A440 Singular Map-Germs. An introduction

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Índice

Previous remarks: These notes corresponds to a short version of the introduction of the module A_{44} (Singualr map-germs) of the matter A_4 (Differential Topology). From the mathematical viewpoint, it is necessary to have some basic knowledge for Groups and Commutative Algebra, General Topology and Basic Differential Topology.

As usual, furthermore this introduction materials are organized in four sections. Subsections or paragraphs marked with an asterisk ($*$) display a higher difficulty and can be skipped in a first lecture.

0.1. Introduction

Regularity hypotheses are too strict about "objects" (PL or PS manifolds, varieties, e.g.) and maps between them. To include shape and behaviour changes, it is necessary to incorporate singularities as "natural" parts of evolving models. The simplest examples for "behaviours" on vector spaces appear already for linear maps $\varphi : V^n \to W^p$ between vector spaces, including endomorphisms $\varphi \in End(V)$ of the whole space, which are classified by the rank of the representative matrix.

A linear map $\varphi : V^n \to W^p$ between two vector spaces on a field $\mathbb{K}$ of dimensions n and p, respectively, can be classified in terms of

- The double conjugacy class by the $A = R \times L$-action (right-left) of $GL(n) \times GL(p)$ acting by double conjugacy $K^{-1} M_\varphi H$ for each pair $(H, K) \in A$ of reglar transformations acting on the pn-dimensional space M_φ of matrices representing φ after fixing a basis.
- The action preserving the graph $\Gamma(\varphi) := \{(v, w) \in V \times W \mid w = \varphi(w)\}$.

Both actions are extended initially to Euclidean, Affine and Projective spaces with the usual formalism for each one of them. More generally, they can be adapted to the tangent map given by the differential $d_x f : T_x N \to T_{f(x)} P$ (represented by the Jacobian matrix) between tangent spaces for a differentiable map $f : N \to P$ between manifolds $A : 11$. A similar argument is applied in the GAGA framework for the differential $d_x f : \Theta_{X,x} \to f^* \Theta_{Y,y}$ (or preferably, its dual) of the map $f : X \to Y$.

Along this module, we extend the precedent construction to the spaces $C^r(n, p)$ of maps $f : (\mathbb{K}^n, \underline{0}) \to (\mathbb{K}^p, \underline{0})$ representing germs $[f] : (N, x) \to (P, y)$ of class C^r for the usual geometric cases corresponding to $r = \infty$ (Differential Geometry), $r = \omega$ (analytic case), and $r = rat$ (algebraic case). We will suppose that the base field is given by the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, to ease their geometric or analytic interpretation.

If we take in account the geometric approach (linearization), the first non-trivial problem consists of characterizing the tangent space $T_f C^r(n, p)$ at $f \in CPr(n, p)$. The extension of the linearization approach to higher order derivatives is performed in terms of k-jets spaces $J^k(n, p)$ introduced in the chapter 2 of A_{41} (Basic Differential Topology).. The main issues concern to

- the *classification of map germs* up to C^r-equivalence corresponding to $r = \infty$ (diffeomorphism), $r = \omega$ (bianalytic transformations) and $r = arat$ (birational transformations);
- the *identification of canonical forms* w.r.t. some $\mathcal{B}$-action of C^r-equivalences;
- the description of *general deformations* including singular map-germs of the same "type";

- the development of computationally effective methods for each one of the above models..

The foundations and main results of this theory were developed between late fifties and eighties of the 20th century; some of the most prominent authors are R.Thom, J.Matter, V.I.Arnold and C.T.C.Wall between others.

Since we are working with maps $f \in C^r(n,p)$, groups of C^r-transformations are given by (decoupled vs coupled) pairs of actions extending the right or $\mathcal{R}$-action developed for function germs in the precedent module A_{43}. The corresponding actions are the natural extensions of those appearing for linear maps $\varphi : V^n \to W^p$ between vector spaces. In other words, they are given by the $\mathcal{A} = \mathcal{R} \times \mathcal{L}$ or right-left action of C^r-equivalences acting on the source and target spaces, and the $\mathcal{K}$-equivalence "preserving the graph".

Along this module we are mainly interested on the cases $r = 0$ (topological equivalence), $r = \infty$ (differential equivalence) and $r = \omega$ (analytical equivalence for map germs $f : (X^n, x) \to (Y^p, y)$) between C^r-spaces. These actions provide the starting point for the topological, differential and analytical classification. The "basic strategy" consists of extending the "degree reduction" of f performed in the precedent module. Now, the reduction involves to k-jets $j^k f$ of f for enough large k, where $j^k f = (j^k f_1, \ldots, j^k f_p)$ being $(f_1, \ldots, f_p)$ the components of f.

To "give shape" to this idea, one needs to develop explicit simultaneous reduction strategies for all the components f_i of f. A first approach (too rigid from the topological viewpoint) consists of simultaneous reductions depending on the value of the "determinacy degree" k . A first naive approach consists of interpreting the components f_i of f as the generators of an ideal, and suppose all of them have the same degree:

1. For $k = 1$, we would have the linear classification corresponding to the linear map given by the differential $d_x f$ locally represented by the Jacobian matrix of f. By the Implicit function Theorem, this case is an extension of arguments given in Linear Algebra. They can be interpreted as "arrangements" of hyperplanes (lines in the 2D case). Possible degenerations are "controlled" by nested subspaces as elements of a Flag Manifold.

2. The following case corresponds to $k = 2$ where hyperplanes are replaced by hyperquadrics. It involves to the simultaneous classification of pencils (easy), nets (more involved), webs (non-trivial), ... of quadrics. The variety of secant spaces to the projective space $\mathbb{P}^N$ of quadrics in $\mathbb{P}^n$ and their possible degenerations in terms of the variety of "complete quadrics" provides a geometric interpretation [Fin83].

The problem becomes much more intrincate when we allow higher degree polynomials, where mixed multi-degrees can be present giving a lot of "pathologies". This simple remark motivates the development of more powerful topological strategies involving not only determinacy, but C^r-*stability* by the action of C^r- equivalences for $r = 0$ (topological case), $r = \infty$ (differential case) and $r = \omega$ (analytical case) as the main cases to be studied.

