A420 An introduction to Fibre Bundles

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Índice

This draft is a short version of the introduction to the module A_{42} (Fiber Bundles) of the matter A_4 (Differential Topology).

Prerequisites.- It is necessary to have some basic knowledge of Differential Geometry A_1 and Algebraic Topology A_2. Analogous to the Differential Geometry approach for Banach spaces, we consider some differential operators on function spaces with minimal elements of Functional Analysis.

As usual, in addition of this introduction and a final chapter with Complements , materials are organized in four sections. They contain a list of exercises for self-verification of understanding of materials. Subsections or paragraphs marked with an asterisk ($*$) display a higher difficulty and can be skipped in a first lecture.

0.1. Introduction to the module A42

A general motivation for superimposed structures on a base space B is locally given by the existence systems of C^r-equations E_b "depending" on the base point $b \in B$, and the need of giving criteria for existence of solutions, and explicit methods for their resolution.

- Typical values for r correspond to $r = 0$ (topological equations), $r =)\infty$ (differentiable equations), $r = rat$ (Algebraic Geometry) or $r = \omega$ (Analytical Geometry).. In correspondence with these examples. the base space B can be a topological space X, a smooth manifold M, or an algebraic or analytical variety X.
- From the global viewpoint, systems of local equations are modelled as local sections $s : U \to E \mid_U$ of a bundle, a sheaf or a fibration, where E represents the total space obtained as a "recollement" of fibres or stalks.
- Resolution of systems of C^r-equations has two parts linked to existence (or alternately, obstruction) and characterization of sets of solutions, in terms of properties of the base space B (initially a manifold M or a variety X).

In the topological or smooth categories, local systems of solutions for C^r-systems of equatons can be extended to yh global case, by using partitions of unity. This result is no longer true in the algebraic or the analytic case. There can appear "obstructions" to prolongation. even in the complex case. They are expressed in terms of the non-vanishing of the first cohomology group. So, vanishing cohomology groups is key to warrant the absence of any kind of "obstructions".

Locla systems of equations can involve to "higher order" derivations or differentials. Their resolution is formulated in terms of higher degree cohomology groups, whose vanishing plays again a central role for existence of prolongation issues. So, the introduction of cohomology groups is the key to express for the space of solutions for systems E of equations. The cohomology is an homotopy type invariant (to be interpreted as some kind of "deformation"). Hence its rank (or the dimension if they are vector spaces) provides topological invariants. So, the main results linked to the corresponding C^r-categories are

- *Vanishing theorems* for the cohomology which are linked to the non-existence of "obstructions" to the existence of solutions or the prolongation of local solutions in the analytic case.
- Identification of *obstructions* to solvability in terms of the cohomology of the base space B.
- Explicit computation of the *dimension of spaces of solutions* in terms of intrinsic invariants, with the Index Theorems as the central paradigm of this module.

- *Relations between invariants* corresponding to maps $f : N \to P$ between smooth manifolds or morphisms $\Phi : E \to F$ between superimposed structures (bundles, sheaves, fibrations).

As always, we start with the regular case, before extending to more general cases. These extensions are justified because the "dependence" w.r.t. the base point $b \in B$ can be regular or not. Irregular cases display "jumps" or discontinuities in "fibers" which are modelled as singularities and developed from the module A_{43}. In the regular case, the C^r-equivalences in the base space are "lifted" to the total space $E = \cup_{b\in B} E_b$ of the superimposed structure ξ on B by using local triviality conditions.

- For $r = 0$ (topological case), regular transformations on the base space B are given by homeomorphisms (bijective and bicontinuous transformations); they are useful to compare discrete structures, and superimposed symbolic structures (graphs, e.g.).
- for $r = \infty$, regular transofrmaitons are given by diffeomorphisms (differentiable homemorphisms with inverse differentiable); they provide the support to compare different kinds of superimposed geometric structures (Riemannian, Conformal, Symplectic, Contact) or other G-structures;
- for $r = \omega$, regular transformations are given by bianalytic maps, which allow to compare the local behaviour around singularities of varieties or analytic maps between analytic spaces;
- for $r = alg$, regular transformations are given by birational maps (rational maps with rational inverse) for $r = rat$, which allow finite-dimensionality reduction in terms oflocal k-algebras.

Fiber Bundles, Sheaves and Fibrations provide a support for connecting local and global aspects in terms of superimposed locally trivial C^r-structures on C^r-varieties. The simplest case corresponds to *vector bundles* which are given by a 4-tuple $\xi = (E_\xi, \pi_\xi, B_\xi, F_\xi)$ where E_ξ is the total space, B_ξ is the base space $\pi_\xi : E_\xi \to B_\xi$ the projection map, and F_ξ the generic fiber, fulfilling the following conditions:

- There exists an open covering $\mathcal{U} = (U_i)_{i\in I}$ such that $\pi_\xi^{-1}(U_i) \simeq_{C^r} U_i \times F_\xi$ for all $i \in I$.
- The above C^r-equivalence is restricted to an isomorphism $\pi_\xi^{-1}(b) \simeq_b \{b\} \times F_\xi$ between the specific fiber $\pi_\xi^{-1}(b)$ at b and the generic fiber F_ξ which depends on the base point $b \in B$ (is not canonical).

The restriction to $b \in B$ of C^r-equivalences $(\phi_j \circ \phi_i^{-1}) : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ on the overlapping of the base space, induce an automorphisms of the specific fiber $\pi_\xi^{-1}(b)$ at the base poitn $b \in B$, which is represented by a regular matrix $g_{ij} \in Aut(F_\xi)$ belonging to some classical subgroup G og $GL(r;\mathbb{R})$. The simplest case corresponds to a vector bundle ξ of rank $r = dim(F_\xi) \simeq \mathbb{R}^r$, where $Aut(F_\xi) = Aut(\mathbb{R}^r) = Gl(r;\mathbb{R})$ can be represented by regular $(r \times r)$-matrices, i.e. with non-vanishing determinant. Typical examples already seen in A_{12} are tangent and cotangent vector bundles, and their tensor products.

Local triviality conditions $\pi^{-1}(U_i) \simeq_{C^r} U_i \times F$ for the open sets U_i of a covering $\mathcal{U}$ of the base space B, and the algebraic structure of fiber $F_b := \pi^{-1}(b) \simeq_{C^r} F$ (generic fiber), are the key to reduce a difficult topological problem (relative to C^r-equivalences) to an easier algebraic problem in terms of isomorphisms between simple algebraic structures (vector spaces. groups, rings, modules) corresponding to fibres on different points.

In particular vector bundles can be interpreted as the result of matching "lifted local systems" of linear equations given on open subsets $U_i \subset B_\xi$ to the total space E_ξ of the bundle ξ. This interpretation explains the ubiquity and the utility of Fiber Bundles (including vector and principal bundles) or more general "fibrations".

The chapter provides a support for Differential Geometry A_1, Algebraic and Analytical Geometry A_3, or Conformal Geometry, between other for the static case. Furthermore, it can be applied to Symplectic and Contact Geometries on the Phase Space P as basic patterns for Analytical Mechanics. Algebraic Topology A_2 and some topics of Functional Analysis, and their applications to Theoretical Physics.

More formally, in the continuous framework the lifting is given by local sections, i.e. C^r-maps $s : U \to E_\xi$ such that $\pi_\xi \circ s = 1_U$ (identity map on U). Typical examples are vector and co-vector fields (differential forms). Intuitively, a C^r-vector bundle (resp., a fibration) is a collection of vector spaces (resp. C^r -varieties) parameterized by the C^r-variety B_ξ, which is the base space.

