A340 Singular Function Germs

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Índice

Previous remarks: These notes correspond to an introduction to the module A_{43} (Singular Function Germs) of the matter A_4 (Differential Topology). Along this chapter one makes a short presentation of basic notions and results arising from the topological analysis of function germs. It is necessary to have basic knowledge of Muyltivariate Analysis, Basic Algebra, Group Actions and General Topology which will be used along the module.

Subsections or paragraphs marked with an asterisk ($*$) display a higher difficulty and can be skipped in a first lecture.

0.1. Introduction to the chapter A43

Functions $f : X \to R$ defined on a space X and taking values in r provide a quite general method to obtain properties about functionals evaluating the behaviour of evolving "objects" represented by a topological space X. In geometric contexts, X is usually a PS manifold, an algebraic or analytical variety X. In the discrete case, X can correspond to a symbolic representation of an object or the whole scene, given by a skeleton, a graph or lattice, e.g. In Experimental Sciences, Probability and Statistics provide a nexus between discrete and continuous representations.

In the continuous case, classical approaches for R are given by real $\mathbb{R}$ or complex numbers $\mathbb{C}$; other choices for the discrete case correspond to natural $\mathbb{N}$, integers $\mathbb{Z}$, binary $\mathbb{Z}_2$ (or more generally, $\mathbb{Z}_p$ for prime p), or rational numbers $\mathbb{Q}$. In this module we restrict ourselves to the real and complex cases, and we focus towards the local case. Typical examples for functionals are given by measures or energy functions, e.g.

We denote by means $C^r(n, 1)$ the set of functions $f : \mathbb{K} \to \mathbb{K}$ of functions of class C^r, i.e. continuous and with continuous derivatives till order r. Some of the most important cases correspond to smooth functions for $r = \infty$, analytic functions for $r = \omega$, i.e. having a (non-necessarily unique) convergent Taylor development at each point, and algebraic functions given by polynomials (truncation of analytic functions) in n variables.

Local Cartesian coordinates around a point $\underline{p}$ (usually taken as the origin or coordinates) are denoted by $\underline{x} = (x_1, \ldots, x_n)$ for the real case, and $\underline{z} = (z_1, \ldots, z_n)$ for the complex case, where $z_j = x_j + iy_j$ for $1 \leq j \leq n$. In some cases, it is more convenient use Spherical or cylindrical coordinates. All of them provide a description of objects in terms of "level surfaces" $f^{-1}(r)$ for regular values $r \in R$ of f. The topology of the level surface changes at critical values $c \in R$, whose inverse image is no longer a submanifold, but a subvariety. In this module one extends the basic "examples" of Morse theory to more complex singularities of the function germ $[f]$ for finitely determined $f \in C^r(n, 1)$

An intrinsic approach to the study and modelling of "objects" (manifolds, varieties, spaces) X is not enough for applications, requiring the interplay between internal structure and external environment. Tangent and normal bundles provide a solution for the smooth case, which is useful for basic kinematic issues. A more practical approach consists of "acting on objects" and measure the "reactions" (deformations, motions, e.g.) on X.

The simplest models for actions on X are given by C^r-functions $f : X \to \mathbb{K}$ defined on X taking values in a field $\mathbb{K}$. Along this module A_{43} one supposes that $\mathbb{K}$ is the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. More general cases have been developed in the precedent matter A_3 (Algebraic Geometry).

In the same way as for any other Geometry or Topology, the *main problems* are to classify objects and morphisms (maps not everywhere defined). This classification is performed up to C^r-equivalence. Hence, the choice of C^r-category involves to objects, maps and regular transformations. The most importante cases for this matter A_4 (Differential Topology) are the following ones:

- *Topological category* corresponding to $r = 0$ where objects are topological spaces X, morphisms are continuous maps $f : X \to Y$, and C^0-equivalences are homeomorphisms, i.e. bijective and bicontinuous maps.
- *Smooth category* corresponding to $r = \infty$ where objects are manifolds M, morphisms are smooth maps $f : N \to P$, and C^∞-equivalences are diffeomorphisms, i.e. smooth homeomorphisms with inverse smooth, also.
- *Analytic category* corresponding to $r = \omega$ where objects are analytic varieties X, morphisms are analytic maps $f : X \to Y$, and C^ω-equivalences are bi-analytic maps, i.e. analytic homeomorphisms with inverse analytic.

Algebraically based geometric models have been developed in the matter A_3 (Algebraic and Analytic Geometry) where the classification is performed up to birational equivalence, i.e. rational maps with rational inverse. From the algebraic viewpoint, the classification is performed on the field $k(X)$ of rational functions defined on the variety X. Anyway, the hypersurface $f(\underline{x}) = 0$ in $\mathbb{R}^n$ or $f(\underline{z}) = 0$ in $\mathbb{R}^n$ provides the starting point for connecting with the geometric viewpoint developed in Differential Geometry A_1 or the GAGA framework A_3. This remark is naturally extended to local complete intersections in [Loo84][1]

A basic difference between the smooth and the analytic categories concerns to the existence of once a Taylor development (not necessarily convergent) in the smooth category, in contrast with the possibility of several locally convergent Taylor developments (one per branch) for the analytic case. If the base field is given by the complex numbers, analyticity conditions are equivalent to Cauchy-Riemann equations, which can be considered as an extension of Symplectic Geometry to the non-linear case. The equivalence between analytical and holomorphic functions has deep consequence for linking local and differential properties.

In regard to the precedent modules, the approach developed along the modules A_{41} (Basic Differential Topology) and A_{42} (Fiber Bundles) of the matter A_4 (Differential Topology) has a predominantly global character. However, in this module A_{43} (Singular function germs) and the following one A_{44} (Singular Map Germs), one adopts a local viewpoint.

A fusion of both local and global approaches is performed in the module A_{45} in the GAGA framework A_3. Furthermore, most contents has a static character; the extension to dynamic aspects (including interaction issues) is performed in the module A_{46} (Dynamical Systems). The simplest dynamical models are linked to conservative systems given by the gradient ∇f of a potential function f, which are naturally extended to the symplectic gradient $\nabla_{\mathbf{J}_n} f := \mathbf{J}_n f$.

[1] E.J.N. Looijenga: *Isolated singular points of complete intersections*, Lect.Notex Vol.77, London Math Soc, 1984.

Unfortunately, global methods based on superimposed structures (vector or principal bundles, sheaves, topological fibrations) are not easily extendable to spaces of functions. The most immediate general analogue is given by spaces $J^k(n,p)$ of k-jets, i.e. formal truncated Taylor developments corresponding to the action of differential operators on $C^r(n,p)$ of order $\leq k$.

In particular, the information associated to 1-jets (or its dual version) is essentially the same that the information contained in sections of the tangent (resp. cotangent) bundle, whose formal products give the tensor algebra constructed on any manifold M or variety X. An advantage of k-jets consists of the capability of managing differential operators of order ≥ 2 which can not be described as tensors.

The choice of a local framework for this module implies that all invariants for classification have a local character, i.e. they are linked to properties of the local ring $\mathcal{E}_n$ (for the smooth case) or $\mathcal{O}_n$ (for the analytic case) of regular functions at each point $x \in X$. A basic idea consists of extending relations between critical points and critical values for Morse functions $f : M \to \mathbb{R}$appearing in the chapter 5 of A_{41} to non-Morse functions [2]

Relations between local and global invariants for hypersurface germs are only described for some particular cases involving hypersurfaces, and are given in differential terms (Milnor, Tjurina, e.g.) or in analytic terms (Mond, Damon, Gaffney, between others) in regard with their applications to some problems appearing in GAGA. Some applications to other scientific or technological areas are described in the last three chapters of this module.