Stable maps play a fundamental role in the differential case. roughly speaking, if $s(n, p)$ is a function representing the codimension in a sufficiently large jet space $J^k(n, p)$, and $n < s(n, p)$, then stable maps are dense in $C^\infty(n, p)$. This result is due to J.Mather (1970) and determines a region called the "nice dimensions". In particular, it explains the "good behaviour" by deformations of "ordinary singularities" appearing in Classical Algebraic Geometry. Furthermore, it opens the door for the analysis of moduli spaces in these regions, and a more detailed analysis for boundary or outdoor regions beyond the nice dimensions.

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To summarize, the study of the singularities of maps is the natural extension of the methods presented for functions A_{43}, where one replaces function germs by map germs $[f]$ of maps $f \in C^r(n, p)$. The main cases correspond to the differentiable case $f \in \mathcal{E}(n, p)$ or analytic $f \in \mathcal{O}(n, p)$. Local Algebra provides a unified language to address both cases simultaneously. Remark that analytic maps can have several branches, and that differentiable maps are not necessarily analytic ones (flat functions).

The fundamental problems to be solved in regard to classification issues are characterization (in terms of invariants), classification (including canonical forms), construction of non-necessarily universal "foldings" (versal deformations) and analysis of the different types of stability. To do this, different $\mathcal{B}$-equivalence relations are introduced where $\mathcal{B}$ extends the $\mathcal{R}$-equivalence on the right, and the $\mathcal{L}$-equivalence on the left, in terms of the $\mathcal{A}$-equivalence where $\mathcal{A} = \mathcal{R} \times \mathcal{L}$ (right-left equivalence), or the $\mathcal{K}$-contact equivalence (linked to the graph oreservation).

The classification is effective for finitely determined maps, i.e. map-germs $[f]$ which are $\mathcal{B}$-equivalent to its k-jet $j^k f \in J^k(n, p)$. This strategy can be understood as a non-linear dimensionality reduction (NLDR), which is ubiquitous in applications, including some recent developments in Artificial Intelligence. In our case, NLDR techniques are relative to a "projection" on a graded algebra, which allows the introduction of hierarchies for adjacencies between $\mathcal{B}$-orbits.

The central algebraic result [Bru86] for k-determinacy (i.e. the equivalence w.r.t. a truncated Taylor development) is the unipotency of the group action "stabilizing" the $\mathcal{B}$-orbit. The explicit computation of the corresponding unipotent group is a very hard problem. By this reason, one computes the nilpotent Lie algebra $\mathfrak{U}_f$ of the unipotent group U_f for the finitely determined map-germ $[f]$ extending the approach performed for function germs developed in the precedent module A_{33}. Explicit computations are performed by hand for some examples. It would desirable to have a computational version.

0.1.1. Stability

From the 19th century, it is well known that generic projections of algebraic curves $\mathcal{C} \hookrightarrow \mathbb{P}^3$ (resp. surfaces $S \hookrightarrow \mathbb{P}^5$ with center a line ℓ) give ordinary singularities on $\mathbb{P}^2$ (resp. $\mathbb{P}^3$. Futhermore, "small perturbations" of generic projections of curves and surfaces give ordinary singularities, again.

Similar arguments can be applied for generic singularities of non-regular maps between spaces of the same dimension by using methods based on "small perturbations" (obtained by integrating vector fields, e.g.). These methods are widely used in Applied Sciences and Engineering. However, they have not a "universal validity" and require a rigorous foundation. The first systematic treatment is due to H.Whitney (1955), who formulated a temptative conjecturee about the density of stable maps in $C^r(n,p)$. R.Thom gave the first counteerexample (1959), and posed several conjectures which required more formal developments of equivalence relations on $C^r(n,p)$ for classificaiton issues.

The classification of maps $f \in C^r(X,Y)$ by C^r-equivalences is performed in terms of some subgroup of the group of homeomorphisms preserving a point acting on source and target space of f. An explicit description of the resulting $\mathcal{B}$-orbits in $C^r(n,p)$ is a hard problem for $p \geq 2$. However, the introduction of "constraints" linked to geometric structures makes the problem more affordable and shows their interest for a lot of applicatioins in Natural Sciences and Engineering.

In particular, diffeomorphisms preserving a metric, symplectic, or a contact structure can be applied to different problems in Riemannian Geometry, Analytical Mechanics or Dynamics, by preserving orthogonality, Lagrangian or Legendrian manifolds as solutions of the corresponding systems of equations. All of them can be applied to phenomena linked to waves propagation and interactions in non-necessarily homogeneous or isotropic media. So, singularities of solutions would correspond to "irregularities" in the distribution of matter, their evolution rates and interacting forces or momenta.

The precedent remarks are linked to "observability" of phenomena in terms of data. In several modules of the part II, one makes a more systematic approach for the modelling the "emergence" of data structures from Data Mining B_{11} (Computational Mechanics of Continuous media), the generation of PL-structures for linking them in B_{12} (Computational Algebraic Topology), and their "regularization" in PS-structures to simplify their management in B_{13} (Computatoinal Differential Topology).

In all cases, observability is linked to "some kind of stability" in despite of qualitative shape changes or sudden "events". In other words, one detects "stable patterns", even instantaneous transitions can not be detected. Typical examples appear in traffic scenes, where the kinematics of traffic flow uses techniques of motion analysis B_{23} of computer Vision B_2, to provide the inputs for Automatic Navigation B_{32} in robotics B_3. They are crucial for simulation and animation B_{44} in Computer Graphics, and for the generation of interactive evolving $3D+1d$ scenes for semi-automatic learning in recent AI developments.

The simplest non-trivial stable models for C^r-maps were developed by H. Whitney [Whi55] [1], who proves that folds and cusps appear for $f \in C^\infty(2,2)$ and proper smooth maps $\mathbb{R}^2 \to \mathbb{R}^2$ can be approached by stable maps. This article can be considered as the starting point for the singular map-germs.

R.Thom (Bonn, 1959) disproved the original Whitney's conjecture about the density of stable maps, and posed the need of identifying the values of (n,p) such that the stable maps $S^r(n,p)$ are dense in $C^r(n,p)$ with the Whitney topology. The characterization of the region corresponding to allowed values for (n,p) fulfilling the density property was performed by J.Mather (1965-1975), who introduced different notions of stability (topological, differential, infinitesimal), and a formal construction for the tngent space $T_f C^\infty(n,p)$ in terms of the so-called "homological equation" (by V.I.Arnol'd).