This very general description eases the extension to applications of the discrete or probabilistic framework, Theoretical Physics and Engineering areas including Fluids Mechanics B_1, Computer Vision B_2, Robotics B_3 and Computer Graphics B_4 and mutual interrelations through transversal areas such as AI and Advanced Visualization, e.g.. [1]

The *Simplest examples of Fiber Bundles* are given by the Cartesian product $X \times Y$ of two topological spaces; these examples are called "trivial" because they are globally given as a Cartesian product, i.e. do not display any kind of "twisting". The simplest model of non-trivial twisting is given by an infinite Moebius band on the central circle $\mathbb{S}^1$ corresponding to the "equator".

More general Fiber Bundles and fibrations are obtained by matching local trivializations by using compatibility between local charts (U_i, ϕ_i) in the base space B_ξ. Thus, the total space inheritates a C^r.structure, also. Similar strategies can be adapted to the discrete case and to parametric Statistics (useful for the Geometric Information Theory).

[1] The adaptation to the discrete framework will be developed in several modules of the part II of these notes.

To fix ideas, we initially assume that the base space B is initially a smooth manifold M, where we have developed examples given by tangent bundle $\tau_M = (TM, \pi, M, \mathbb{R}^m)$ and the cotangent bundle $\tau^*_M = (T^*M, \pi, M, \mathbb{R}^m)$ whose properties have been described in the module A_{12} (Linearization) of Differential Geometry A_1. Their extension to Tensor Bundles has been developed in the module A_{13}.

This approach is immediately extended to the Algebraic and Analytic Geometry A_3. by replacing local diffeomorphisms on open subsets equivalent to $\mathbb{R}^m$, by birational or bianalytic equivalences by using the affine structure instead of (pseudo-)Euclidean spaces. Similarly, vector and covector fields are replaced by derivations and differentials on quotients A/I of a ring A by an ideal I.

In the same way as for other areas of Geometry and Topology, *classification issues* occupy a central place. Absolute classification involves to "intrinsic" data of the base space B given usually by a smooth manifold M or a C^r-variety X; intrinsic means that it is independent of the embedding or the immersion. In a complementary way, the relative classification involves to superimposed structures on maps $f : X \to Y$ on the bases of fibrations. An important problem is to find relations between intrinsic invariants (for the base space, e.g.) and "extrinsic characters" (depending on embedding, e.g.). Typical "examples" are given by

- The first and second fundamental forms (for intrinsic and extrinsic geometry, respectively) of a smooth surface in $\mathbb{R}^3$ in the Differential Geometry of Surfaces which are related through Gauss-Peterson-Mainardi-Codazzi formulae (see chapter 3 of A_{10} for more details).

- Plucker formulae for algebraic curves linking the genus with projective character linked to a projection (see the chapter 6 of the module A_{31} for more details). These formulae were extended to complex algebraic surfaces which are generic projections of a regular surface in $\mathbb{P}^5$ by Salmon, Cayley, Zeuthen and Pieri (modern proof by R.Piene, 1977). Along the decade of 1930s, Roth has given more than 50 formulae involving relations between intrinsic and extrinsic characters, but some of them are wrong. A complete proof for complex threefolds with ordinary singularities in $\mathbb{CP}^4$ has not proven, still [2].

Intuitively, extrinsic approaches have in account "interactions with the environment" which are linked to immersions or submersions. The interplay between intrinsic and extrinsic approaches is crucial for applications in Physics and Engineering. In geometric terms, the normal bundle is the support for modelling interaction with the environment. In presence of singularities, the normal bundle must be replaced by the conormal bundle introduced in A_{33}.

Some related problems concern to the $3D + 1d$ modelling of evolving solid deformable objects, and their corresponding applications developed in B_{36}

[2] See the module A_{36} (Algebraic and analytic Complex Threefolds) for an introduction, including my own proofs for some of these formulae.

(Three-dimensional Video) in B_3 (Computer vision), B_{45} (Animating the Human Body) and B_{46} (Animats) of B_4 (Computer Graphics). Vector bundles or, more generally, fibrations provide simplified models for all of them. In all these cases "events" are modelled in terms of singularities of C^r-maps (or more general fields), for which one must describe the corresponding tangent spaces [3].

For Vector Bundles on smooth manifolds, "differential characters" are given by "characteristic classes" of vector bundles (measuring their "twisting" e.g.). Formally, they are cohomology classes $H^q(M;R)$ of the base space (by the topological triviality of the fibers), where R is usually $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Their extension to eventually singular algebraic or analytic varieties is more complex, and requieres some elements of Stratification Theory A_{45}.

The non-vanishing of characteristic classes express the "obstruction" to solve system of equations or to extend local solutions corresponding to the superimposed structure ξ. From the dual viewpoint, they can be geometrically interpreted as dependence loci of initially independent sections of the vector bundle (including the vanishing or indeterminacy of some of them). A general treatment of characteristic classes is performed in the chapter 8. Schubert cycles provide a geometric interpretation in terms of loci where different sections become dependent between them [4]

An important tool for classification of vector bundles is given by (relations between) C^r-invariants. For $r = 0$ one finds the simplest topological invariants (such as Betti numbers or Euler-Poincaré characteristic, e.g.), whereas for $r = \infty$ (resp. $r = \omega$) one has differentiable (resp. analytic) invariants given by the number of l.i. smooth ve holomorphic q-forms. As always, smooth structures are simpler than analytic ones, and consequently, must be explained before to take them as an "ideal model". In this way, "characteristic classes"provide a quite general approach to compute invariants of the real, complex,or cuasi-complex structure, e.g.

This introductory chapter gives a coarse idea about the basic strategies for the calculation of invariants associated with C^r-structures, as well as the outline of some applications to Theoretical Physics and Engineering. The initial motivation for the study of these structures comes from considering the simultaneous resolution of systems of equations (differential, algebraic, analytical) given on a base space B (initially a PS-manifold M; later, an algebraic or analytic manifold X) that depend on the base point $b \in B$.

In the framework of *Local Differential Analysis*, the systems to be considered are locally given by differential systems $\mathcal{S}$ that are solved by using analytic (integro-differential methods), algebraic (Lie groups) or numerical methods. The extension of local methods to the global case (Differential Topology) leads to give integrability conditions for distributions $\mathcal{D}$ of vector fields $\mathbf{v}_i(x)$ or systems $\mathcal{S}$ of differential forms $\mathbf{w}_j(x) = \mathbf{v}_j(x)^*$ defined on an initially smooth manifold M and, later, on an algebraic or analytical variety X.

[3] See details in the mdoules A_{43} (Singular function germs) and A_{44} (singular Map Germs).
[4] A classical version is developed in the module A_{34}

(*) Next steps are linked to the study of modules over a ring D of differential operators. Some motivations arise from the study of linear PDEs by using topological and algebraic methods (Algebraic Analysis initially developed by Sato). This theory, labelled as D-modules, appears as a natural extension of the Grothendicek approach applied to polynomials on D. It was initially developed by Kashiwara at the early 1970s. Some general issues have been developed in the chapter 6 of the module A_{33} (Sheaves, Cohomology, Schemes). Some connections with PDEs appear at the end of the mdoule A_{46} (Dynamical Systems).

Some more basic motivations concern to the simultaneous consideration of several trajectories $\gamma_i(x)$ (integral curves of vector fields); the finite collection of vector fields generate a distribution $\mathcal{D}$ on a manifold M. Similarly, one has a finite collection of constraints which are given as the integral hypersurfaces of differential forms ω_j; they generate a differential system $\mathcal{S}$ on M.

However, "most" distributions $\mathcal{D}$ and systems $\mathcal{S}$ are non-integrable, because the Frobenius Theorem imposes high codimension constraints to be fulfilled. A basic strategy consists of identifying "near" integrable distributions $\mathcal{D}$ or systems $\mathcal{S}$, acting as "organizesr" for the dynamics. Deformations of changing tensors to be tracked on manifolds M of more general varieties X. All of them are initially represented by deformations of local sections of a tensor bundle [5].

In the simplest cases, partial integrability is "translated" to locally trivial decomposition of "foliatioins" (in their horizontal and vertical parts) which express partial or global "decoupling" for solutions of distributions and systems. Due to the presence of singularities, their space-time evolution can display any kind of irregular behaviours, whose hiearchies are analyzed in terms of singular map-germs A_{44}.