The reduction from global to local issues involves to the study of regular functions, and maps $\varphi : X \to Y$ which are replaced by $\varphi^* : \mathcal{E}^n_{Y,y} \to \varphi^* \mathcal{E}^n_{X,x}$ in the smooth case, or $\varphi^* : \mathcal{O}^n_{Y,y} \to \varphi^* \mathcal{O}^n_{X,x}$ (in the analytic case), where $\varphi^*(f) := \varphi \circ f$ for any regular (smooth vs analytical) function defined on the germ (X, x).

The unified approach allows the reformulation of differential and analytic properties in terms of A-modules (derivations or differentials, e.g.) where A is the local ring corresponding to $\mathcal{E}^n$ (smooth case) or $\mathcal{O}^n$ (Analytic case).

So, the algebraic language (based in modules and their formal sums as graded algebras) provides a unified treatment to describe local properties involving varieties and morphisms. By using multi-germs instead of germs one can manage different processes holding in a simultaneous way. Their extension to the simplest dynamical systems (given by once an equation) is performed by analyzing the zero locus of the function-germ (as the equilibrium locus of the system). A nontrivial problem consists of relating generic perturbation of the function with properties of solutions for the corresponding dynamical system.

The basic idea consists of interpreting qualitative changes (corresponding to changes of state involving materials or the chapter, or phase transitions involving the behavior, e.g.) as singularities of functions in a first step. Next, as

[2] This approach is a particular case of relations between the Ramification and Discriminant Loci appearing in the GAGA context which has been developed in the module A_{32} (Quasi-Projective Varieties).

singularities or maps A_{44}, and try of identifying their "stratified nature" A_{45} to understand the space-time evolution of different phenomena (including interactions). Even under conditions of continuity, matter is neither homogeneous nor isotropic; therefore, the behavior is not uniform. Thus, the analysis of singularities corresponding to the response w.r.t. different scalar fields provides information about a matter or a structure initially unknown.

In this way, singularities provide models for evaluating, representing, constraining, and controlling qualitative changes in the characteristics or behaviour of the system. Typical examples are given by changes of state, phase transitions or dynamic bifurcations in relation to the propagation and interaction of wave fronts with matter. In these notes an approach based on products of fields (scalar, vector or covector) is adopted.

The ubiquity of the fields in Physics, Engineering, Chemistry, Biology or Economic Theory, and their use in Engineering, makes the study of singularities essential to understand any type of phenomenon that presents a space-time evolution with changes of state or transitions of phase.

To fix ideas and by complementarity with the materials presented in Algebraic Geometry A_3, in this module A_{43} and in the next ones we focus our attention on the singularities of function germs A_{43}, of map germs A_{44} and of dynamic systems A_{46}. In A_{45} one adapts results of the smooth case to the singular case, which requires more advanced local analytical methods, with a special attention to the complex case: so, one recovers multiple connections with the Algebraic Geometry of Complex Varieties.

Anyway, the *fundamental problems to solve* remain the same: Characterization and Classification, but now of germs. In particular, a central problem is the identification and analysis of hierarchies for canonical forms represented by adjacencies for $\mathcal{B}$-orbits for some of the usual equivalence relations. Different kinds of (topological, algebraic, analytic) invariants and relations betwen them provide keys for effective classification, at least from a theoretical viewpoint.

0.1.1. Reducing the classification problem

A first reduction for singularity classification problems consists of replacing the study of a map $f \in C^r(N,P)$ at a point by the *germ of a map* $f \in C^r(n,p)$ on $(x, f(x)) \in \Gamma(f)$ (graph of f) [3]. This local reduction allows us to express the germ as equivalence classes $[f]$ of $f \in C^r(n,p)$ where $n = dim(N)$ and $p = dim(P)$. If we identify the germ $[f]$ with its representative f, we can denote by $f : (\mathbb{K}^n, \underline{0}) \to (\mathbb{K}^p, \underline{0})$ to the germ of an application with $f(\underline{0}) = \underline{0}$

The reduction to germs eases the qualitative management of changes or transitions between adjacent types of singular points, whose properties only

[3] Two functions $f, g \in C^r(n,p)$ defined in two open U, V have the same seed in $x \in U \cap V$ if and only if there exists an open $W \subset U \cap V$ such that $f\mid_W = g\mid_W$.

Therefore, the germ is an equivalence class corresponding to the "projective limit" in Topology.

depend on the local algebra at $x \in X$. Initially, on considers the local ring $\mathcal{E}_x^n$ or $\mathcal{O}_x^n$ with its the graded structure linked to successive quotients $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ of consecutive powers of the maximal ideal $\mathfrak{m}$

(*) In more advanced settings, we will consider the graded structure linked to I^k/I^{k+1} representing e.g. the Jacobian ideal. An advanced motivations arises from the considefation of several "simultanoeus events" (corresponding to disapparition of "objects" e.g.) along a video sequence, e.g.. When there are several singular points we speak of *multi-germ* with the corresponding to a finite number of semi-local rings [4]. This topic and the analysis of non-isolated singularities are addressed at the end of module A_{44}.

To fix ideas, initially we only work with germs of isolated singularities that we assume centered on the origin $\underline{0} \in \mathbb{K}^n$ corresponding to the source and target spaces for the map germ $f \in C^r(n,p)$. The use of the algebraic language is key to provide a synthetic proof of most of the topological results in an intrinsic way, that is, independent of the chosen coordinate system. By using the algebraic language it is possible to extend the analysis of hypersurfaces with isolated singular point at the origin (corresponding to singular function-germs) to complete intersections with isolated singular points [Loo84]

As first conclusions,

- the problem of *characterizing germs* of maps $f \in C^r(n,p)$ is usually expressed in terms of Local Algebra that affects both the local ring in the germs of the spaces of departure and arrival, as well as the derivations used to represent the differential properties.
- The topological cases $r = 0$, smooth $r = \infty$ and analytical $r = \omega$ present specific characteristics that are highlighted throughout each of the two modules.
- In the smooth and analytic cases, Local Algebra provides a common language that facilitates formulation and transfer (when possible) between results.
- The introduction of local invariants associated to the singular point (when they are isolated) or invariants of associated graded algebras provides criteria to describe the basic types and their canonical forms.

Their extension to the non-isolated case will be performed at the second part of the module A_{44} in regard to some deep connections with GAGA.

0.1.2. Equivalence relations

Groups of C^r-equivalences $\mathcal{B}$ act on $C^r(n,p)$ in a deocupled way (direct group of actions on the source and target spaces of f), or in a coupled way

[4] A local ring has once a maximal ideal $\mathfrak{m}$ (corresponding to a branch); a semi-local ring can have a finite number of maximal ideas.

(preserving the graph Γ_f or higher contact conditions) for the map $f \in C^r(n, p)$. The description of $\mathcal{B}$-orbits by these actions provides an equivariant approach to the space of map-germs.

