relations with classical Morse Theory are weak. To start with, the support is not a smooth manifold M but an eventually singular variety X. The decomposition of X is not performed w.r.t. the gradient field ∇f of a Morse function, but w.r.t. the discriminant locus $Discr(F)$ of a vector map F. The reconstruction of X is not a cellular complex (nor even a CW-complex as in Geometric Topology A_{24}), but a collection of Tangential and Normal Morse Data to be matched by using the information of an evolving singular support. A systematic treatment for these issues is developed by Goresky and R.Macpherson from the early 1980s [Gor88].

A typical example to have in mind correspond to the well known family of curves $y^2 - tx^2 - x^3 = 0$. Away from $t = 0$ it can be represented as a nodal curve which degenerates in cusp curve for $t = 0$. Hence, the resulting surface in the (x, y, t)-space has a double curve of a surface whose orthogonal sections to the axis Ot display an evolving simple singularity. For each $t < neq 0$, the tangent cone is given by two transversal lines which coalesce in a double tangent for $t = 0$. The stratification is given by once a point (the origin), separating two half-lines (positive and negative parts of the Ot axis) and the complementary pieces of the surface. Tangent Morse Data (TMD) along the singular case are given by a (half)line, whereas Normal Morse Data (NMD) are given by two lines coalescing in a double line at the origin. According to the Thom's approach, the corresponding attaching maps are given by local submersions. More details are given in the §1,4

Classical Morse Theory provides a method for computing the homology of a compact smooth manifold M from the cellular decomposition linked to a Morse function. Similarly, the Stratified Morse Theory introduces methods and tools for computing the Intersection Homology of a singular variety X with a "good stratification" (fulfilling Thom-Whitney conditions, e.g.) from TMD and NMD which play a similar role to cell complexes in the classical case. An advantage of this approach consists of its compatibility with changes of state in M or phase transitions in $P = TM$ given as singularities of any kind of fields (vector maps, distributions of vector fields, systems of differential forms) making part of the tensor algebra with the corresponding local symmetries.

All of them are considered as different kinds of tensors defined on the eventually singular support X. Thus, behaviors involving them can be described in terms of (maps between) filtered algebras; in our case, they are approached by using the dual (tensor algebras) of generalized flag manifolds by using simple properties linked to representations of the symmetric and alternating groups (in correspondence with Invariant Theory in Algebraic vs Differential Geometry). The introduction of supersymmetries on the associated superalgebras has been performed in an overlapping of Theoretical Physics and Algebraic Geometry along the late 1980s. A small introduction can be read from the chapter 7 of the module A_{33} (Sheaves, Cohomology, Schemes). The last section of this chapter is devoted to illustraate several adaptations of this approach to some Engineering areas which are developed in the part II of these notes.

0.2.3. From estimation to generation

All the above paragraphs have a theoretical character and follow a top-down approach. When one tries of adapting some of the above ideas to other scientific and technological areas there appear a lot of non-trivial problems. Experimental data can have a static, mobile or interacting nature; thus, a first hierarchy for evolving data involves to the weak stratification of Mechanics in their Geometric, Kinematic and Dynamic levels. Typical examples involve to traffic scenes or interacting agents. The problem is how can learn the underlying models?. In a more formal setting, how to generate different hierarchies in a semi-automated way which can be understood by an artificial device?

In AI (Artificial Intelligence) the SOM (Self-Organizing Maps) [Koh97] provide an unsupervised approach to learning subspaces L^k by successive submersions reducing information. Along the part II we develop several tools for the automatic generation of meaningful subspaces which are applied to Computational Mechanics of Continuous Media B_1, Computer Vision B_2, Robotics B_3 and Computer Graphics B_4. To develop this proposal it is necessary to specify a local stratification (involving morphological and functional aspects), consider multiple paths on the support (integral curves of vector fields belonging to a distribution $\mathcal{D}$), and introduce multiple constraints (covectors) involving them (integral hypersurfaces belonging to a system $\mathcal{S}$). More details appear in the section 4 of this introductory chapter.

In bottom-up approaches, problems appear from the beginning because one must work with vectors linked to discrete differences of data corresponding to partially unknown models to be identified. Very often their distribution is non uniform, including different densities with different distributions. Data mining provides tools for clustering, but one must design and implement strategies for clustering according to different similarity criteria. Each one of them gives a "hierarchy" depending on clustering criteria for "quantities".

One can use topological, metric, differential or analytic criteria for clustering. In our approach we prior vector-based approaches which are generated in an automatic way by using first-order variation rates between data of the same type; their space-time variation provides the inputs for Hidden Markov fields as a first step for data-driven models. Linear forms defined on vectors provide covectors, whose evolution in the ambient space-time provides a formal analogue of PS differential forms in the statistical framework. In our approach, we introduce a formal description of higher order central momenta to recover more meaningful information than linear forms of Hidden Markov Fields (see the section 3 of this chapter for additional information).

Multivariate Statistics of k quantities involves to the simultaneous estimation of (a discrete version of) k-dimensional subspaces which are ideally represented by multivectors. Thus, in our approach statistical Grassmannians play a central role in multivariate statistics. The computational management of redundant information is performed in terms of matroids, which are given by rectangular matrices of size $k \times N$ where one stores eventually redundant information to be cleaned in later stages. Again, these rectangular matrices can be interpreted in terms of Grassmannians $Grass(k, N)$ which provide structural models for managing vector and covector information.