In advanced applications to Physics and Engineering the decoupling in horizontal and vertical components, involves to partially integrable distributions and systems representing traceable paths under constraints which are managed in terms of tensors. So, Differential Topology deals with the study of properties of the space-time "evolution" of tensors on PS-manifolds to be extended to varieties X in the absolute and the relative frameworks.

0.1.1. Goals and applications

The focus of this module A_{42} follows a models-based or top-down approach. Tensor fields are the natural extension of multilinear maps in Linear Algebra. Tensors provide the general approach to non-liner multi-variable models having in account multiple weighted trajectories and constraints. So, tensor fields are ubiquitous in all scientific and technological areas, including AI with TensorFlow as paradigm in recent developments of Deep Learning.

By this reason, the section 4 of this introduction outlines selected applications related to some Engineering areas labeled as B_1 (Continuous Media Mechanics), B_2 (Computer Vision), B_3 (Robotics) and B_4 (Computer Graphics).

[5] See the modules A_{45} (Stratifications) and A_{46} (dynamical systems) for more details.

All of them are developed in the part II of these notes. Furthermore mathematical modeling, the main problem is the estimation and tracking of tensor fields, which is performed by Tensor Voting procedures.

Obviously, multilinear approaches are not enough for a lot of geometric properties whose evolution can require higher order derivatives which are not behaved as tensors. Jets spaces $J^k(n, p)$ and their corresponding fibrations provide the right framework for their treatment [6]

Beyond the mathematical considerations that are developed along the whole module A_{42}, there are currently a growing number of applications from fiber bundles to very diverse areas of knowledge. The best known and sophisticated applications are related to Theoretical Physics; Yang-Mills nonlinear PDEs provide the framework for the standard model (unification of electromagnetic, weak and strong interactions) and the hard core of the Fiber Bundles that was developed in the mid-1960s.

The interaction of Theoretical Physics with Differential Topology has been one of the most important sources for the joint development of both. Fields-based approaches for Vector Bundles and connections on Principal Bundles play a fundamental role for all these issues. Propagation models based on symmetries play an important role for replication of basic patterns; breaking symmetries will be introduced later. In view of the "universal" character of G-homogeneous spaces, a challenge is the extension of fields and connections to other knowledge areas. In the part II of these notes, we adapt some of these ideas to Mathematical Engineering; it is worth highlighting applications to areas as diverse as

- An *assistance to design and 3D modeling* (including CAD/CAM models) from a finite set of views by using groups of rigid transformations whose foundations are developed in the module B_{11} (Computational Geometry). A more detailed approach by using deformations of rational varieties given as product of weighted rational curves (Algebraic Geometry) is developed in the B_{41} (Modelling) of B_4 (Computer Graphics).

- The simulation of smooth deformations for PL-structures in the CMCM (Computational Mechanics of Continuous Media) developed in B_1. The goal is to identify dynamical behaviors of flexible structures in terms of PDE with diverse applications (aircraft bodies or fuselage, for example).

- The development of dynamical models in *Biomedicine*, including biomechanical devices (from exoskeletons or reciprocators to cardiac devices, e.g.), the modeling of filamentous structures in artificial muscle tissues.

- The functional modeling and simulation in ANN (Artificial Neural Networks) in terms of tensor products of bundles, e.g. In our approach, detectors and descriptors in AI are modeled in terms of sections of bundles, whereas classifiers are modeled in terms of isomorphism classes of bundles.

[6] Some basic "examples" have been introduced in the chapter 3 (Intrinsic properties of Surfaces) of the module A_{10} (Differential Geometry of Curves and surfaces).

- The development of local vs global models in *Economic Theory*, including topics such as micro and macroeconomic adjustment, International Trade or Financial Economics, e.g;. in terms of Foliations or more general Fibrations linked to structural systems of equations for each subarea of Economic Theory.
- Simplified models in *Meteorology* to facilitate understanding of the dynamic interaction between atmosphere, sea and land, as fiber bundles on three-dimensional PS-manifolds.

Unfortunately, there is no a unified treatment of all the above topics in the literature. Along these notes, we will limit ourselves to some applications in an Information Society Technologies (IST) framework. The first integration of Differential Geometry and Parametric Statistics has been performed in GIT (Geometric Information theory) by S.T.Amari et al [Ama16]. In the part II of these notes, one sketches several extensions to Topological (TIT), Kinematic (KIT) and dynamic Information Theory (DIT), by using the corresponding discrete veersions for each one of them.

Some transversal Engineering areas to all these topics and the matters B_i (mentioned above) are given by Advanced Visualization and Artificial Intelligence. A highly non-trivial challenge is the development of reliable models and software tools for automated tasks in both transversal areas. Advances in this dom are being performed in AI terms by using automatic generation of multimedia contents in Deep Learning.

Some of the most interesting examples are related to Machine Learning Systems that began to develop in the 1990s. In particular, the formulation of simultaneous learning of "evolving quantities" (applicable in all the above " examples ") is formulated in terms of learning k -dimensional subspaces in the SOM framework [Koh97] [7].

In our approach, learning subspaces becomes a problem of convergence in a superimposed statistical Grassman bundle, or more general, Flag bundles. Both of them can be easily extended to their corresponding algebraic versions in terms of Modules. For Grassmann Bundles see [Gro71] [8]. Flag bundles are modelled in a similar way.

0.1.2. From absolute to relative cases

The commented examples of the precedent paragraph, show how in despite the abstract nature of most materials presented here, there is an enormous amount of interactions between scientific and technological aspects that are transferred to everyday applications in the real world. An "example" is given by Adaptive Subspaces for Self-Organizing Maps [Koh97], which can be understood

[7] T. Kohonen: *Self-Organizing Maps (SOM)*, 2nd ed , Springer-Verlag, 1997

[8] A.grothendieck and J.Dieudonné: *Eléments de Géometrie Algébrique I*, GMW 166, Springer-Verlag, 1971.

as an AI-based reformulation of the Gauss map. It can be understood as a precedent of more advanced Learning Manifolds techniques commonly used in Deep Learning.

To achieve this goal, it is convenient to remember some basic notions relating several C^r frameworks. To fix ideas we will restrict ourselves to the topological case $r = 0$ and the smooth case $r = \infty$, because the last one is roughly speaking"dense" (Baire sets) in C^r for $r \geq 1$. Some connections with the ubiquitous Data Mining in Engineering arise from the superposition of PL-structures to discrete data distributions, and the evaluation of differences between PL and PS-structures which are developed in this module.

From a theoretical viewpoint, the weakest situations for a C^r-bundle structure corresponds to $r = 0$, i.e. a m-dimensional topological space X endowed with a covering $\mathcal{U} = (U_i)_{i\in I}$ such that U_i is C^0-equivalent (homeomorphic) to $\mathbb{R}^m$ via φ_i, where the compatibility condition $\varphi_j \circ \varphi_i^{-1}$ is a C^0 -equivalence (homeomorphism) between the open sets $\varphi_i(U_i \cap U_j)$ and $\varphi_j(U_i \cap U_j)$ for $\mathbb{R}^m$.

In the relative case, classification of C^r-maps $F : X \to Y$ is performed in terms of C^r-equivalences acting on the source and target spaces, or, alternately, on the graph of the C^r-map. This gives the right-left action or the contact classification. This sounds very nice, but the problem is that, up to very simple cases, there are no effective criteria for constructing C^r-equivalences for usual values of $r = \infty$ (diffeomorphisms), $r = \omega$ (Bianalytic equivalence) or $r = rat$ (birational equivalence) corresponding to Differential, Analytic or Algebraic Geometry. Thus, we introduce additional constraints.