The extension of the original viewpoint (due to H.Whitney and J.Mather) to families of maps $f_t \in C^\infty(2,2)$ includes different kinds of "degenerations" for the discriminant curve corresponding to profiles of apparent contours (for smooth transparent surfaces). They are useful to image understanding purposes corresponding to different views of the same object (linked to partially overlapping views from near camera's localizations) or to track mobile objects for video sequences (real or apparent motion).

A general strategy to obtain equivalence relations able of detecting and relate deep properties of map germs $f \in C^r(X,Y)$. Typical "examples' of subgroups of homeomorphisms $r=0$, are given by differentiable maps $r=\infty$, analytical maps $r=\omega$ and birational maps $r=alg$. Each one of them provides the support for Differential Geometry, Analytic and Algebraic Geometry, respectively. We start, by remembering a coarse characterization of stability.

Let X be a topological space, and let $\sim$ be an equivalence relation on X. An element $x \in X$ is said to be stable if the equivalence class of x under $\sim$ is an (open) neighbourhood U of x.

There are different notions of stability going from the coarsest topological descriptions to finest infinitesimal vs analytical stability. All of them are based on the existence of a neighborhood of $\mathcal{B}$-equivalent C^r-maps for $f \in C^r(n,p)$. The most important cases are labeled as

- Topological stability for $r=0$ stability (General Topology).
- Differential stability for $r=\infty$ (Differential Geometry).
- Analytical stability for $r=\omega$ (Analytical Geometry).
- Algebraic stability which is reformulated in terms of Moduli Thoery (Algebraic Geometry).

Relations between C^r-invariants are very meaningful to understand relations between different approaches to the classification problem. They allow to reinterpret relations between $\mathcal{B}$-orbits in Differential Topology and properties of

[1] H.Whitney: "On Singularities of Mappings of Euclidean Spaces. I. Mappings of the Plane into the Plane", *Annals of Mathematics* Vol. 62, No. 3, 374-410, 1955.

moduli spaces in GAGA, e.g.. This kind of results is an extension of formulae linking Milnor number for hypersurfaces and algebraic invariants. Typical examples appear in regard to the use of projective characters of algebraic surfaces with topological invariants of $C^r(2,3)$ in the complex case .

All materials developed along this module correspond to the local case, with some sporadic extensions to the multi-local case in terms of multgerms. The main problem is to give effective criteria to warrant the above stability criteria.

0.1.2. Deforming singular map-germs

Deformations appear in a lot of mathematical areas and their applications to other knowledge areas. If one starts with an algebraic, analytic or differential local expression, variatioin of coefficients provide a first coarse approach to the description of the simplest deformations. More formally,, one can apply scalar, vector, covector or tensor fields to local descriptions involving the morphological or functional aspects linked to the “shape” or different kinds of “behaviours” in the ambient space, including interactions with themselves or the environment.

Deformations can involve to local or global aspects of the support (manifolds or varieties, e.g.) or the superimposed structures defined on them. The simplest cases correspond to first-order deformations which are performed in the tangent space for

- smooth manifolds in terms of T_xM for the preservation of Riemannian, Conformal or Symplectic structure, e.g.;
- algebraic or analytic varieties in terms of $\Theta_{X,x}$ including the analysis of possible dgenerations and different kinds of “completions” in regard to classification issues;
- first order approaches to map germs in terms of $T_fC^r(n,p)$ for differentiable $r=\infty$ or analytic $r=\omega$ map-germs, as the central topic for this module A_{44}.

This viewpoint is extended to the algebraic case, where first-order one-parameter deformations linked to a parameter ε are formally described described in terms of $k[\varepsilon]/\varepsilon^2$. called “dual numbers” in the formal approach to deformations (in terms of local algebras).

In addition of the support X, deformations are applied to superimposed structures which affect to the resolution of systems E of equations corresponding to the local form (given by sections, typically) of bundles, sheaves, or fibrations, between other. In particular, deformations appear in regard to

- *PS-manifolds* linked to tensor fields A_{13} for deformable multilinear structures. They are crucial to ealuate the “rigidity” of Riemannian manifolds A_{16} with constant curvature, e.g..

- *PL-structures* in Geometric Topology A_{14} in regard to coverings or, alternately, (resolution of) graded complexes, where the Cohen-Macaulay character of determinantal varieties plays an essential role to recover varieties with the same character.
- *Algebraic Curves* to formalize old arguments of "families of curves" (used by Italian geometers), and in regard to Moduli theory in the context of schemes.
- *Algebraic Surfaces* A_{35} for analytic classification issues (Kodaira-Spencer) extending the Enriques's proof). in terms of infinitesimal deformation given by the (co)normal space to $\Theta_{S,s}$.
- *Quasi-Projective varieties* A_{32} in regard to the description of the tangent space $\Theta_{X,x}$ of families parametrized by a connected complex manifold S on the first cohomology group of $\Theta_{S,s}$ extending the approach performed for surfaces. The corresponding constructions is given by the Kodaira-Spencer map introduced originaly for surfaces.
- *G-structures* linked to principal bundles $G \to B_G \to X$ in A_{42}, where B_G consists of all G-frames on X, to evaluate if a deformation of an integrable G-structure is integrable or not.
- *Unfoldings* of simple singularities for singular function germs A_{43} in regard to the "evolution" of critical loci for $A-D-E$ singularities, to be extended to foldings of the Discriminant Loci for families of map germs $f \in C^r(n,p)$.

Furthermore, they appear in a lot of areas of Natural Sciences (Physics, Chemistry, Biology) and Engineering (Mechanical, Civil, Materials, e.g.). Some of them are revised in the four modules of the part II (Geometric and Topological methods in engineering) of these notes devoted to Computational Mechancis of Continuus media B_1, Computer Vision B_2, Robotics B_3 and Computer Graphics B_4, which have been introduced in the chapter A_{406}.