So, the canonical forms are representatives of the $\mathcal{B}$-equivalence classes or orbits. Their local structure allows to compute the tangent space $T_f C^r(n, p)$. Therefore, the identification of singularities types provides the support for thee computaiton of numerical invariants for each $\mathcal{B}$-orbit. In this module we only study the case corresponding to the $\mathcal{R}$-action (action to the right of the homeomorphisms), that is, on the starting space given by

$$\mathcal{R} \times C^r(n, 1) \to C^r(n, 1) \;\mid\; (h * f)(\underline{x}) := f(xh^{-1}) \quad \forall h \in \mathcal{R} := Dif_0(\mathbb{K}^n)$$

In this case, the *fundamental problem to solve* is the $\mathcal{R}$-classification of the germs of functions that, initially and from the geometric point of view, correspond to hypersurfaces $f(\underline{x}) = 0$ with isolated singularities at the origin. Other relationships of interest for germ classification problems are the following ones:

- The $\mathcal{L}$-equivalence (acting on the arrival space) is applicable to the study of “path” spaces over a manifold that can be interpreted as trajectories or integral curves of vector fields.
- The $\mathcal{A}$-equivalence corresponding to the left-right action is of interest for a simultaneous study of paths and constraints (corresponding to vector fields and differential forms, e.g.) that are considered decoupled (direct product of actions).
- The $\mathcal{K}$-equivalence or contact equivalence is of interest to study the effect of coupling conditions (feedback between systems in scientific or technological applications, for example) that are evaluated on the graph Γ_f of $f \in C^r(n, p)$.

Each one of them can be restricted to C^r-equivalences leaving invariant a tensor, in other words, it can be restricted to ta Classical Group [5]. Some typical examples concern to the Special Linear Group leaving invariant the volume form, and the Symplectic vs Contact Groups leaving invariant the symplectic 2-form ω and the contact 1-form α. The analysis of symplectomorphisms (resp. contactmorphisms), i.e. diffeomorphisms leaving invariant ω (resp. α), plays a fundamental role in the study of Lagrangian (resp. Legendrian) varieties in modern Analytical Mechanics (V.I.Arnold and his school).

Unfortunately, the analysis of singularities linked to eventually degenerating volume forms has been disregarded. The ideal volume preservation is the key for “perfect fluids” (Liouville Theorem), and it can be expressed in a differential or an integral way. The last one is the most natural for Variational issues. From

[5] See the chapter 5 of the module A_{12} (Linearization) of the matter A_1 (Differentiable Manifolds).

a geometric viewpoint, last ones can be considered as the problem of finding minimizers of an integral functional

$$F(S) \ := \ \int_{x \in S} F(x, S(x)) dV^m_H x \ ,$$

where $S(x)$ represents the m-dimensional linear tangent subspace to a "flow" $S(x)$ (a surface in the classical case) at the point $x \in X$, and dV^m is the volume element linked to a m-dimensional Hausdorff measure ds^2_H or, alternately, to a distance on a Grassmann manifold. Last one can be extended to the statistical framework and several applications will be developed along the part II, jointly with their extensions to Flag Manifolds to include completions of possible degenerations.

A volumetric measure is given by a determinant whose global evolving version correspond to the canonical divisor in A_{3who} or the determinant line bundle $det(E)$ of a vector bundle E in A_{42}. Now, the novelty consists of a functional determinant can can have singularities, to be classified by the corank. The rank stratification (Thom-Boardman) induces a degeneration in the above integral functional, corresponding to "collapsing flows".

This simple remark provides a structural connection between global aspects [Tho56] and local aspects [Mil68]. Volumetric degenerations are managed in terms of nilpotent operators linked to dissipative phenomena. Their estimation is performed in terms of degenerations of Kullback-Leibler divergence (see the subsection §4,1 for more details).

The overlapping of topological, differential and algebraic techniques on these objects (seeds of functions or of hypersurfaces) reveals the richness of these objects. This module presents different approaches to the problem depending on the tools used. Special attention is paid to the case of simple singularities of functions for which eighteen equivalent characterizations are available (at least according to A.Durfee) (we do not review all of them, obviously).

The second part of this module combines the approaches of R.Thom on the one hand and V.I.Arnold on the other. Firstly, basic algebraic results are presented that allow recovering topological invariants for the case of isolated singularities. Morse singularities are non-degenerate critical points of functions $f : M \to \mathbb{R}$, that is, the Hessian matrix of f is non-degenerate, so the critical points are classified by the signature .

The next case in difficulty corresponds to corank 1 singularities , that is, such that $rk(Hess(f)) = m-1$. In this case, it is possible to easily construct the most general possible (universal) deformations of f -called unfoldings, déploiements- and control their possible evolution in terms of the pathologies that occur when projecting onto a space of parameters Λ. This study was initially carried out by R. Thom who, in view of the sudden changes in the topology of the solutions, gave it the name of Catastrophe Theory.

Currently, this study is part of the Bifurcation Theory. The review and reformulation of this theory by V.I.Arnol'd and his school has revealed the ubiquity of the ADE classification in multiple areas of Mathematics, showing unsuspected structural relationships previously scattered in the literature. The list provided by Arnol'd was extended by A.Durfee to 18 types, but recent advances in algebras of deformations, quantization of gravitational theories, and mirror symmetries make it difficult to know the number of equivalent characterizations.

0.1.3. Some geometric aspects

For any $f \in C^r(n,1)$, i.e. a function $f : \mathbb{K}^n \to \mathbb{K}$ of class C^r, the set of points $(\underline{x}, y) \in \mathbb{K}^n \times \mathbb{K}$ such that $y = f(\underline{x})$ can be interpreted as a hypersurface in $\mathbb{K}^{n+1}$. The most known cases correspond to plane curves in $\mathbb{K}^2$ and surfaces in $\mathbb{K}^3$ have been studied in the Differential Geometry of Curves and Surfaces for the real case in A_0, and in Algebraic Geometry for the complex case. In the last case, there appear singularities on the support X even for simple rational curves.

Up to veery simple cases (uniform rectilinear motion), evovling objects or maps display singularities for the kinematic or dynamic behavior, whose first order approach is given by the differential of local equations. Thus, singular loci (corresponding to the vanishing locus of the differential) display "qualitative changes" in the shape or the behaviour. This simple remark explains the ubiquity of singularities in all scientific and technological areas, and the need of characterizing, classifying and studying their evolution in the space-time to understand characteristics and evolution of complex systems.

Along the module A_{43} we are focused towards singularities of hypersufaces defined by a finitely dtermined germ function $f \in C^r(n,1)$. We paid special attention to singularities of curves and surfaces, and their evolution in terms of deformations as "generic" as possible.

The simplest example corresponds to Morse functions where singularities are non-degenerate critical points[6]. Excellent properties of the Morse case are due to the simple behaviour of the tangent space at critical points due to the non-degeneracy of the Hessian matrix in the smooth case.

The good behaviour at critical points can be extended to ordinary singularities with "different tangents" in the GAGA framework. The simplest example corresponds to a nodal curve, where the introduction of the slope m for the tangent at each branch provides a desingularization in the phase space which is locally parameterized by (x, y, m) fulfilling a contact condition.

A similar reasoning can be performed for hypersurfaces or "complete intersections" (locally defined by the intersection of transversal hypersurfaces (roughly speaking), by using the Nash transform). Unfortunately, the Jacobian ideal is not usually a "permissible" centre for blow-ups (is not regular in general). In particular, in presence of higher order contact between branches at a singular point, the Nash blowing-up does not provide a non-singular model (the Jacobian variety is usually singular).