A C^r-morphism $\Phi : \xi \to \eta$ between two vector bundles is a pair (ϕ, f) where $f : B_\xi \to B_\eta$ and $\phi : E_\xi \to E_\eta$ are C^r-maps such that $\phi(F_\xi b) \subseteq F_\eta(f(b))$, and $\pi_\eta \circ \phi = f \circ \pi_\xi$. The description for Principal Bundles $\mathcal{P} = (P, \pi, B, G)$ is similar, but by replacing linear maps between vector spaces by homomorphisms between groups. The equality between composition of maps can be represented by the commutativity of the following diagrams:

$$\begin{array}{ccc} E_\xi & \to & E_\eta \\ \downarrow & & \downarrow \\ B_\xi & \to & B_\eta \end{array} \quad \text{and} \quad \begin{array}{ccc} P_\xi & \to & P_\eta \\ \downarrow & & \downarrow \\ B_\xi & \to & B_\eta \end{array}$$

corresponding to vector and principal bundles. Morphisms between fibrations can be thought as relations between "behaviors" (locally given by systems of equations) on two related base spaces. In general the restriction map of ϕ the fibers is not injective or surjective, in such way that we have a short exact sequence

$$0 \to Ker(\phi \mid_{F_\xi} (b)) \to F_\xi(b) \to F_\eta(f(b)) \to Coker(\phi_{F_\eta}(f(b))) \to 0$$

If $Ker(\phi \mid_{F_\xi} (b)) = 0$ one saya that ϕ is a monomorphism (or injective morphism) at $b \in B_\xi$. If If $CoKer(\phi \mid_{F_\xi} (f(b))) = 0$ one saya that ϕ is an epimorphism (or surjective morphism) at $f/b) \in B_\eta$. In the smooth case, the above maps are called immersions and submersions, respectively, and provide criteria for constructing submanifolds, e.g. In both cases, the Jacobian matrix representing the differential is regular, i.e. has maximal rank. The natural duality between $Ker(\varphi)$ and $Coker(\varphi$ for vector spaces is translated to superimposed structures corresponding to fiber bundles.

In general, the morphism $\varphi : \xi \to \eta$ between bundles "linearizing" the map $f : B_\xi \to B_\eta$ is not necessarily regular, and the lack of regularity of the morphism $\Phi = (\phi, f)$ is "measured" in terms of loci where the Kernel or the Cokernel is null. These conditions can correspond to changes of state in the base space or "phase transitions" in the total space of the vector bundles. In terms of dynamical systems they correspond to bifurcation phenomena, In terms of group actions they correspond to breaking symmetries. Their combination gives phenomena of G-equivariant bifurcations. Last ones are very common in all the modules of the part II of these notes.

Very often, the condition of vector or principal bundle is too strict and ideal, because the structure of the fiber is too strict, and can not be fulfilled from experimental data. A relaxation of topological constraints involving the fiber leads to the notion of C^r -fibration (see A_{24} for a more extensive treatment) or "sheaf" (see A_{33} for more details and references).

Intuitively, a C^r-fibration is a "family" of C^r-varieties that varies continuously with respect to a C^r-base space B. More formally, we denote a C^r-fibration by means a 4-tuple $\xi = (E_\xi, B_\xi, F_\xi, \pi_\xi)$, where E_ξ is the total space, $B\xi$ the base space, F_ξ the fiber of ξ and $\pi_\xi : E_\xi \to B_\xi$ the projection maps verifying that

- the *topological local triviality* $\pi^{-1}(U_i) \simeq_{C^r} U_i \times F$ for the fibered structure, where $(U_i)_{i\in I}$ is a covering $\mathcal{U}$ of open trivialization subsets of the base space B; and
- a non-canonical *algebraic isomorphisms* $\pi^{-1}(b) \simeq \{b\} \times F$ (i.e. depending on the base point $b \in B$) between the general fiber F and the specific fibre $\pi^{-1}(b)$, which is the restriction of the precedent C^r-equivalence to the base point $b \in B$.

The notion of "local section" $s : U \to \pi_\xi^{-1}(U) \simeq_{C^r} U \times F$ is the key to "lift" the information from the base space B_ξ to the total space E_ξ. A local section is characterized by the condition $\pi_\xi \circ s = 1_U$ (identity map on U),, and allows relating the C^r-properties of the fiber, and their regular transformations. If fibrations have discrete fibers, changes of sheets correspond to "different determinations" and are described by monodromy groups, which is initially thought as a local representation of symmetric and/or alternating groups. Last ones play a fundamental role for connecting continuous and discrete approaches which are developed in the part II of these notes.

The local triviality conditions introduce a local decoupling between the base space B and fiber F of a fibration that is very useful for maps with "similar" behavior. In topological fibrations it is only required that the type of homotopy be kept constant, so that the dimension of the fiber can "jump" and change its shape: a point and any contractible space have the same type of homotopy, e.g.

This vague notion adapts to the management of information both in the continuous case and in the discrete case. It is also compatible with changing symbolic representations. In particular, as trees (graphs without cycles) are contractible in any dimension. Hence, they have the same type of homotopy, which allows the development of expansion-contraction strategies using appropriate graph cuts, e.g.. In more advanced stages, we introduce superimposed structures on graphs (including hypergraphs, e.g.) to incorporate "emebdded behavioiurs" at different depth levels.

0.1.3. Fields-based approaches

If the base space B is a PS-manifold M, one has an accurate description of (scalars, vector or covector) fields or more generally, tenros fields as the local sections of a bundle, and their space-time evolution. This description provides the key to relate the properties of M with those of the total space E_ξ of the overlapping structure $\xi = (E_\xi, \pi_\xi, B_\xi, F_\xi)$ over the base space B_ξ. Typical examples are given by vector fields, differential forms or tensor fields; all of them are examples of local sections of vector bundles.

Sections and projections are typical operations of Projective Geometry, and correspond to ubiquitous ascent and descent strategies in many areas of Mathematics and their applications for inverse and forward problems. The insertion and deletion algorithms provide the corresponding simpler computational tools to relate processes holding in spaces with different dimensions. Ideally, if we take a field-based approach to connecting local and global aspects,

- a *scalar field* f is given by a C^r-assignment that takes each base point $\mathbf{b} \in B$ in the value that a C^r-local function $f : U \to \mathbb{R}$ takes in $b \in U \subset B$; In bundle terminology, it is said to be a " linear bundle " or line bundle with range 1. Some typical examples can be the height, depth, or energy of a particle system, among others.

- A *vector field* $\mathbf{v}$ is given by a C^r-assignment that takes each base point $\mathbf{b} \in B$ in a vector $\mathbf{v}(b) \in F_\xi(b) \simeq \mathbb{R}^m$; if the base space B is a manifold M. The simplest example is given by a tangent vector t_bM a M at the base point $b \in M$ to a curve γ through $b \in M$ and has a contact of order ≥ 2 with M in b. The set of tangent vectors generates the tangent vector space T_bM to M in a base point $b \in M$. A r-dimensional distribution $\mathcal{D}$ over M is a collection of r vector fields that are " generically " independent. Typical examples are given by the simultaneous consideration of trajectories for different moving points.

- A *co-vector field* $\mathbf{h} = \mathbf{v}^*$ represented by a linear form (corresponding to an evolving hyperplane) is given by a C^r -assignment that to each point base $\mathbf{b} \in B$ takes you in a co-vector $\mathbf{h}(b) \in F_\xi(b)^* \simeq (\ mathbbR^m)^*$, that is, a linear form $varphi : F_b \to \mathbb{R}$ defined on the fiber F_b. If B is a smooth manifold M and $F_b = T_bM$ is the tangent space, then a covector is a 1-differential form ω defined on M, that is, a linear constraint given by a hyperplane at each point. A s -dimensional $\mathcal{S}$ system on M is a collection of s differential forms that are " generically " independent. Typical examples are given by the external restrictions in a scene or the internal ones of the mechanisms in charge of generating movement, e.g.