Other applications to deformations will be developed in the next modules corresponding to Stratifications A_{45}, and Dynamical Systems A_{46}, whose discrete versions are developed in the part II. The diversity of objects, maps, and superimposed structures gives a large diversity of methods to be applied. Hence, it is very difficult to give an overview of deformations and their applications. By this reason, we limit ourselves to a topologicla approach in the differential framework.

From the topological viewpoint, equivalence classes appearing in the classification under C^r-equivalence of objects, maps, superimposed structures and morphisms between them, can be onsidered as some kind of C^r-deformatioins. Their local description in trms of locla algebras, poses the probleem of developing a formal theory of deformaitons involving local algebras, which is sketched in the chapter 4.

In this module, we adopt a more down-to-earth viewpoint for map germs in infinitesimal terms for the their tangent space $T_f(\mathcal{B}f)$ to the $\mathcal{B}$-orbit $\mathcal{B}f$

of a finitely-determined map germ f. Its characterization by J.Mather gives a differential equation calle the *homological equation* (J.Mather) for the tangent space $T_f(\mathcal{B}f)$. It can be applied to analyze the questions of stability and versal deformations in regard to classification issues which are commented in the next paragraph.

0.1.3. Classification issues

Classification of map-germs $f \in C^r(n,p)$ up to $\mathcal{B}$-equivalence uses a mixture of topological, algebraic and differential methods. We restrict ourselves to finitely-determined map germs, which allow a reduction to finite-dimensional cases in terms of k-jets spaces $J^k(n,p)$, where k is the order of finite determinacy. Initial $\mathcal{B}$-actions defined on $C^r(n,p)$ in terms of C^r-equivalences, are adapted to their k-jets $J^k(n,p)$ by using k-jets of diffeomorphisms (or their natural extensions for bianalytic transformations).

Local methods for singular map germs are the natural extension of the Local Algebra methods presented for function germs A_{43} to differentiable maps $f \in \mathcal{E}(n,p)$ or analytic maps $f \in \mathcal{O}(n,p)$. Each one of them is a local module on the local ring $\mathcal{E}(n,1)$ of differentiable functions, or analytic functioins $\mathcal{O}(n,1)$, respectively. In particular, Commutative Algebra provides an initial unified language to address both cases simultaneously [2]. We will denote by means $\mathfrak{m}_n = (x_1, \ldots, x_n)$ the maximal ideal w.r.t. local coordinates centred at each point, corresponding to differentiable or analytic map-germs at the origin $\underline{0} \in \mathbb{K}^n$

Some of the most *main algebraic problems to be solved* are

- algebraic characterization in terms of $\mathbb{K}$-algebras;
- classification of k jets up to $\mathcal{B}$-equivalence;
- identification of canonical forms and computation of invariants;
- construction of “foldings” (versal deformations); and
- analysis of the different types of (in)stability.

For this, the algebraic approach based on double conjugation is extended to the topological approach. More explicitly, different $\mathcal{B}$-equivalence relations are introduced that affect the source or target spaces, or alternatively, the graph Γ_f of f. More explicitly, $\mathcal{B}$ is the $\mathcal{R}$-equivalent to the right, the $\mathcal{L}$-equivalent to the left, the $\mathcal{A}$- equivalence where $\mathcal{A} = \mathcal{L} \times \mathcal{R}$ or the $\mathcal{K}$-contact equivalence (graph preservation).

This viewpoint can be extended to the global case, by taking a morphism $\varphi : E- \to F$ of superimposed structures on the C^r-map $f : N \to P$. The

[2] Methods of Homological Algebra are very useful in more advanced developments, including projective and injective resolutions of modules, also.

superimposed structures are locally interpreted in terms of systems of equations such as distributions $\mathcal{D}$ of vector fields, differential systems $\mathcal{S}$ of covector fields, or their multilinear products in terms of tensors, e.g. This viewpoint has been developed in the module A_{42}.

The incorporation of higher order differential operators is initially formulated in terms of the space of k-jets J^kE given by k-jets j^ks of sections $s : U \to E\mid_U$ of $E \to N$ as usual. One can develop a formalism similar to the one that appears in Sheaves theory A_{33} for covariant aspects. Alternately, from thee GAGA viewpoint, one can use sheaves of principal parts $\oplus_{k\geq 0}(\mathcal{I}^k/\mathcal{I}^{k+1})$ as the analogue of Taylor development for an Ideal $\mathcal{I}$ of a Module [3]

The classification is an extension of the one developed for functions germs. In particular, the corange of the Jacobian matrix associated with the linearization d_xf in p of $f : (\mathbb{K}^n, \underline{x}) \to (\mathbb{K}^p, f(\underline{x}))$ provides a first criterion; next, consider the codimension of the $\mathcal{B}$-orbit $\mathcal{B}f$ of the germ $f \in C^r(n,p)$ in $\underline{x} \in \mathbb{K}^n$. To fix ideas, we can assume that $(\underline{x}, \underline{y}) \in \Gamma_f$ is $(\underline{0}, \underline{0}$.

Beyond an endless catalog of different "pathologies" that are increasingly "more improbable", the classification of singularities reveals the morphological and dynamic hierarchies that occur in Analytical Mechanics. Arnold's approaches on the one hand, and Marsden-Weinstein's on the other, show deep connections related to the structure of the projections, on the one hand, and the locally symmetrical character of the moment map, on the other. Issues related to equivariant stratifications for analytic varieties are addressed in the next module. Therefore, in this module the focus is on Arnold's approach and his extensive area of influence (beyond the Moscow and Leningrad schools).

In a very simplified way, we can think of the Lagrangian submanifolds (resp. Legendrian) as maximum dimension solutions of the Hamilton-Jacobi equations in even dimension (resp. odd) which are the structural equations of Analytical Mechanics. This first approach is useful for kinematic issues (involving simple motions or waves propagation, e.g.), but it is not enough for more advanced dynamical issues involving the interaction between agents (in terms of forces and momenta, e.g.).

From the dynamic point of view, its elevation to the cotangent bundle with its symplectic structure (contact resp.) provides a more natural representation in terms of wavefronts, as well as a visualization of the singularities that appear when projecting said varieties onto space. base. [4]

[3] A more sophisticated approach is developed in the module A_{45} (Stratifications) in terms of $\mathcal{D}$-modules.