[6] See the chapter 5 of the module A_{41} (Basic Differential Topology) for details and results.

Hence, the first case to be analysed corresponds to the overlapping of several branches for corank one singularities including Morse singularities as an almost trivial particular case. They are classified by the codimension. When one has only finite number of different types, one says that singularities are "simple". They appear in a lot of matehamtical reas, including the classification of semimiple Lie algebras, where there appear as A-D-E singularities, also.

For simple singularities one can give an explicit description of universal deformations,, i.e. such that any other deformation factors out through the universal deformation. In more general cases, one has only "versal" deformations (which are not unique). Instead of looking at the tangent space of a manifold M as in Differential Geometry, a basic geometric strategy consists of looking at the "tangent space" $T_f(\mathcal{R}f)$ to the $\mathcal{R}$-orbit $\mathcal{R}f$ of f, where $\mathcal{R} := Diff_0\mathbb{R}^n$ of diffeomorphisms fixing the origin $\underline{0} \in \mathbb{R}^n$ in the real smooth case. In this case, the $\mathcal{R}$-action on $C^r(n,1)$ is defined as follows:

$$\mathcal{R} \times C^r(n,1) \to C^r(n,1) \mid (f,h)(x) \mapsto (f \circ h^{-1})(x) := f(h^{-1}x) \; \forall x \in U \subset \mathbb{R}^n$$

and $\forall h \in Diff_0(\mathbb{R}^n)$. The canonical form of f is given by the simplest expressions for the elements of $\mathcal{R}f$ in the space of function germs, i.e. equivalence classes in $C^r(n,1)$, which is denoted as $\mathcal{E}_n$ for $r = \infty$ (smooth case) and as $\mathcal{O}_n$ for $r = \omega$ (analytic case). A typical "example" is given by all monic polynomials of degree $k+1$ in one variable x whose canonical form is $f(x) = x^{k+1}$ for any $k \geq 0$ and universal deformation given by

$$f(x,\lambda) = x^{k+1} + \lambda_1 x^{k-1} + \ldots + \lambda_{k-1}x + \lambda_k$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \Lambda \subseteq \mathbb{R}^\ell$ (space of parameters). In more general cases, the space Λ of parameters is not a Cartesian space, but a variety. The above examples corresponds to the A_k-singularity and it is the simplest one for the classification of non-trivial singularities. Thus, it reappears in different ways along this module. Several extensions of this example will appear along this module, jointly with some applications. The most popular are known as "catastrophes" in the Thom's terminology, which provide non-trivial models for conservative phenomena.

From a topological viewpoint (as an extension of the geometric approach), the role of general deformation is performed by the Milnor fibration for isolated singularities in the complex case. The existence of (uni)versal deformations is justified by finiteness results about the quotient $k[x]/Jac(f)$ (called the Milnor algebra), where $Jac(f)$ is the Jacobian ideal of f generated by $\partial f/\partial z_i$ for $1 \leq i \leq n$.

In local algebra, it plays a similar role to the tangent space but adapted to the case of finitely generated k-algebras. The topological study of an isolated singularity is performed by taking a small ball S_ε, cut a small environment of the isolated singularity and study the topology of the intersections.

The analysis of the local topology around the isolated singular point is performed in terms of a small tubular neighborhood of radius ε, whose "nerve" is the hypersurface with a degenerate critical point. Let us remark that, contrarily to the approach performed in the module A_{41}, the "nerve" of the tubular neighborhood is an eventually singular variety with an isolated singularity at the origin. The basic idea for the study of the topology of the corresponding *Milnor fibre* consists of relating real and complex aspects as follows [Mil68]:

1. One supposes the fiber $X_0 = f^{-1}(0)$ at 0 is the germ of an analytic map $f : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, \underline{0})$ has an isolated singularity at the origin [7].
2. The condition of isolated singularity at the origin is equivalent to the *Milnor algebra* $\mathcal{O}_n/Jac(f)$ has finite dimension, i.e.

$$\mu := dim_{\mathbb{C}}[\mathcal{O}_n/Jac(f)] < \infty$$

   where $Jac(f)$ is the Jacobian ideal of f, $\mathcal{O}_n$ is the local ring of (germs of) holomorphic (equivalently analytic) functions and $\underline{0} \in \mathbb{C}^n$, and μ is the *Milnor number* of f
3. Study the differentiable structure for the $(2n-1)$-dimensional real manifold $K_\varepsilon := \Sigma \cap X_0$, where $\Sigma = \mathbb{S}_\varepsilon(\underline{0}, \mathbb{C}^n)$ is a sphere of radius ε.
4. After associating the smooth manifold K_ε to the singularity (X, x), one must analyze the part of of the algebraic structure of the singularity determines the topological structure.
5. The precedent item is solved by using the homeomorphism between the pairs $(\mathbb{D}_\varepsilon, X_0 \cap \mathbb{D}_\varepsilon)$ and the cone on $(\mathbb{S}_\varepsilon, K_\varepsilon)$.

The last item of the precedent steps is the most advanced one and it is called the Theorem of Conical Structure (see Theorem 2.10 in [Mil68]). It can be understood as some adaptation of the construction of tubular neighborhoods in the smooth case (see module A_{41} for details) to the singular case. This result has deep implications in regard to the topology of singularities, extending old results of O.Zariski (1932) in terms of the topology of Knots which have been clarified along the early seventies by Le Dung Trang (more details in the Chapter 3).

0.1.4. Some applications

The most classical applications of singularities of function germs appear in Geometrical Optics (Newton, Huygens, e.g.) and some basic problems of Mechanics (Hooke, e.g.). In particular, the description of different kinds of (tangent

[7] This construction is immediately extended to isolated singularities in complete intersections [Loo84].

vs normal) envelopes shows from the beginning singularities which is necessary to identify and localize to avoid optical aberrations in lenses design, or maximal stress zones. Both kind os tangent and normal envelopes provide the foundations for the Intrinsic and Extrinsic approaches in any Geometric framework. The analysis of different types appearing in their space-time evolution makes part of the Differential Topology. A very good description can be found in [Bru9?] [8]

The connection with ODEs and PDEs appears later in regard with the integral formulation (Euler. Lagrange) and the differential formulation (Hamilton, Jacobi) of the Analytical Mechanics at the end of the 18th century and the first half of the 19th century. The analysis of envelopes of solutions for ODEs is a classical topic where geometric and analytic methods are overlapping with important milestones as the Cauchy's method of characteristics. A more synthetic and systematic unification is performed in the framework of Lie actions from 1880s. A basic distinction for some applications to Physics and Engineering considers two complementary approaches:

- *Differential approach:* The need of incorporating more complex behaviors that those linked to quadratic functionals (such as the Newtonian total energy), motivates the introduction of Hamiltonian functions $H : TM \to \mathbb{R}$ of degree ≥ 2. They provide more flexible patterns than those appearing in the Classical Morse Theory. Formal properties of Hamiltonian scalar fields are described in terms of Lie-Poisson algebras involving their corresponding Hamiltonian vector fields. Canonical types for function germs are adapted to Hamiltonian vector fields.
- *Integral approach:* In a complementary way, instead of looking at critical points of Morse functions $f : M \to \mathbb{R}$ on a compact manifold, one can consider critical points of integral operators corresponding to the minimization of measurements (length, area, volume) or geometric characteristics (different kinds of curvatures, e.g.). From a geometric viewpoint, they can be considered as the search for minimal values of a geometric flow involving volume or curvature elements. The "simplest example" correspond to geodesics [9]; More details involving other situations (minimal surfaces and curvature flows, e.g.) are developed in the module Computational Kinematics B_{14} and their applications.