- A C^k-*tensor field* $\mathbf{t}$ of type $(r.s)$ is given by a C^k -assignment that at each base point $b \in B$ assigns a formal product of r vector fields and s differential forms with coefficients given by scalar fields. Intuitively, it corresponds to consider r trajectories and s (eventually evolving) constraints at each base point $b \in B$. The application to traffic scenes with r mobile agents and s evolving constraints provides almost obvious examples of tensors (see §4,3 for more details).

The classical description of different types of fields is local. Let us remember that a local section of τ_M or τ_M^* is always locally integrable. However, this result is not longer true when one takes several vector of covector fields. One needs strong integrability conditions (Frobenius theorems). In particular, the Global Topology of Varieties poses additional constraints for integrability, which are interpreted in terms of "singularities, existence of " holes " or " tunnels " in dimensions 2 and 3, e.g.. Therefore, the global topology (holes or tunnels, orientation) of the base space B_ξ of the fibration plays a fundamental role.

In Cartesian space $\mathbb{K}^n$ there are no " holes " or tunnels (any cartesian space it is contractible to a point). If the base space B does not have the " homotopy type " of a point, the appearance of these significant events (holes or tunnels, non-orientability) is translated analytically into the existence of multivalued functions or topologically in the existence of to k -dimensional cycles that cannot be contracted to a point .. This " duality " is already found in Riemann's writings and provides one of the keys to understanding the relationships between homology and cohomology (see the module A_{22} for more details and references).

0.1.4. Elements for an intra-history

The interrelation between local and global aspects was well known since the middle of the 18th century (Fagnano, Euler, Legendre). So, for example, at the beginning of the 19th century the impossibility in Differential Geometry to express the length of the arc of an ellipse was well known, giving rise to the development of elliptic functions that are doubly periodic (analytical approach).

Another more sophisticated example is the formal manipulation of the integrals of complex differential forms (H. Abel, 1826) in terms of non-integrable differential forms on the torus $\mathbb{T}^2 = \mathbb{S}^1\ times \mathbb{S}^1$ or, later, of the connected sum of tori (B.Riemann, 1857). This " example " is the first sample of a " global obstruction " to the resolution of an analytical problem due to the non-trivial topology of the support manifold M.

The introduction of cuts for surfaces (B.Riemann, 1856) along non-contractible cycles to a point converts a multiply connected surface to simply connected and, consequently, to multivalued functions in univalued functions. For this reason, the topological and analytical approaches are interrelated, providing support for a local and global information transfer; This idea is due to B. Riemann (1826-1866).

At the end of the 19th century and the first decade of the 20th century, both approaches (analytical and topological) were extended to any type of surfaces, with special attention to the case of algebraic surfaces (Picard) [9].

The formal study of the integrals $\int_{\gamma_i} \omega_j$ of differential forms ω_i on a complex curve $\mathcal{C}$ on the non-contractible cycles γ_j of the Riemann surface $X = M_{\mathcal{C}}$ that represents it becomes a central topic for the algebraic and transcendent approaches to algebraic curves. Throughout the 20th century, the analytical and topological approaches were extended to varieties of arbitrary dimension, both from the local point of view (Analytical Functions of Several Complex Variables, for example) and global (Vector Fiber Cohomology, for example).

In this module A_{42} the global topological approach predominates w.r.t. the algebraic one, that is, we are closer to the global transcendent approach than to the local algebraic approach (see A_{31} for a comparison). Anyway, the most relevant contributions of the second half of the 20th century correspond to the development of results that relate local and global aspects:

- The *local aspects* appear linked to the " singularities " of the distributions $\mathcal{D}$ of vector fields, or of the systems $\mathcal{S}$ of differential forms. item The *global aspects* appear linked to the topology of the base space B of the bundle (or more generally a "fibration") that imposes restrictions on the prolongation or dependency between sections of the distributions or systems .

The information exchange between local and global aspects has given rise to some of the most brilliant developments of the second half of the 20th century, with the Index Theorems (Atiyah-Singer), e.g. For applications, local singularities appear as " events " in the base space B (as changes of state, for example) or in the total space of some of the associated fibrations (as phase transitions in the tangent space TM or in the cotangent $T * M$, e.g.).

Systems $\mathcal{S}$ of differential forms ω_i allow to represent and manipulate the simplest EDP systems (Equations in Partial Derivatives) given on varieties. For this reason, the topological invariants associated with the $\mathcal{S}$ systems provide information about the existence or not of solutions for those systems. When $\mathcal{S}$ can not be solved in an exact way, the cohomology (dual of homology) allows to identify the submanifold over which the system is not soluble (obstruction to the integration of the system) and to work with said submanifold in a formal way.

[9] The technique based on cuts is also used to convert a graph into a tree in Discrete Mathematics and Computational Geometry B_{11}.

Note.- In Engineering it is sometimes said that such problems are "ill-posed" (Hadamard's terminology) and it is wrongly concluded that they are intractable. According to this argument, it would be impossible to predict the behavior of most systems appearing in Nature (which are non-integrable), in particular the system formed by the Sun, the Moon and the Earth. Also, most biomechanical models would be "ill-posed" in Hadamard's terminology. Therefore, we will not use this terminology.

If the system $\mathcal{S}$ generated by s differential forms $\omega_1, \ldots, \omega_s$ is integrable [10], the existence of a solution is represented locally by s " independent functions " f_i (whose differentials df_i are linealry independent) at an initial point $b_0 \in B$. Under regularity conditions, the solution locally parameterized by $(f_1,\ ldots, f_s)$ is extended to the points of a small environment $U \subset M$ (by using standard prolongation of solutions).

In the absence of integrability conditions (for distributions or systems) or regularity conditions for the tangent map, the existence of solutions or their local extension is not always possible. This may be due to "pathologies" involving the local dependency between the sections $s_i : U \to \pi_\xi^{-1}(U)$ for $1 \leq i \leq s$ or to singularities of the B support. Following a strategy of increasing complexity, we first consider the case in which there are independent r solutions (representing the generic range of the fiber system or dimension). Next, the " bonding " conditions between local data (transition functions) are shown and, finally, the " pathologies " corresponding to the support or behavior in the fiber are analyzed.

In the first developments, one assumes initially that the base space B is a smooth manifold M; the existence of local sections induces a structure of smooth manifold over the total space E of any other vector bundle or principal bundle ξ on M, by using the local inverses of π. This description is extended to C^r-fibrations where conditions about the fibre are not so good ones. So, the C^r-global structure on E_ξ is obtained by pasting the C^r -local structures defined on $\pi^{-1}(U) \simeq_{C^r} U \times F$ through local sections.

A lot of results for vector bundles on smooth manifolds were extended to bundles on algebraic or analytical varieties X. The extension is formulated in terms of sheaves of modules on a ring, where initial extensions of local sections given by derivations Θ_X and differentials Ω_X^1 have a natural structure as $\mathcal{O}_X$-modules. In more general cases (fibrations appearing in Engineering, e.g.), the fibers can be discrete spaces (such as those corresponding to finite group actions or covering spaces, e.g.) or continuous ones with the corresponding algebraoc structures (vector spaces, groups, rings, modules on a ring or distributions of different types of fields, e.g.)

Since the fiber of a C^r-fibration is not in general a topologically trivial space, the extension of the results from bundles to fibrations requires a more careful study. From the topological viewpoint it is carried out using Higher Order Homotopy A_{23}, which extends the topology based on closed path spaces

[10] The integrability conditions for field distributions and systems of forms have been developed in the modules A_{11} and A_{14}, respectively, of matter A_1 (Differential Geometry)

to a topology based on sphere spaces. On the other hand, the extension of these results to Hilbert spaces $\mathcal{H}$ is trivial, since $\mathcal{H}$ with the usual metric is a Riemannian manifold with null curvature

Let us remark that the projectivization of a Hilbert space has a metric (like Fubini-Study) that gives rise to a curved manifold with singularities in quantum phase transitions. For more general functional spaces in regard to higher order differential operators, it is necessary introduce additional elements to restrict the variability of solutions. In this module we only consider the case corresponding to Fredholm spaces (bounded linear operators between two Banach spaces) that occupy a central place in the Index Theorems (chapter 8 of this module).