[4] This visualization is especially significant for applications related to Robotics (automated navigation of autonomous vehicles) or Computer Vision (recognition of objects from their envelopes), e.g.

0.1.4. Some applications

Singular map germs are ubiquitous in almost all knowledge areas. If we adopt the mechanical viewoint, they correspond to

- *changes of state* in the ambient space X initially given by a PS manifold, later by a stratified space;
- *phase transitions* in some model for the ; space P (Poincaré) given by the tangent bundle $\Theta_{X,x}$ (or its dual); and
- *dynamical changes* involving complex interactions expressed in terms of higher order differential operators in the Euler space E modelled as

All of them are locally described by singular map germs $[f] : (N, \underline{x}) \to (P, \underline{y})$, which are locally described in terms of $f : (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, \underline{0})$ for local systems of coordinates centred at each pair $(\underline{x}, \underline{y}) \in \Gamma_f \subset N \times P$ of points at source and target spaces, where Γ_f is the graph of f.

The first order approach is given by the differential $d_x f$ which is locally desribed by the $p \times n$ Jacobian matrix $Jac_0(f)$. For higher order derivatives, it is convenient to adopt a description based on k-jets $j^k f$ defined by $j^k f_1, \ldots, j^k f_p)$, where $f_1, \ldots f_p$ are the components of f. This formalism is naturall extended to suewrimposed structures given by morphisms $E - - \to F$ (vector bundles, e.g.) on $f : N \to P$ (locally represented by systems of equations, e.g.). To describe $J^k E$ it suffices to take the k-jets $j^k s$ of local sections $s : U \to E\mid_U$ (and similarly for $J^k F$).

Data appearing in most applications to Natural Sciences and Engineering are discrete and irregularly distributed. Hence, it is necessary to develop strategies for clustering data at diffrent levels of detail (LoD), removing redundant data, filling incomplete information (by means propagation models, e.g.), and superimposing continuous structures to ease their management. Graded complexes appearing in Algebraic Topology A_{22} and Geometric Topology A_{24} provide natural candidates to generate a PL or a PS-model linked to data.

They are computationally managed in terms of functionals defined on them (cohomological methods) which have been developed before (see A_{42} for the PS case). The most difficult problems concern to the "emergence" of PL-structures from sparse, incomplete and noisy data. By following an increasing difficulty order, clustering process can be illustrated by the following applications[5]

- Maps between graded complexes (simplicial vs cuboidal meshes, e.g.) linked to clouds of points arising from 3D laser scans for bottom-up modelling in Computational Algebraic Topology B_{12} and its smoooth regularization in B_{13} (computational Differential Topology).

[5] All of them are developed in the part II (Geometric and Topological Methods in Engineering) of these notes.

- Simultaneous kinematic tracking of multiple agents in evolving traffic scenes B_{23} (Motion Analysis) in B_2 (Computer Vision), and their decentralized control in Automatic Navigation B_{32}.
- Control of dynamical interactions B_{34} (Robot Dynamics) between atomic vs molecular configuratioins in Sample Preparation or Pharmaceutical Synthesis by using segmentation, separation and synthesis strategies.
- Animation of Characters and Scenes B_{44} including radiometric properties (involving illumination and color) for planar vs volumetric representations in terms of uniparametric families of maps for $n = p = 2$ vs $n = p = 3$.

Each one of them poses a lot of issues in regard to the modelling of the above mechanical hierarchy involving Geometric, Kinematic and Dynamical aspects in terms of k.-jets of maps $F : X \to Y$ or morphisms $\varphi : E- \to F$ on them. Despite of this diversity, continuous modelling based on singular map-germs provides "almost universal" models which can be adapted to ideally conservative (regular maps) and more realistic dissipative phenomena (described in terms of multiple events or singular multigerms).

The identification of local symmetries for materials, propagation and intraction siimplifies the modelling. The classical distinction between regular and irregular algebraic actions (breaking symmetries) is formalized in terms of regular vs unipotent actions. The last ones (or its linearized nilpotent action) are the "responsible" for the finite dimensionality reduction of map germs to their k-jets [Bru86]. to understand how it works, one must remember some antecedents beyond the Catastrophe Theory which has been developed in the precedent module A_{43} .

A very detailed classification of topics presented in the previous paragraphs was developed by V.I.Arnol'd and his school in the 1970s and in the early 1980s. In particular, the description of the simple singularities associated with the propagation of wave fronts on ordinary and edged manifolds facilitated a reinterpretation in terms of root systems for the ABCD series of the classical groups and the reflections associated with the symmetry groups. crystallography of the Platonic solids. This reinterpretation has been presented in module B_{43}.

A surprising extension of this approach carried out by O.Scherbak (1985) and A.B.Givental (1988) showed the connection between the singularities of the projections of the symplectic and contact space with the Euclidean reflection groups (not only the crystallographic ones as in Arnold's initial formulation). This result has profound consequences that have not yet been sufficiently explored.

The differential approach to Analytical Mechanics initially described on a PS-manifold M (PS: Piecewise Smooth) has two has two paradigms labeled as Symplectic and Contact Geometry. Roughly speaking, they correspond to the preservation of the symplectic form ω for evendimension $2m$ and the preservation of the contact one-form α for odd dimension $2m - 1$, respectively. From the early decades of the 19th century, Analytical Mechanics is formulated on an "extended space" (called Phase space P by Poincaré) in terms of ODEs involving generalized coordinates $\underline{q}$ for the support (corresponding to several control points) and momenta $\underline{p}$ as local coordinates in the Phase space $P = TM$.

The dual approach puts the accent on evolving constraints represented by differential forms ω_i as local sections of the cotangent bundle τ_M^* with fiber $T_{\underline{q}}^* M = Hom(T_{\underline{q}}M, \mathbb{R})$ (i.e. they are *linear* forms on the tangent vector space generated by vector fields ξ_j.