In absence of external forces, integral and differential approaches are equivalent between them. The simplicity of differential formalism has contributed to the popularity of the differential approach. However, the integral approach in terms of volume or curvature flows is more meaningful in a lot of applications. In the global cas,e the Moment Map plays a fundamental role for the study of invariant functions on G-orbits. This remark motivates the adaptation of some

[8] Bruce and P.Gibllin: *Curves and Singularities (2nd ed)*, Cambridge Univ. Press, 199?

[9] In the chapter 3 of A_{10} (Differential Geometry of Curves and Surfaces) one can see eight equivalent characterizaitons of geodesics.

ideas for an equivariant approach to the classifications of function germs, which will be developed in the Chapter 6 of this module.

Coming back to the differential approach,, let us remember that the total energy Newtonian functional is extended to more general Hamiltonians $H : TM \to \mathbb{R}$ define don the Phase space $P = TM$ (total space of the tangent bundle τ_M of a smooth manifold M) fulfilling the Hamilton-Jacobi equations. Last ones describe the space-time evolution of a point $(\underline{qq}, \underline{p})$ in the phase space P according to the symplectic gradient $\nabla_J H := \mathbf{J}\nabla H$; in other words, the motion is ideally given by a conservative system [10].

The solutions of maximal dimension of Hamiltonian systems of equations in the Phase space (or its dual) are called *Lagrangians* (in absence of external constraints) or *Legendrian subvarieties* (in presence of "contact" constraint w.r.t. a hypersurface). Both of them display singularities which extend the singularities appearing in Geometrical Optics of basic Mechanics already described from the early years of the 18th century.

Thus, to control the space-time evolution of a system including possible degenerations, it is necessary characterize the singularities appearing in evolving integral subvarieties of $P = TM$, their projections on the base space M, and their possible evolution according to the hierarchy linked to adjacent types. If singularities are "simple" (finite dimension for the moduli space), adjacencies are well known, and their evolution can be described in terms of nilpotent vector fields preserving the $\mathcal{R}$-orbit (see chapters 7 and 8 of this module).

In presence of G-*actions* (linked to a classical group or first integrals, e.g.) one must analyze the behavior of $\mathcal{R}$-orbits w.r.t. the G-action. In this case, one has a double conjugacy whose study is a an extension of the double action of General Linear Groups acting on the source V and target space W of a linear map $\varphi : V \to W$. Roughly speaking, the $\mathcal{R}$-action allows to reduce to canonical types involving topological transformations on the source space, whereas other G-actions provide information about other properties linked to integrability issues in regard to dynamical systems, e.g.

The basic idea consists of "reducing the complexity" of singularities by using local and infinitesimal symmetries corresponding to functions and their tangent spaces. They have been used from the last years of the 19th century to solve ODEs in the Lie algebras framework. For finite-dimensional groups each independent symmetry lowers in a unity the dimensionality. The development of a similar reasoning for PDEs is more difficult because the corresponding infinitesimal symmetries for variational problems has infinite dimension (E.Noether). PDEs are local sections of jets spaces. Thus, classification issues of singular function-germs in $J^k(n, 1)$ for a variational functional is relevant for the integral approach to Mechanics.

The *homogeneous singularities* are related to the action of $\mathbb{C}^*$ and respond

[10] In more advanced settings, one must add dissipative components which will be modelled in our approach by using nilpotent operators.

to a locally conical structure of the solutions of dynamical systems appearing in different classical mechanics or engineering problems. However, the restrictions associated with this type of action are too strict and unnatural in different applications. nThis has motivated the introduction of *quasi-homogeneous singularities* related to the action of finite subgroups (in particular, symmetric or reflection groups), the study of quasi-homogeneous potentials (and related inversion problems), the analysis of the (co)homology associated with this type of functions or super-conformal field theories (Kreuzer and Skarke, 1992), e.g.

Quasi-homogeneous singularities appear in *Theoretical Physics* (mirrors on superstrings, non-linear particle dynamics with colliding singularities) to try to explain from an algebraic point of view the contraction or collapse of topological structures superimposed on manifolds.

The connections between these aspects and their applicability to the quantization of gravitational phenomena is an advanced research topic that presents deep connections with complex three-dimensional manifolds that are presented in module A_{36}; for details and references related to the A-D-E classification of simple singularities see [Fan13] [11]

More recently, one can find applications of Singular Functions Germs to other scientific or technological areas. To fix ideas, we will restrict ourselves to the areas appearing in the part II of these notes, i.e. Computational Mechanics of Continuous Media B_1, Computer Vision B_2, Robotics B_3 and Computer Graphics B_4 (see the fourth section of this chapter for more details). Recent developments of AI (Artificial Intelligence) provide the nexus between all of them. Let us illustrate with an “example”:

In addition of extensions of classical energy-entropy functionals, some of the most relevant ones are linked to the *loss function* of a model. Roughly speaking, a loss function tracks the error between the predicted (or expected in a statistical framework) y the current value of the output of a system, from an initial state. With this description, one can develop an analytic approach (in terms of convergence vs divergence), a geometric approach (in terms of a “volume form”), or a statistical approach (in terms of distribution functions, e.g.). All these approaches are unified in the AI framework for learning tasks corresponding to objects or behaviours.

The difference $a = y - f(x)$ (called “residual” in P& S) between the observed and predicted values of a function f, is the key for a formal characterization of the the *Huber loss function* defined as

$$L_\delta(y, f(x)) = \begin{cases} \frac{1}{2}(y - f(x))^2 & \text{for } |y - f(x)| \leq \delta, \\ \delta \ \cdot \left(|y - f(x)| - \frac{1}{2}\delta\right), & \text{otherwise.} \end{cases}$$

largely used in robust regression strategies, with application in a lot of knowledge areas.

[11] H.Fan, T.Jarvis and Y.Ruan: “The Witten equation, mirrors symmetry and quantum singularity theory”, *Ann of Maths* 178, 1-106, 2013

The joint management of multiple constraints with only an output suggests a determinantal approach, where the output is expressed as a the determinant D of a square matrix with the corresponding stratification by the rank [12]. For non-linear coefficients, the rank stratification is linked to the Jacobian matrix $Jac(D) = (\partial D/\partial w_{ij})_{1\leq i,j\leq n}$ where w_{ij} are the local coordinates for the (initially unknown weights) of the matrix.

(*) In the AI framework, one supposes parametric data to be clustered in terms of the behaviour of initially smooth functions (the hyperbolic tangent, typically), whose weights are fitted by minimizing a "loss function". In practice, the smooth functions are replaced by ReLU (Rectified Linear Unit) as activation function. ideally, the loss function has only "puere" critical points corresponding to the geometry of functional space (as it occurs for Morse functions, e.g.).

(*) In practice there appear "spurious" critical points (labelled as "artifacts" of learning processes), corresponding to "degenerate" singularities. Typical "examples" appear in the A-D-E classification (simple singulariites), where degenerations are interpreted as "overlapping" of intermediate singularities in the corresponding unfolding soace.