0.1.5. Vector bundles from Geometry to Engineering

The theory of Vector Bundles (VB in the successive) provides the first global version for systems of equations defined on a Smooth Manifolds M. Later, along the late 1940s to the 1960s. Its extension to the GAGA framework (Algebraic Geometry and Analytical Geometry) began in the sixties within the framework of the Locally Free Sheaves Theory. These extensions are not trivial ones (modules are not necessarily free); they are motivated by the need to solve systems of internal equations on eventually singular varieties, whose dependency loci present a “stratification”.

The algebraic extension to non-locally free superimposed structures allows to describe “pathological behaviors” for solutions of systems, and identify it there are deformations to “visualize” simpler structures for such solutions, as in classical PDE or ODE. Thus, they are key to explain “qualitative” changes involving the Geometry, Kinematics and Dynamics for interacting agents.

From a more formal point of view, the first difficulties is linked to the fact that the kernel or cokernel of a map between vector bundles on smooth manifolds are not necessarily a vector bundle. In more down-to-earth terms, there appear jumps in the dimension of the spaces of solutions or the fulfilled constraints [11], nor even when the support it is a smooth manifold M. Hence, it is necessary to give a weaker version of the notion of bundle by replacing it with the notion of sheaves in GAGA or fibration in more general topological cases.

The GAGA approach is developed mainly in the module A_{33} (Sheaves, Cohomology, Schemes) of the matter A_3 (Algebraic Geometry) and is a natural extension of the VB approach (since any bundle is a locally free bundle) . To simplify the formalism, to fix ideas we prior a vector bundle based approach along most chapters of this module [12]

Despite the extension to the GAGA approach uses a description of tangent and co-tangent structures in terms of (non-necessarily free) modules of local derivations $\Theta_{X,x}$ and differentials $\Omega^1_{X,x}$, with their corresponding structure of

[11] M.W.Hirsch gives some counterexamples

[12] The sheaf-theoretical version of the Riemann-Toch theorem (due to Serre for curves, Hirzebruch and Grothendieck for varieties) has been develloped in A_{33}.

"sheaf". Classification issues are solved as usual in terms of Cohomology Theories; in particular the Čech cohomology (linked to a covering $\mathcal{U}$ with "good properties") provides a common support for the GAGA framework [13].

In the smooth case, one proves that Čech cohomology and DeRham cohomology are isomorphic between them. The proof is based on the introduction of a Spectral Sequence, i.e. a double graded complex relating both "cohomological resolutions". Roughly speaking, they can be understood as a simultaneous extension of classical Differential and Integral Calculus from Cartesian spaces to bundles on smooth manifolds (see the chapter 3 for more details).

The relationships between local (typically differential) and global (typically comprehensive) aspects is one of the most important sources of results and motivations for a large number of problems. The obstruction to solving systems of equations or to prolonging definite local solutions over an open is expressed using "characteristic classes" which are cohomology classes of the base space, representing the loci where sections are dependent between them (the module is not free).

Nevertheless, in this module we will restrict ourselves to the simplest cases corresponding to linear maps between bundles (extending linear operators to the global case). A more complete approach requires a reformulation in terms of jet bundles; some basic aspects will be introduced in the module A_{45} (Stratifications).

Along most chapters we prior applications to Theoretical Physics de to the interplay developed from the 1950s. The most ccomplete reference is [Nak90]. This mutual influence has motivated the development of applications to Engineering areas, and the challenge of adapting to other knowledge areas such as Theoretical Biology or Economic Theory, e.g. In view of irregularities for the base space and flows on them, it is convenient to use weaker structures than vector bundles or sheaves; a typical example is given by pre-sheaves.

From the beginning, Vector or more general Fiber Bundles have been applied to Geometry and Analysis in regard to the description of structural properties involving linear operators. They are related with the simplest deformation or propagation models on C^r-varieties. Some more difficult challenges concern to non-linear models, such those appearing in non-linear diffusion-reaction models, which are crucial to understand more complex interaction models. The interplay between discrete and continuous structures is developed in the part II of these notes.

Furthermore, linearization strategies A_{12} (and the corresponding multilinear reformulation A_{13}) have been extended to Differential A_{14} or integral Calculus A_{15} on Manifolds M. A challenge in Differential Topology and their applications is to understand relations between local and global aspects having in account "irregularities" in matter distribution and their modifications in Flow Analysis.

Vector and Principal bundles provide a structural initial framework to relate local and global aspects. However, they can be "too rigid" ones Things beco-

[13] For more advanced cohomology theories (étale, crystalline, e.g.) see A_{33}.

me easier when one introduces the action of some (discrete, finite- or infinite-dimensional) group G, because it provides a support for extending local properties by virtue of the group (or linked algebra) action.

Most initial applications of Vector or Principal Bundles have been developed in a parallel way to the *Standard Model* in Theoretical Physics, i.e. the unification of Electromagnetism, Weak and Strong interactions, along the 1960 and 1970s. More recently, some extensions of Bundles have been introduced in several Engineering areas including

1. *Computational Mechanics of Continuous Media* B_1 including basic aspects of Computational Kinematics B_{14}, or Hydrodynamics and Elasticity in Computational Dynamics B_{15} in terms of Tensor Fields.
2. *Computer Vision* B_2 involving bundles adjustment for 3D Reconstruction B_{22}, Image and scene flow in Motion Analysis B_{23} or a reformulation of Deep Learning for automatic recognition B_{24}.
3. *Robotics* B_3 involving motion analysis in terms of principal bundles B_{31}, control issues for Automatic Navigation B_{32}, Symplectic Geometry for Robot Kinematics B_{33}, to start with.
4. *Computer Graphics* involving geometric propagation models for 3D objects B_{41} and scenes B_{42}, Radiometric models as "superimposed layers" for objects and scenes B_{43} or local vs global techniques for Animation and Simulation B_{44}.

Applications to other areas such as Economic Theory or Biomedical Sciences are more scarce. A preliminary version for applications to Economic Theory has been developed in the last chapters for each module of Differential Geometry A_1. They include some developments in several areas related to Micro- and Macro-Economics, International and Financial Economy. Biomedical applications will be developed in the last chapter of each module of the matter B_1 (Computational Mechanics of Continuous Media), because one needs additional computational resources.

In the precedent section we have displayed some impressive applications of Fiber Bundles to Theoretical Physics, including some recent contributions of TQFT (Topological Quantum Field Theory) to unify Gravitation and Quantum Theory in an extension of Differential Topology. Recent developments suggest a joint treatment of "finite collection of sets and maps" instead of individual sets and functions (on operators on functoin spaces). Even if these developments appear in regard to Theoretical Physics issues, the application of the same ideas to Engineering is almost obvious.

Thus, one can ask abut the possibility of extending basic principles going from Set Theory to more advanced structures (as Fiber Bundles or Sheaves, e.g.) to other technological areas. In abstract terms, topological field theory is viewed as a functor, not on a fixed dimension but on all dimensions at the same time. Thus, their basic principles can be applied for the management of fluctuating systems where the dimensionality is changing. Nevertheless, to fix ideas, we will adopt a more down-to-earth approach, nearer to an extension of basic set-theoretical formalism.

In particular, the idea of managing "finite collections of sets" has some antecedents in the simplest vectorial case (subspaces as points of a Grassmannian or nested subspaces as points of flag manifolds, classes of varieties as points of a moduli space, e.g.). However, when one takes morphisms between these finite collections of sets, there are some deeper aspects which deserve more attention.

Some remarks appear in a sporadic way in some excerpts of Grothendieck's reflections, which receive a more systematic treatment in [Bae97] [14]. It is almost obvious that the study of simultaneous behavior of several "smart agents" interacting in some region can be reformulated in these terms.