The use of "projections" (in fact submersions for the curved case) minimizes the redundancy. Linear Optimization issues are solved in the Grassmannian representing subspaces to be learned in terms of geodesics w.r.t. the Cramer-Rao distance on the space of parameterized pdf (probability density functions). Nonlinear Optimization requires more advanced techniques based on the use of higher order central momenta as functionals (not as scalar functions which is the most usual approach).

In the top-down approach first-order hierarchies involving variation rates of "clustered quantities" (multi-vectors and multi-covectorss) are reformulated in terms of conormal sheaves (as the extension of tangent bundles of the smooth case) or in terms of Jet Bundles from a functional viewpoint. The problem is how can be estimated; in more down-to-earth terms, how to estimate "subspaces" (as naive version of subalgebras, e.g.) or differential functionals on them. It is almost obvious that one must use r-th order finite differences w.r.t. a "central value" (mobile k-means), and introduce clustering criteria for the corresponding central momenta to find "hidden structural relations". Let us remark that finite differences require some kind of parametric representation of (geometric, radiometric, physical, chemical, economic) data. Thus, we will restrict ourselves to locally parameterizable spaces, including their corresponding probability density functions (pdf).

In our approach, the usual numerical approach to the computation of central momenta is replaced by a symbolic computation of the corresponding tensors. Tensors can be generated in an automatic way by using Tensor Voting strategies. The idea is very simple and can be described in a sequential way as follows: O

1. Identify the minimal number of evolving vectors and covectors which are meaningful for the process
2. Introduce a generator of hypotheses with varying (stepped decreasing and increasing) weights for the structure tensor
3. Adapt Sampling Consensus strategies for each tensor,
4. Validation (acceptation vs rejection) of the proposed tensor.
5. If the tensor is rejected, transform it according to contraction and expansion techniques for adjacent tensors according to the poset structure of Tensor Algebra.

Tensor Flow methods in Deep Learning [Goo16] provide the support to estimate transformations between tensors for each "regular piece" of tensors representing "aggregate quantities" represented by multivectors under evolving constraints (ovectors or differential forms in the smooth case). The variability of the dimension of vectors and covectors suggests the use of pairs of flags representing their aggregates.

To simplify the management and develop decoupling strategies, we restrict initially to complete flag manifolds linked to vectors and covectors. Their combinatorial management is performed in terms of representations of the symmetric group (for their structure as poset) and the alternating group (for internal duality relations). Thus, possible degenerations involving dependence relations between sections (locally controlled in terms of limits of tangent spaces in Whitney stratifications, e.g.) are managed in terms of complete models for symmetric and skew-symmetric forms representing quadrics or elements of a Grassmannian. Their completion (different kinds of compactification) are interpreted again as flag manifolds representing contact conditions on each one of the above spaces.

0.2.4. A short description of the module

Furthermore the introductory chapter, the module A_{45} contains the following chapters:

1. A *Differential approach to Stratifications* which starts with Ehresmann's Fibration Theorem and develops the Thom's approach based on corank stratification, relations with Schubert cycles of Grassmannians and its applications to manifolds having finite CW-complexes

2. *Algebraic aspects* where one exploits differential aspects in terms of Fitting ideals for determinantal varieties, and one constructs an algebraic version of relative tubular neighborhoods for adjacent strata.

3. *Analytic stratifications* where one introduces Whitney's conditions to avoid constraints linked to proper maps, in terms of geometric conditions involving limits of linear subspaces. This approach provides some connections with deep issues in GAGA (Algebraic Geometry Analytic Geometry) developed in A_{33}.

4. *Extending geometric stratifications* to semialgebraic and semi-analytic varieties, which allow the extension to boundaries of adherences, which provides the support for attaching maps as new topological framework.

5. *Stratifications in GAGA* with a reformulation of Polar Varieties as the main tool for an intrinsic study of the Discriminant Loci of maps and higher contact conditions.

6. *Stratified Morse Theory* where one uses Tangential and Normal Morse Data as basic pieces to describe singular stratifications. Homology Intersection Theory provides the framework to compute invariants.

7. *Stratified fibrations* where one extends the original Ehresmann's approach to stratified C^r-maps specially in regard to equimultiplicity and equisingularity for families of maps.

8. *Some applications to Natural and Social Sciences* where stratifications involve to the ambient space for Dynamics, which is given by the conormal space C_P for the Phase space P (corresponding to Kinematics) as a natural generalization of TP. Equivariant stratifications provide a support for some interesting propagation models in different kinds of (bi)stable, excitable, oscillatory and active media which will be developed with more details in A_{46} (Dynamical Systems).

9. *Stratifications in Engineering* with a special regard to stratifications in Computational Mechanics of Continous Media B_1, Computer Vision B_2, Robotics B_3 and Computer Graphics B_4. A more detailed treatment of these matters will be developed along the part II of these notes.

0.3. References for this chapter

0.3.1. Basic bibliography

The most complete references for the approach developed along these notes are [Gor88] and [Shi97]. One includes some relevant additional articles because some ideas are not included in these texts.

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0.3.2. Software resources

Only open source libraries are included. Any suggestion is welcome

- Visualization Toolkits (VTK)
- OpenCV

Final remark: Readers which are interested in a more complete presentation of this chapter or some chapter of this module A_{45} (Stratifications), must write a message to franciscojavier.finat@uva.es or to javier.finat@gmail.com.