0.2. Outline of the module A43

The above remarks are a small sample of the deep relations between topological, differential, algebraic and analytic invariants. Thus, it is necessary to make a small introduction to the main techniques to be used along this module. We have followed a strategy going from coarse topological properties to finer structures concerning to algebraic and analytical properties for static issues. Differential aspects re-appear in different ways to unify diverse viewpoints, including dynamical aspects.

A typical example for the interplay between static and dynamical aspects in different contexts appear from the beginning with the simple harmonic oscillators or the Hooke's law $\ddot{q} = -\omega^2 q$. where quadratic perturbations given by quadratic functionals $H(q,p) = q^2 + p^2$ do not modify the topology of solutions. Nevertheless its simplicity, this example displays deep relations between static and dynamic aspects, stability and genericity which are in the nucleus of larger developments. The problem becomes less trivial when one substitutes the equilibrium locus of the parabola by other higher degree Hamiltonian functions on the Phase space $P = TM$ (see the section 3 of this chapter for more details).

The analysis of mechanisms and lenses along the last years of the 17th century are in the origin of Singular Function germs in regard to the study of envelopes for mobile points on gears, and the behavior of caustics in Geometrical Optics. An analysis of contributions performed by Hooke, Newton, Huygens and

[12] The extension to $(n \times p)$-matrices is also a determinantal variety which is given by the vanishing of the determinant of maximal minors of the matrix, with its corresponding rank stratification.

Barrow has been reviewed in modern terms and extended to other applications by V.I.Arnold in [Arn90].

With some exceptions arising from the Russian school (Arnold and his team in Moscow and Leningrad) and some people of the Liverpool school (Bruce, Giblin), the development of applications of Singular Function Germs to other scientific and technological areas is some scarce and sparse. One aim of this module and other ones of the part II of these notes is contribute to a better knowledge of foundations with a view to their applications in Engineering. In the last section of this introductory chapter one can find some snapshots to encourage young mathematicians and engineers to develop these remarks.

In addition of this enlarged introduction, this chapter has the following five sections:

1. *Local analysis of functions* where one remembers some basic notions and one introduces some tools of Local algebra to unify the language.
2. *Classifying singularities* where we adopt an increasingly complex strategy going from regular to degenerate singular geerms. The introduction of group actions and their corresponding infinitesimal version provides the key for a systematic approach.
3. *Topological properties* invovling stability, genericity, and stratifications as central topics, ending with some remarks about their interplay.
4. *Some applications to Engineering* with a special regard to the 4 areas developed in the part II (Geometric and Topological methods in Engineering) linked to Computational Mechancis of Continuous media B_1, Computer Vision B_2, Robotics B_3 and Computer Graphics B_4.
5. *Outline of the module* A_{43} where one specifies goal, methods, relations with other knowledge areeas and a short description of chapters, ending with references and some open problems.

0.2.1. Goals of the module

The *main goals* of the module A_{43} are the study of singular function germs in the smooth case $\mathcal{E}_n$ and the analytic case $\mathcal{O}_n$, and their applications to other scientific and technological areas. This study includes their topological properties involving stability, genericity and stratifications of their zero loci. The formal interpretation is performed in terms of (uni)versal deformations, i.e. “as generic as possible” in the space of map germs.

To achieve this goal, one uses a collection of techniques arising from several mathematical areas with a special attention to Local Algebra (it provides a common language for differentiable and analytic functions) and the Topology of spaces of functions, with its natural “stratification” linked to jets spaces $J^k(n,1)$ for finitely determined singular function germs.

A relevant novelty w.r.t. the two precedent modules consists of the incorporation from the scratch of singularities as a structural part. This approach provides a support for more comprehensive computational approach by avoiding their treatment as “exceptions” to be avoided.

The first part of this module (the first four chapters) develops some topics presented in module 1, although from a more algebraic approach to facilitate a unified treatment with geometric aspects (typical of GAGA) that are presented in module A_{45} if Differential Topology A_4 and in module A_{33} of Algebraic Geometry A_3. The algebraic approach is complemented with an analysis of the different types of stability linked to the action of groups of C^r-equivalences on the space of functions.

In the simplest case, the approximation using Morse functions allows us to show generic evolutions of singular phenomena towards simpler ones with quadratic singularities, controlling the evolution in terms of the topological characteristics of said singularities. Reconstruction of array topology by cell addition makes it easy to visualize and control evolution for compact arrays.

The extension of the arguments presented in the framework of Morse Theory to non-compact manifolds, orbifolds, or, even more difficult, to function spaces with “good metric properties” requires more advanced tools that are only outlined in chapter 4. The introduction of CW-complexes provides a tool to address the above cases in a framework that extends the finite cell gluing of the compact (Morse) case. In this case, the tools are closer to Homotopy Theory and its relations with (co)homology theories typical of Morse Theory [13].

However, the reintroduction of deformations or, alternately, variational principles allows to recover a typically variational approach, showing some applications. The basic idea consists of computing critical points for integral functionals. The novelty of this module consists of including degenerate singularities for functionals.

The second part of the module A_{43} combines the approaches of R.Thom on the one hand and V.I.Arnold on the other, jointly with some applications to other scientific or technological areas. Firstly, basic algebraic results linked to determinacy and unfolding are presented. They allow recovering topological invariants for the case of isolated singularities of function germs.

The simplest case correspond to Morse singularities, i.e. non-degenerate critical points of functions $f : M \to \mathbb{R}$, that is, the Hessian matrix of f is non-degenerate, so the critical points are classified by the signature of the quadratic form which is $\mathcal{R}$-equivalent to the original function germ.

The next case in difficulty corresponds to corank 1 singularities, that is, such that $rk(Hess(f)) = m - 1$. In this case, there is once a variable x where non-

[13] An introduction from the point of view of Geometric Topology has been presented in the module A_{24} (Geometric Topology) of the matter A_2

Morse behaviour can appear for a finitely-determine singular germ. The basic strategy consists of constructing the "most general" (universal) deformations $F : \mathbb{R}\times\Lambda \to \mathbb{R}$ of the original $f : \mathbb{R} \to \mathbb{R}$. The corresponding map $F \in C^r(1+\ell, 1)$ is called the unfolding (deploiements in French language) of f

The unfolding F is the key to "control" all possible "degenerations" in terms of singularities of the second projection $\pi_2 : \mathbb{R} \times \Lambda \to \times\Lambda$ on the space of parameters Λ. The study was initially carried out by R. Thom for A_k singularities given by $f(x) = x^{k+1}$ with unfolding $F(x, \underline{\lambda}) = x^{k+1}+\lambda_1 x^{k-1}+\ldots+\lambda_{k-1}x+\lambda_k$. In view of the sudden changes in the topology of the fibres corresponding to the inverse images of the projection on the space Λ of parameters, Thom introduced the name of Catastrophe Theory.

Currently, Catastrophe Theory makes part of the Bifurcation Theory in Dynamical Systems A_{46}, but in this module one adopts a simpler approach in topological terms by looking at topological changes appearing in their unfoldings. The review and reformulation of this theory by V.I.Arnol'd and his school has revealed the ubiquity of the ADE classification in multiple areas of Mathematics, showing unsuspected structural relationships previously scattered in the literature.