Furthermore and in a more specific way in regard to contents of this moduile, some initial applications of Bundles to Engineering appear in IST areas from the early nineties. According to the organization of materials of the B_i matters in my web page, one has grouped them around several topics which are related to Computational Mechanics of Continuous Media B_1, Computer Vision B_2, Robotics B_3, and Computer Graphics B_4 (in a broad sense). In all of them there is an overlapping of advanced physical tools and mathematical models which provide a support for extending some basic aspects of fiber bundles which have been sketeched above.

Most materials presented in this section are developed with more detail in the forementioned matters B_i, but there is no a systematic treatment, still. Thus, our presentation has some fragmentary character, waiting for smarter unifyting approaches. Each one of sketched applications is developed as a specific matter with more details (involving models and computational tools) in my web page [15], where one can find more details and references. Thus, one has preserved the language and specific notation corresponding to the introduction of B_i matters.

The introduction of fiber bundles in several Engineering areas is motivated by the need of extending local results to a global framework. The *local* approach to propagation and interaction phenomena uses different kinds of fields which are nothing else than local sections of fiber bundles. So, scalar, co-vector and tensor fields defined on a C^r-variety X can be matched together to give global reuslts involving more complete representations of dynamic phenomena. In the simplest case corresponding to a manifold M these objects are well known from the Differential Geometry of Manifolds [16]

Hence, the first problem to solve is *how to match local data* corresponding to fields in *global C^r-structures* in a coherent way, i.e., compatible with state changes or phase transitions holding on the original space or their "first-order"

[14] J. Baez: "An introduction to n-categories", in E. Moggi and G. Rosolini (eds): *7th Conference on Category Theory and Computer Science*, eds. , LNCS 1290, Springer Verlag, Berlin, 1997, pp. 1-33.
[15] https://www.mobivap.es/miembros/javier-finat/
[16] Details and references along the matter A_1 in my web page.

variations. Differential Geometry and Topology provides the most structured framework which is initially based on (maps or morphisms between) smooth manifolds and vector bundles. Thus, along this section one can consider differentiable frameworks as the most intuitive framework to apply these techniques to Engineering areas.

0.2. Reducing the Classification Problem

The central topological problems are the classification and characterization of C^r -varieties X. This module does not address the characterization problem (too difficult). The classification problem follows a classical dichotomy which is based on

- *Morphological strategies* based on the structure of the base space denoted as B in the topological case, M in the smooth case or Xin the GAGA framework. Several strategies have been developed in Algebraic Topology A_2, Algebraic Geometry A_3 and the precedent module A_{41}
- *Functional strategies* based on superimposing additional structures (vector vs principal bundles, fibrations, sheaves) $\mathcal{F}$ on the base space and to study morphisms between structures. The existence of a collection of C^r-invariants for the topological, differentiable, algebraic and analytic cases provides test beds for their extension to structures superimposed on the manifolds.

The "lifting" of C^r-invariants from B to the total space E of the superimposed structure or inversely, the "descent" from E to B, rquires additional results involving the "contraction of the fibre" F. There are different strategies involving products and integration along the fibre. All of them are formally expressed in terms of the cohomology of the superimposed structure as a tensor product of cohomologies of the ase B and the fibre F (Kunneth theorems).

Anyway, all classification results based on cohomology are of "negative type", i.e. if two manifolds or superimposed structures have different invariants, they cannot be C^r-equivalent between them, but the converse is not true [17]. In despite of this limitation, an advantage of the cohomology based approach consists of providing " effective computations" for homotopy type invariantes.

0.2.1. Comparing topologies

From a geometric viewpoint, if we restrict ourselves to C^r-topologies, the simplest case of association of cohomological invariants to a smooth manifold M corresponds to the cotangent bundle τ^*_M as a dual version of the linearization of

[17] Some similar problems have already appeared in Algebraic (co)Homology A_{22}, where one uses additional criteria (Reidemeister torsion, e.g.) for finer classifcaiton criteria

the structure of the base space M. The DeRham cohomology has been developed in the modules A_{14} (Exterior Differential Calculus) and A_{15} (Integration in Varieties) of the matter A_1 (Differential Geometry.

The above methods are extended in this module to the Ordinary and compact DeRham support cohomologies of a bundleon a PS-manifold M. The existence of a local trivialization for the cotangent bundle τ_U^* , allows to extend the Exterior Differential Calculus defined on $U \subset M$ to each $\pi^{-1}(U) \simeq_{C^r} U \times \mathbb{R}^m$, " by matching local data.

Kunneth-type theorems allow the computation of topological invariants of the total space E in terms of tensor products of cohomology classes of the base space B (usually a C^r-variety) by cohomology classes of the fiber F. The homotopy type invariance of the bundle classes obtained for the bundle is a consequence of the similar result for any cohomology (as invariant of the homotopy class).

Spectral sequences (initially developed by Leray) allow comparing the results obtained using different cohomologies or comparing results associated with different structures (real vs complex, for example). The basic idea for spectral sequences consists of considering a doubly graded complex linked to the tensor product of the two structures to be compared, and fix the "total degree" of tensor product. Next, one must construct the "homotopy operator" (preserving the total degree) which relate each pir of consecutive elements by combining differentiation and integration transformations [18]. If the fiber is "almost topologically trivial" (as for vector or sphere bundles, e.g.), then all we need is to know how compute the cohomology of the base space.

There are different procedures to calculate the cohomology of manifolds or more generally of " triangulable " topological spaces [19]. In an intuitive way, cohomology provides a general method to identify when a system of equations is solvable and, if so, a strategy for its resolution (by means of complex graduates). Therefore, cohomology provides the appropriate language that extends the point of view presented above as motivation for bundles and their extensions (as solving systems of equations).

In particular, the vanishing of the cohomology (of the base space B or of the fiber F) gives solvable systems in the C^r-category. Contrarily, the non-vanishing of the cohomology results in an "obstruction" to solve the system or to extend the solutions calculated for a small neighborhood of a point. The existence of an additional C^r -structure (corresponding to each one of theC^r -Geometries) means that a system can be solvable in one category (differentiable, for example) and not be so in another (the algebraic, e.g.). This explains the usefulness of cohomological techniques not only for solving systems of equations, but for comparing C^r-structures.

[18] A very good exposition appears in the part II of [Bot83]

[19] A topological space is triangulable if it is homeomorphic to a simplicial complex

0.2.2. Comparing classifications

The coarsest classification is the topological one, that is, up to homeomorphisms. It is developed in General Topology with weak criteria (compactness, connectedness, separation). In Algebraic or Geometric Topology A_3 one develops more effective methods, which are based on the superposition of topological structures such as generalized loop spaces, different types of complexes, and/or homological vs cohomological relationships between both of them [20]. The basic combinatorial nature of the resulting structures facilitates explicit calculations in terms of the homotopy or (co) homology groups that have been presented in modules A_{21} (Basic homotopy) and $A22$ (Homology and Cohomology), respectively.

Whatever the environment category, as the classification up to homeomorphisms (or any of its C^r-subgroups) is stricter than the classification up to homotopy type, the cohomology classes of bundles only provide criteria of "negative type" for the classification. In other words, if two superimposed structures have non-isomorphic cohomology, then they cannot be homeomorphic and, therefore they are not C^requivalent. However, the converse is not true; that is, equality between C^r-invariants does not mean that the C^r-structures are equivalent. Thus, one must specify which is the corresponding C^r-structure linked to each characteristic cohomology class.

Beyond the classification of manifolds or varieties, we are interested in classifying superimposed structures given by bundels of fibrations. This problem involves to an extension of "relative classification" of maps between manifolds or varieties. In more formal terms, this involves to maps between bundles $\Phi : \xi \to \eta$ given by pairs (ϕ, f) of maps making commutative the diagram

$$\begin{array}{ccc} E_\xi & \to & E_\eta \\ \pi_\xi \downarrow & & \downarrow \pi_\eta \\ B_\xi & \to & B_\eta \end{array}$$

i.e., $\pi_\eta \circ \phi = f \circ \pi_\xi$. Similar constructions (including direct and inverse images) have been developed in the module A_{33} (Sheaves, Cohomology, Schemes), which provide a unified language for all usual Geometries.