The initial purpose of old Analytical Mechanics was the study of solutions of structural motion's equations given initially (in absence of external forces) by the Hamilton Jacobi equations:

$$\begin{pmatrix} \dot{\underline{q}}(t) \\ \dot{\underline{p}}(t) \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial \underline{q}} \\ \frac{\partial H}{\partial \underline{p}} \end{pmatrix} =: \nabla_J H \ ,$$

where $H : TM \to \mathbb{R}$ is a Hamiltonian function (in fact a one-form) and $\nabla_J H$ is called the symplectic gradient, giving a *Hamiltonian field*. The simplest Newtoniana "example" for conservative systems is given by the total energy $E_{tot} = E_{pot} + E_{kin}$ (whose derivative gives the Newtonian force) whose solutions give iso-energy levels. For quadratic expressions, soltuions can be easily describe. In this way, one can explain propagation phenomena, involving light (Huygens, Newton), sound (Barrow) or, a little bit later, the simplest equations of continuous fluids (Euler).

The problem becomes non-trivial but tractable in presence of external forces, where one introduces "torques" to balance the system. However, the problem becomes almost untractable in presence of several Hamiltonian fields interacting between them or, even worse, for Hamiltonian of degree three or four. In these cases solutions of maximal dimension called Lagrangians for even dimension and Legendrians for odd dimension can display quite complic behaviors on the base manifold, which must be "controlled".

Typical strategies for controlling singularities appearing in the envelopes of geometric elements (tangent vs normal lines, osculatrix circles or spheres, e.g.) are based on the identification, classification and construction of (uni)versal foldings which allow their "stabilization" in a similar way to the case of "simple" singularities developed in the precedent module.

The apparition of singularities in propagation phenomena was well known at the early years of the 18th century in regard to caustics (envelopes of normal lines) or evolvents (envelopes of tangent lines). They were meaningful for the optimal design of optical lenses (avoiding aberratioins) or the optimal design of transmission mechanisms.

The simplest example (appearing in the precedent mdoule), corresponds to the caustic of the parabola $y = x^2$ which is a translation of the cusp'$y^2 = x^3$ called the semicuspidal curve by I.Newton. In this case, the caustic gives the support for the "aberration locus" of the lens, whose singular point must be avoided for observer's loclaization. Alternately, the singular point is used to concentrate energy for incident light on a parabolic mirror for energy applications (solar energy or laser, e.g.)

Geometrical optics and its evolving phenomena linked to wave fronts provide the foundations for the development of applications to

1. Analysis of materials, including chromatography for separation in Computational Differential topology B_{13}.
2. Capture of radiometry in digital images or video sequences for Image Based Rendering (IBR) or Video Based Rendering (VBR) to obtain more realistic 3D Reconstructions B_{22}.
3. Non-linear dimensionality reduction for the Perecpetion-Configurations-Working-Action (PeCWA) cycle linking the coarse hierarchy $\mathcal{P} \to \mathcal{C} \to \mathcal{W} \to \mathcal{A}$ in Robotics B_3, and their pseudo-inverses away the singular loci for each proper morphism.
4. Synthesis of new graphical multimedia contents in Computer Graphics B_4 by using advanced AI developments based in Deep Learning.

Obviously, one must start with “basic examples” by imposing constraints about characters or invariants, in the same way as in the precedent module A_{43} where one bounds corank and codimension for function germs. Furthermore, it is necessary to develop strategies linking continuous models (developed in this module) with discrete experimental data where statistical techniques (for segmentation, clustering, and weighting) play a fundamental role.

The simplest illustrative examples appear in regard to semi-automatic understanding images in computer Vision B_2, where one starts with the simplest cases corresponding to the following values for (n, p):

- $(2, 2)$ in regard to the comparison of two digital images by using their support as a bitmap.
- $(3, 2)$ where the digital image is understood as a projection of a bounded region of the “world” on the image plane for extraction of “features”. A mobile camera generates a uni-parametric family of $(3, 2$ maps, with emerging and vanishing events which are modelled as singular multigerms.
- $(3, 3)$ to compare two reconstructions to detect modifications, propagate lacking information or generate “new” geometric contents (where radiometry is a superimposed layer on the underlying geometric model).
- $(4, 3)$ to detect “events” in evolving volumetric objects in terms of singular map-germs.

More details about these topics will be developed in the chapter 7 of this module; applications to Economic Theory and and biomedical sciences are developed in the chapters 8 and 9, respectively.

0.2. Outline of the chapter

In addition of this introduction and a final section about Complements (Conclusions, Practices, Challenges and References), this chapter has the following four sections:

1. A topological approach to classification.
2. Some connections with Dynamical Systems.
3. Singularities in Mechanics.
4. Singular map-germs in Engineering.

The two first sections have a standard contents, and they can be considered as a natural extension to singular map-germs $f \in C^r(n,p)$ of methods developed in the precedent module A_{43} for singular function germs $f \in C^r(n,1)$.. Main novelties appear in regard to the last two sections, where one introduces some motivations arising from Analytical Mechanics (extending wavefronts of Geometrical Optics) and some Engineering areas (appearing the part II of these notes), respectively.

Mechanics is ubiquitous in Engineering. Thus, all developments involving singular maps in Analytical Mechanics can be applied their corresponding applications in Engineering. Fromm the middle of the 20th century, the development of Control and Optimization strategies pose new challenges wich require a matrix reformulation of classical approaches involving to once a control point to be tracked and controlled.

Central issues such as stability and deformations must be reformulated in matrix terms to give response to problems in different Engineering areas. To fix ideas, we will paid attention only to those matters which are related to modules B_i developed in the second half of these notes, by ignoring developments in Nanomaterials, Biotechnology, or Chemical Engineering, between others. In the last section of this introductory chapter we devote a subsection to each one of the following matters: B_1 (Computational Mechanics of Continuous Media), B_2 (Computer Vision), B_3 (Robnotics) and B_4 (Computer Graphics) [6] .

Topology of singular map germs $f \in C^r(n,p)$ can be motivated from properties of maximal dimension solutions of Analytical Mechanics which are labelled as Lagrangian vs Legendrian manifolds of the Phase space $P = TM$. This viewpoint is a natural extension of singularities appearing in front waves in Geometrical Optics, whose basic aspects have been developed in the precedent module A_{43}. From the Klein's viewpoint (characterization of Classicla Geometries in terms of Groups),

- the *Symplectic Geometry* can be characterized by the preservation of the 2-differential symplectic form ω given locally by $\omega \mid_U = \sum_{i=1}^{m} dq_i \wedge dp_i$ which is equivalent to the preservation of Hamilton-Jacobi motion's equations.