The characterizations of simple singularities provided by Arnol'd was extended by A.Durfee to 18 types, but recent advances related to deformations of algebras, quantization of gravitational theories, and mirror symmetries make it difficult to know the number of equivalent characterizations. For A_k singularities the first description of topological changes in families of function germs was given by R.Thom in the Catastrophe Theory framework, jointly with an analysis of stability for solutions cutting the discriminant loci. More general relations with Classical Groups (described by V.I.Arnold) open the door for applications to Physics and Engineering which are introduced in the section 4 of this chapter.

0.2.2. Methodological issues

In this module a *progressive approach* is carried out with increasing difficulty that goes from the simplest cases to the most complicated. Some of the latter (such as the one corresponding to quasi-homogeneous singularities, e.g.) are developed in more detail in module 4, as it requires more sophisticated algebraic techniques. Due to the abstract nature of foundations, we follow a top-down approach in most developments of this module. We don't suppose a previous knowledge of Local Algebra techniques, which will be introduced from the beginning by giving the appropriate references for more advanced results.

In a complementary way, it is advisable to try of illustrating basic principles with a bottom-up approach to improve the initially theoretical approaches performed along 1960s and 1970s in Physics, Biology, Geology and Engineering. Bottom-up strategies try of explaining the behavior of complex systems from data clustering arising from sensors capturing information at different scales.

Natural hierarchies linked to the scales suggest the introduction of hierarchies, where previous topological models play an important role due to higher "flexibility".

The construction of continuous models from discrete data combines techniques arising from Data Mining in Statistics with techniques arising from Piecewise Linear (PL) models in Algebraic Topology to provide a continuous support for propagation issues. In classical approaches one requires some stronger smoothness criteria for the support and maps which are not fulfilled in practical applications. Thus, it is necessary to develop connections between continuous and smooth frameworks through algebraic and analytic frameworks which allow any kind of singularities and powerful tools for their treatment.

On the other hand, as smooth functions are a Baire set (some kind of weak density) in the space of C^r maps for $r \geq 1$, they provide enough adaptive methods for objects displaying singularities. Unfortunately, most singularities are not even of class C^1. Hence, one must enlarge the smooth to the analytic category. Local Algebra plays a fundamental role for this extension. For computability reasons, it is necessary to replace convergent developments of analytic functions by polynomials. Last ones can be interpreted as truncated Taylor developments. It is necessary to know when this reduction from analytic to algebraic expressions is compatible with regular (diffeomorphisms vs bianalytic) transformations. Finite determinacy provides the key for this issue.

The extension of this viewpoint to propagation phenomena is important for analyzing topological properties of related dynamical systems displaying changes of states or phase transitions. Singular function germs play a fundamental role for classification issues. So, families of curves cutting out the discriminant locus on the space Λ of parameters provide the key for controllability issues. In particular transversal curves to an ordinary cusp in the plan (as discriminant locus in the plane of parameters) display two stable and an unstable points (depending on the sign for the slope at the intersection point), e.g.

A better understanding and extension of the above toy example, requires an analysis of families of hypersurfaces cutting transversally (genericity conditions) to the Discriminant Locus. Hence, to apply the theoretical results at more complicated singularities than Morse one, it is necessary to have propagation models linked to flows, able of adapting to diffusion-reaction models, e.g. The simplest propagation models are described in terms of the evolution of volumetric flows which are locally given by a functional determinant.

(*) From the statistical viewpoint, Kullback-Leibler divergence provides the statistical version of the classical Stokes approach of the smooth framework. The divergence represents the volume element, and the corresponding integral functional provides a structural model for the flow information. The equivalence class (up to birational equivalence) of the volume element is the Canonical Divisor; it is the corner stone for classification issues in the GAGA framework A_{32}

In the smooth framework, one supposes that the volume form is non-null, and as consequence the determinant bundle $det(E)$ is a rank one vector bundle. This hypothesis is no longer true in presence of singularities corresponding to a "degeneration" of the volume form [14]. From a global viewpoint, sections of the bundle E can become linearly dependent by lowering the rank of the matrix representing the volume form $det(E)$.

Hence, the analysis of critical points of the determinant is a non-trivial extension of the Morse's approach (chapter 5 of A_{41}) because the determinant representing locally the flow divergence has a stratification depending on the corank of the matrix [15].

0.2.3. From Local Analysis to Local Algebra

The first part of this module (the first four chapters) develops some topics presented in A_{41} (Basic Differential Topology). Local Algebra of rings of polynomials is easily adapted to the differential or the analytical framework. Their extension to finitely determined map germs $[f] \in C^r(n,p)$ is performed by using the natural structure as $\mathcal{E}_n$ of $\mathcal{E}(n,p)$ in the smooth case (or of the $\mathcal{O}_n$-module of $\mathcal{O}(n,p)$) given by the components $(f_1, \ldots, f_p()$ of f.

The algebraic approach is complemented with an analysis of the different types of stability linked to the action of groups of C^r-equivalences on the space of functions and other Classical Groups. The introduction of group actions eases the description of eventually bifurcating propagation phenomena. A more global approach will be performed in terms of Equivariant Stratifications in A_{45} (Stratifications) and their corresponding Equivariant Bifurcations in A_{46} (Dynamical Systems).

In the simplest case, the approximation using Morse functions allows us to show generic evolutions of singular phenomena towards simpler ones with quadratic singularities, controlling the evolution in terms of the topological characteristics of said singularities. Reconstruction of topology of compact smooth manifolds M by matching cells makes it easy to visualize and control the evolution of simple gradient systems linked to M.

In this module, things are a little bit more difficult. From the geometric viewpoint, the space Λ of parameters is not a smooth manifold, cells collapse around a singularities (giving vanishing cycles in the complex case), singularities display a higher complexity than Morse singularities, and there can appear infinitely many different types in the neighborhood of a singularity, even for families of degree four curves.

The extension of the arguments presented in the framework of Morse Theory to non-compact manifolds, orbifolds, or to function spaces with good metric

[14] A statistical inteerpretation in terms of loss functions has been sketched below at the end of the paragraph §0,1,4.

[15] A global topological analysis in terms of Euler obstruction class appears in the chapter 3 of the module A_{35} (Enumerative Geometry) for analytic varieties (J.L.Verdier and G.Gonzalez-Sprinberg).

properties requires much more advanced tools that are only outlined in the module A_{41}. The introduction of CW-complexes provides a tool to address the above cases in a framework that extends the finite cell gluing of the compact (Morse) case.

In this case, topological tools are closer to Homotopy Theory (in terms of deformations) and its relations with (co)homology theories typical of Morse Theory [16]. However, the reintroduction of deformations or variational principles allows to recover a typically variational approach, showing some applications. The topological formulation is performed in terms of k-jets spaces.

Along the module A_{43} one takes a quite different approach. Instead of looking at the topology of the support (a CW-complex, e.g.) one looks at the topology of spaces of functions. The main object is again the Jets spaces $J^{n,p}$ (formal Taylor developments) which have been introduced in the chapter 2 of the module A_{41} (Basic Differential Topology).

A novelty of this module consists of using algebraic properties linked to modules and algebras of derivations linked to quotients R/I of a ring R by an ideal I, or M/N of a module M by a submodule N, e.g.). Thus, the Commutative Local Algebra plays a fundamental role to unify analytic and algebraic approximations as intermediate between PL and smooth frameworks. In the next module A_{44}, Homological Algebra plays a similar fundamental role for this unification.