[6] An introduction to each one of them can be found in my web site

- the *Contact Geometry* can be characterized by the preservation of the 1-differential contact form α given locally by $\alpha\mid_U \sum_{i=1}^m p_i dq_i$, corresponding to a motion which is constrained to a displacement preserving the contact with a hypersurface in Phase space P.

Obviously, $d\alpha = -\omega$ which gives a structural relation between both Symplectic and Contact Geometries. Thius simple feature has deep implications in common phenomena appearing for the classification of singularities linked to Lagrangian and Legendrian wavefronts in the applications.

The re-formulation of Symplectic Geometry in functional terms is performed in terms of Poisson structures which are defined w.r.t. a Hamiltonian H (Newtonian or total energy functional defined as a map $H : TM \to \mathbb{R}$) or a more general Lagrangian L (the simplest example corresponds to $L = E_{cin} - E_{pot}$ to explain the energy exchange for falling bodies, e.g.). The Poisson formulation in functional terms provides the support for extending the Analytical Mechanics to Quantum Mechanics [7]. An adaptation of the Quantum approach provides the support for discrete models in Engineering which are based on configurations of particles.

0.2.1. A short overview of the module A_{44}

The module A_{44} contains the following chapters:

1. **Topological Properties** of $C^r(n,p)$ with a special regard to "relative genericity" appearing in regard to different kinds of stability, equivalence relations and deformations.
2. **finite determinacy** in terms of their equivalence w.r.t. jets. Reductive groups. Unipotent group actions.
3. **Classification** by using a finite-dimensional reduction of Lie groups. Construction of Versal foldings.
4. **Infinitesimal criteria for Deformations**, by developing Lie algebras actions for an effective control of deformations.
5. **elements of Analytical Mechanics** with Symplectic and Contact Geometries and the main paradigms. Simplectization and Contactization of Geometry. Lagrangian and Legendriansingularities.
6. **Applications to Theoretical Physics** illustrated by some applications of Qauntum Mechanics to Physics of Materials and Physical-Chemistry.
7. **Applications to IST** (Information Society Technologies) with a special regard to the topology of Signals B_1 , events in Computer Vision B_2, Singularities in MIMO systems for Control in robotics B_3, and evolving space-time animation in Computer Graphics B_4

[7] First formulations were given in matrix terms by Heisenberg, Von Neumann et al; see my notes of the module A_5 for details and references.

8. **Applications to Economic Theory:** for competitive vs Cooperative systems appearing in Micro vs MacroEconomics, International Trade and Financial Economics. Modeling Social and Biological Complex Systems. Robust vs Adaptive Control

9. **Applications to Biomedical Sciences** centered on mutations at different scales, including some mathematical aspects of Genetic Algorithms, Evolutionary programming and Self-organziing Maps, in regard to paradigms develope in ML along nineties.

The first four chapters contain the basic notions of this knowledge subarea. The fits starts the transitions towards applications with a reformulation of theoretical aspects in terms of Analytical Mechanics. The last four chapters are focused towards some temptative applications which are usually disregarded in textbooks.

0.2.2. Methodological issues

From the *local point of view of singularities*, it affects the study of orbits by the action of Lie pseudo-groups (initially simple) whose algebraic classification was carried out by E.Cartan as an extension of Killing's classification of the simple Lie algebras. The infinitesimal version of this approach (carried out by Bruce, DuPlessis and Wall in [Bru87]) emphasizes the algebraic properties (nilpotency) of the Lie algebras responsible for determining the k-jets.

Currently, algebras of nilpotent vector fields are known that preserve the tangent space $T_f\mathcal{R}f$ for a simple singularity [8], whose methods are applicable to any type of singularity. However, the tools of (representation of) nilpotent Lie algebras necessary to manage this information have not yet been developed.

From the *global point of view of varieties* (even more interesting), simplectization (resp. contactization), that is, the introduction of a symplectic structure (resp. contact) on the cotangent bundle T^*M of a manifold M suggests that any differentiable manifold should have corresponding symplectic and contact versions (this idea is again due to V.I.Arnol'd). The extension of this idea to the Differential Topology leads to construct a Symplectic Differential Topology (Lagrangian resp.), and to study the Lagrangians (Legendrian resp.) as integral solutions of maximum dimension of the associated symplectic (contact resp.) structure.

0.2.3. The interplay between Geometry and Topology

The Symplectic Geometry can be characterized by the preservation of the 2-differential symplectic form ω given locally by $\omega\mid_U = \sum_{i=1}^m dq_i \wedge dp_i$ which

[8] For an explicit description see Tomás Pérez's Doctoral Thesis (unpublished), on which Arnol'd himself commented to JF that this is the Mathematics of the 21st century (private communication in Trieste, 1991)

is equivalent to the preservation of Hamilton-Jacobi motion's equations. Similarly, the Contact Geometry can be characterized by the preservation of the 1-differential contact form α given locally by $\alpha\mid_U \sum_{i=1}^m p_i dq_i$, corresponding to a motion which is constrained to a displacement preserving the contact with a hypersurface in Phase space P. Obviously, $d\alpha = -\omega$ which gives a structural relation between both Symplectic and Contact Geometries.

A modern synthetic treatment of Symplectic and Contract Geometry is performed in terms of the Moment map. Furthermore its intrinsic character, an advantage of this approach consists of the existence of a locally homogeneous structures, which can be translated to structures of G-orbits. These structures are well known for the reductive case, i.e. for homogeneous spaces G/H such as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (direct sum of Lie algebras).

An advantage of the reductive case consists of providing a "decouling" which eases the explicit computation of geodesics. However, even in the two-dimensional case, the situatioin is not quite elementary: Euclidean and Lobachevsky planes admit a reductive decomposition, whereas the cylinder or the projective line do not admit a reductive decomposiiton [9]

From an infinitesimal viewpoint, the re-formulation of Symplectic Geometry in functional terms is performed in terms of Poisson structures which are defined w.r.t. a Hamiltonian H (Newtonian or total energy functional defined as a map $H : TM \to \mathbb{R}$) or a more general Lagrangian L (the simplest example corresponds to $L = E_{cin} - E_{pot}$ to explain the energy exchange for falling bodies, e.g.).