An almost obvious limitation is the strictly local nature of the approach performed along this module A_{43} and the next one A_{44}. In other words, one ignores global aspects linked to topological characteristics of an eventually curved support and the possible apparition of "holes", "tunnels" r similar phenomena appearing in higher dimensions. They are crucial for the analysis of propagating phenomena, and they can appear in a natural way in a lot of applications to other scientific and technological areas (as "obstacles" or unknown zones in the static case, e.g.).

Along these notes we will use different Group Actions and Divergence Flow as algebraic and analytical tools to ease the transition between local and global issues. Their confluence in Stratified G-equivariant theories has been developed by F.Kirwan along the 1980s; some details appear at the end of the module A_{45} in regard to the minimization of the Yang-Mills functional. Roughly speaking, instead of taking the divergence flow (evolving volume from) one takes the curvature flow (interaction of the support with the environment), and instead of minimizing volume flow along paths one minimizes the curvature functional in the space of connections on a Principal Bundle.

Again, critical points of the YM-functional provides "ooptimal solutions? to solve YM equations (instantons, e.g.). The YM-functional appears in the Standard Model (unification between electromagnetic, weak and strong forces). It would be interesting to extend the YM functional by allowing more general

[16] An introduction from the point of view of Geometric Topology has been presented in the module A_{24} (Geometric Topology) of the matter A_2.

singularities than those appearing in Morse stratified theory.

A "toy model" of this approach consists of looking at simpler curvature flows in surfaces (linked to mean or total curvatures, e.g.) and the corresponding energy functionals (such as the Wilmore energy functional, e.g.). This approach will be developed in Computational Dynamics B_{15} in Computational Mechanics of Continuous Media B_1, Video restoration B_{25} in Computer Vision B_2, more adaptive grasping and handling for manipulation tasks B_{31} in Robotics B_3, and advanced rendering models B_{44} in Computer Graphics B_4.

0.2.4. A short description of contents

The module A_{43} contains the following chapters:

1. *Local Algebra for Function spaces* Basic notions- The classification problem. Nakayama's lemma. Local algebra of a singularity. Weierstrass Preparatory Theorem. Impact phenomena

2. *Stability.* Actions of homeomorphisms and diffeomorphisms. Topological stability and infinitesimal stability. Unfolding.

3. *Milnor Theory.* Revisiting Morse Theory. Isolated singularities of hypersurfaces. Milnor algebra. Tjiurina algebra. Invariants. Relations between invariants

4. *Algebraic Classification* Equivalence relations. Tangent to Function Spaces. Finite determinacy. Normal neighborhood. Canonical forms. Homogeneous singularities. Quasi-homogeneous singularities.

5. *Catastrophe Theory.* Some motivations. Basic types. Unfoldings. The ADE classification of simple singularities. different characterizations. Relations with groups.

6. *Applications to Natural Sciences:* Geometric Optics. Analytical Mechanics. Catastrophes in Biology and Geology.

7. *Applications to Engineering:* Wave fronts in Fluid Mechanics. Evolving contours in Computer Vision. Envelopes in Mechanics. Rendering for scenes and characters.

8. *Applications ot Economic Theory:* Speculative bubbles in microeconomics. Business Cycles. Stagflation in Macroeconomics, Limits of growth in International Economics, following [Ros91]. Volatility and speculative financial markets. Cryptocurrencies.

9. *Applications in Biomedical Sciences:* Basic Physical-Chemical reactions. Pharmacological design. Epidemiological models. Chaotic cycles in overlapping generations models

0.3. References for this introduction

References must be understood as an invitation to complete the reading of these notes. They are not exhaustive, nor the most recent ones. Each reader must reconstruct his/her own "visioin" according to his/her interests and preferences.

0.3.1. Basic bibliography

One includes only manuals or textbooks which are relevant for subjects developed along this chapter. More specific references appear along the text, and at the end of the chapter. Only some general references related to the more "classical" aspects related to the Differential Topology of Applications Singularities framework up to the early 1980s are included.

With these restrictions, most of the literature is dominated by the works of V.I.Arnol'd and his school on the Russian side and C.T.C.Wall and his area of influence on the Western side. For a more complete bibliography, the references of the books by Arnol'd, Poston and Stewart or Wall cited below should be consulted. We include only bibliographical references to ease the introductory character of these notes. Original results can be traced out from these references.

[Arn84] V.I.Arnold: *Catastrophe Theory*, Springer-Verlag, 1984.

[Arn90] V.I.Arnold: *Huygens and Barrow, Newton and Hooke. Pioneers in Mathematical Analysis and Catastrophe Theory. From evolvents to quasi-crystals*, Birkhauser, 1990.

[Arn91] V.I.Arnold: *The Theory of Singularities and its Applications*, Lezione Fermiani, Pubblicazioni della Classe di scienze, Scuola normale superiore, Pisa, 1991.

[Bru92] J.W.Bruce and P.J.Giblin: *Curves and Singularities* (2nd ed), Cambridge Univ.Press, 1992.

[Dim92] A. Dimca: *Singularities and topology of hypersurfaces*. Springer-Verlag, 1992.

[Loo84] E.J.N. Looijenga: *Isolated singular points on complete intersections*, Cambridge University Press, 1984.

[Ros91] J.B.Rosser, Jr: *From Catastrophe to Chaos: A general theory of Economic Discontinuities*, Kluwer, 1991.

[Tho75] R.Thom: *Structural Stability and Morphogenesis*. Benjamin, 1975.

Other more advanced references which will be used along in next sections and chapters are the following ones:

[Arn85] V.I.Arnol'd, A.Varcenko and S.Gusein-Zade: *Singularities of Differentiable Mappings* (2 vols transl, by I.Porteous), Birkhauser, 1985 (French translation in Mir, 1982).

[Bro75] T.Broecker, and L.C. Lander: *Differentiable Germs and Catastrophes*. LMS Lecture Notes Vol. 17. Cambridge University Press, 1975

[Gol73] M.Golubitsky and V.Guillemin: *Stable Mappings and Their Singularities*, Springer-Verlag, GTM 14, 1973.

[Mil63] J. Milnor:*Morse Theory*, Princeton Univ. Press, 1963.

[Mil68] J. Milnor: *Singular points of complex hypersurfaces*, Ann.of Math. Stud.,61, Princeton University Press, 1968.

[Orl92] P. Orlik and H. Terao: *Arrangements of hyperplanes*, GMW 300. Springer-Verlag, 1992.

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[Wal04] C. T. C. Wall: *Singular points of plane curves*, Cambridge University Press, 2004

0.3.2. Software resources

Singular provides a support for a lot of applications based on Local Algebra. To my knowledge, there are no still software tools for classification issues in Jets spaces $J^k(n,p)$. The avaialbility of this software woudl simplify a lot of tedious computations used for classification issues in the smooth or the analyticc frameworks.

Usual software resources are based on symbolic Programming or their extensions to Functional Programming. Connections with OOP (Object Oriented programming) are very scarce, and they are waiting for their development still

- Mapple V: https://www.maplesoft.com/products/Maple/
- SINGULAR: https://www.singular.uni-kl.de/Manual/4-4/
- ... any suggestion is welcome

Final remark: Readers which are interested in a more complete presentation of this chapter (in spanish laguage) or some chapter of the module A_{43} (Singular function-germs),please write a message to javier.finat@gmail.com.