The Poisson formulation in functional terms provides the support for extending the Analytical Mechanics to Quantum Mechanics [10]. An adaptation of the Quantum approach provides the support for discrete models in Engineering which are based on configurations of particles.

Mechanics is ubiquitous in Engineering. Thus, all developments involving singular maps in Analytical Mechanics can be applied their corresponding applications in Engineering. Fromm the middle of the 20th century, the development of Control and Optimization strategies pose new challenges wich require a matrix reformulation of classical approaches involving to once a control point to be tracked and controlled.

Central issues such as stability and deformations must be reformulated in matrix terms to give response to problems in different Engineering areas. To fix ideas, we will paid attention only to those matters which are related to modules B_i developed in the second half of these notes, by ignoring developments in Nanomaterials, Biotechnology, or Chemical Engineering, between others. In the last section of this introductory chapter we devote a subsection to each one of the following matters: B_1 (Computational Mechanics of Continuyous Media),

[9] This remark is immediately extended to Grassmannians or Flag manifolds; see the module A_{24} (Geometric Topology).

[10] First formulations were given in matrix terms by Heisenberg, Von Neumann et al; see my notes of the module A_5 for details and references.

B_2 (Computer Vision), B_3 (Robnotics) and B_4 (Computer Graphics) [11] .

0.2.4. Some illustrative examples

In the case of Symplectic Geometry, ordinary vector fields are replaced by Hamiltonian fields and the Lie bracket is replaced by the Lie-Poisson bracket of Hamiltonian functions. This approach is found developed in different manuals both from the local or infinitesimal point of view (Arnol'd) and algebraic within the framework of the Geometry of the Moment Application (Guillemin and Sternberg, Marsden and Ratiu, Weinstein) [12]. In the same way, it is necessary to replace the study of the submanifolds of differentiable manifolds by the study of the Lagrangian (intersection of) submanifolds of symplectic manifolds (Weinstein).

Unfortunately, the currently available knowledge about Lagrangian submanifolds of symplectic varieties or Legendrians of contact varieties is very scarce. We hardly know anything beyond some homogeneous varieties such as Grassmannian or projective space, to cite two classic examples of both situations. Again, the most valuable information relating to more complicated spaces such as k-jet spaces is due to Arnold's, but the corresponding results affect areas of research that go beyond the objectives of an Introductory Course like this.

The fourth section of this chapter is devoted to display some examples of singularities arising from the discriminant loci of smooth maps (whose componentes are polynomials of low degree). A more systematic treatment of these examples will be developed in the matters B_1 (Computational Mechanics of Continuous Media), B_2 (Computer Vision), B_3 (Robotics) and B_4 (Computer Graphics).

Another application which is transversal to all the above matters concerns to the use of Singular Map-Germs in regard to some basic issues of AI in the Deep-Learning framework. An artificial Neural Network (ANN in successive) is composed by successive layers, whose cells are connected between them. In artificial models, very often one takes "fully connected" layers, i.e. each weighted cell of the k-th layer is connected with all cells of the the $(k+1)$-layer.

Weights of ANN are initialized in a random way, and they are corrected by following different strategies along the (supervised vs non-supervised) learning procedures. In the Deep Learning framework, it is not necessary wait at the end of processing, and their very strong parallelism allows a learning in two or three consecutive layers by using an adaptation of old CNN (Convolutional NN) and RNN (Recurrent NN). In this way, it is possible to perform very cmplex recognition tasks.

If we look only at two consecutive layers, one has a discrete version of maps $\mathbb{R}^n \to \mathbb{R}^p$, where n and p are the number of non-correlated cells in the source

[11] An introduction to each one of them can be found in my web site

[12] For a elementary introduction see chapter 7 of module A_{12} (Linearization) of the matter A_1 (Differential Geometry)

and target layers of a CNN. If we want to incorporate the orientation, it is convenient to take

In more modern terms, the Symplectic Geometry can be characterized by the preservation of the 2-differential symplectic form ω given locally by $\omega\mid_U = \sum_{i=1}^{m} dq_i \wedge dp_i$ which is equivalent to the preservation of Hamilton-Jacobi motion's equations. Similarly, the Contact Geometry can be characterized by the preservation of the 1-differential contact form α given locally by $\alpha\mid_U \sum_{i=1}^{m} p_i dq_i$, corresponding to a motion which is constrained to a displacement preserving the contact with a hypersurface in Phase space P. Obviously, $d\alpha = -\omega$ which gives a structural relation between both Symplectic and Contact Geometries.

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0.3. References

The references are not exhaustive. They must be understood as an invatation to the reader to explore this subsject, according to his/her interests to construct his/her own representation of this knowledge subarea.

[13] First formulations were given in matrix terms by Heisenberg, Von Neumann et al; see my notes of the module A_5 for details and references.

[14] An introduction to each one of them can be found in my web site

0.3.1. Basic bibliography

Only some textbooks are included. For an enlarged bibliography, see the subsection §5,4. References for meaningful research articles are included as footnotes. The most important absence is linked to the series of articles written by J.Mather between 1965 and 1970, which appear as footnotes.

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0.3.2. Software resources

To my knowledge, nowadays there are no textbooks for a computational treatment of Singular map Germs. Singular provides a support for some algebraic issues in the framework of Computational Algebra. Beyond some basic cases related with Singular Function Germs involving $f \in C^r(n, 1)$, the number of relevant contributions for $C^r(n, p)$ for $p \geq 2$ is very scarce. To my knowledge, there is no reference in the OOP framework.

It is necessary an effort to develop software at least for corank 2 and low codimension map germs. These developments would must have in account preliminary contributions which are curreently being develope in B_{13} (Computational Differential Topology) inside the matter B_1 (computational Mechanics of Continuous Media). Their foundations can be found in

[Her13] M.Herlihy, D.Kozlov, S.Rajsbaum: *Distributed Computing Through Combinatorial Topology*, 2013.

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It is necessary to develop more specific software for a computational treatment of most aspects developed in this module. To my knowledge, the only reference is Singular. Additional informaiton about other software packges is welcome.

Final remark: Readers which are interested in a more complete presentation of this chapter or some chapter of the module A_{23} (Topology of Graded Complexes), must write a message to franciscojavier.finat@uva.es or to javier.finat@gmail.com