The relative (co)homology allows to compare superimposed structures corresponding to the restriction to a subspace A with those of the ambient space X. They can be understood as a topological extension of the study of extrinsic properties of a submanifold in the smooth case. . This reaosning scheme is naturally extended to more general C^r-maps $f : Y \to X$ in a natural way.

To fix ideas, let us remember that in the smooth case the relative cohomology $H^*(A, X; \mathbb{R})$ makes it possible to compare equivlalence classes of co-chains of multilinear maps on "aggregate quantities'. Formal manipulation is performed in terms of graded algebras linked to the subspace A with those of the

[20] Their foundations have been developed in the modules A_{23} (Cell complexes) and A_{24} (Geometric Topology) of the matter A_2 (Algebraic and Geometric Topology)

ambient space X. In both cases (homology and cohomology) involving (X, A), the fundamental result is the Excision Theorem that allows us to " separate " the "non-essential" part (open contractible subset) from the most meaningful part corresponding to the behavior at the boundary. This idea is extended to superimposed structures (bundles vs fibrations) in this module.

The extension of the relative co-homology corresponding to maps $f : (A, X) \to (B, Y)$ with $f(A) \subset Y$ to pairs to maps (almost always defined) is almost obvious. The extension of these applications to the study of the relative behavior of tensor fields given by sections of bundles ξ or η on the corresponding base spaces $f : (A, X) \to (B, Y)$ is pure routine. However, it is very meaningful for some advanced aspects of Robotics B_3 involving the Automatic Navigation of Autonomous Vehicles B_{32}.

An almost obvious application of this point of view is to take A (resp. B as the space occupied by a set of mobile agents $a_i(t)$ for $t = 0$ (resp. $t = 1$. The simultaneous consideration of evolving trajectories and constraints in a traffic scene with the corresponding "relative weights" is represented by means a tensor on X, whose type (r, s) varies according to "events" involving trajectories (integral curves of vector fields) and constraints (integral hypersurfaces of covector fields). This approach is developed in the module B_{32} (Automatic Navigation) of the matter B_3 (Robotics).

0.2.3. Vector Bundles on Homogeneous Spaces

A homogeneous space is given by a quotient G/H of a Lie group G by a closed Lie subgroup H. First examples are linked to classical (cartesian, affine, projective) Geometries with the (Euclidean, Pseudo-Euclidean, Hermitian) metrics linked to Classical Groups. Other important examples with non-trivial topology are given by (real vs complex) spheres, Stiefel manifolds (parameterizing spaces of references), Grassmannians (parameterizing k-dimensional subspaces, or Flag Manifolds (representing collections of nested subspaces).

The homogeneous nature of G/H allows to study its properties at an base point $b \in G/H$, and translate them to any other base point $b' \in G/H$ by the action of G. In this way, the hierarchies associated to a "cellular decomposition" can be replicated at every element. Furthermore, the inverse image $g^{-1}S$ of the tautological bundle S via the generalized Gauss map $g : M \to G/H$ on the Homogeneous Space, is isomorphic to the tangent bundle τ_M on M. More precisely, the isomorphism classes of vector bundles can be described in terms of homotopy classes $M, G/H$, and inversely. Less simple situations correspond to the extension of this approach to generalized symmetric spaces [21].

With the above notation, if G/H is a homogeneous space, then $(G, G/H, H)$ is a fiber bundle, where G is the total space, G/H is the base space and H is the fiber. The resulting structure is a "principal bundle", where the fiber is given by a group instead of a vector space. By using the locally structure as a

[21] See the module A_{24} (Geometric Topology) for more details.

product, one can compute the topology of the principal bundle $\mathcal{P} = (P, \pi, B, G)$ by using tools (Kunneth theorems) arising from the (co)homology and homotopy theory for the base space and the fiber. Equivariant bifurcation problems linked to breaking symmetries will be introduced in the modules A_{45} (Stratifications) and A_{46} (Dynamical Systems).

In the PS framework, explicit computations for the cohomology of classical groups were developed by Maurer and E.Cartan in terms of invariant differential forms [22]. The homotopy groups of Classical Groups was computed by R.Bott, and requires additional elements of Spectral Sequences to compute Higher Homotopy Groups [23].

If we look at the base space G/H, cellular decompositions of homogeneous spaces G/H (see module A_{23} for an extensive treatment) and their extensions to symmetric Riemannian spaces, play a fundamental role to obtain an explicit description for structural symmetric models. Cellular decompositions provide a support for propagation models by using "replication strategies" for g-orbits. This geometric construction can be extended to the analytic framework by using differential operators (generalized Laplacians, e.g.) and integral operators (appearing in Variational Calculus, e.g.).

From the differential viewpoint, advanced Morse theory (linked to variational problems provides a nexus between the diagrams of symmetric Riemannian spaces G/H and homological properties of loop spaces which are linked to a cellular decomposition (Schubert cells) of homogeneous space [24]. Extended variational calculus will be generalized at the end of this module in terms of bigraded variational complexes, extending the treatment of Emmy Noether based on infinitesimal symmetries for PDEs linked to the minimization of integral functionals.

The cellular decomposition of the Grassmannian $Grass(k, n)$ of k-dimensional subspaces of a n-dimensional vector space V was initially described by H.Schubert (around 1880) in terms of incidence conditions of subspaces w.r.t. aa complete flag. This decomposition plays a fundamental role in Enumerative Geometry (module A_{35} of Algebraic Geometry) and along the current module. Schubert cellular decompositions for G/H are a natural extension of the cellular decompositions of $Grass(k, n)$ which ease query processes of optimal solutions on spaces of matrices.

Furthermore the ubiquity of Stiefel and Grassmann manifolds in Theoretical Physics, they are increasingly used in several Engineering areas such those appearing in the part II of these notes. Artificial Intelligence is a "transversal topic" to all of them. Self-Organizing maps (SOM) develop learning methods for adaptive subspaces in supervised vs unsupervised strategies. The adaptive con-

[22] See chapter 5 of the module A_{14} (Differential Forms) of the matter A_1 (Differential Geometry of Manifolds)

[23] See module A_{24} (Geometric Topology) of the matter A_2 (Algebraic and Geometric Topology)

[24] Basic Morse theory has been developed in the Chapter 5 of the precedent module A_{41} (Basic Differential Topology) of the module A_4 (Differential Topology)

trol maximizes the fitting of subspaces to and optimal subspaces, which can be guided in terms of incidence conditions parameterized by appropriate Schubert cycles. Beyond this simple remark, in view of the variability of the dimension, in the module A_{45} we develop a stricter hierarchy which is based on Flag Manifolds as universal space for varieties with “good stratifications”.

0.2.4. Topological classification

The first step is the proof of the role played by the tautological bundle S on $Grass(k,n)$ as a universal bundle for (homotopy classes of) Gauss maps $g : M \to Grass(k,n)$. By using basic properties involving the topology of Classical Groups A_{24} one can construct similar classifying spaces for real, complex and symplectic smooth compact manifolds for classical groups constructed for $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (quaternions) connecting classification problems involving Real vs Complex Geometry and Kinematics.

Following this approach, furthermore the classical approach in terms of generalized Schubert calculus. This viewpoint can be illustrated in terms of the higher homotopy groups of the corresponding Classical groups $SO(n)$, $SU(n)$ and $Sp(n)$ constructed on the real $\mathbb{R}$, unit complex $\mathbb{C}$ and unit quaternions $\mathbb{H}$ numbers,. Their representations provide the benchmark for developing algebraic aspects of the four interactions appearing in the Nature. Obviously, they provide ideal or “toy” models to be recombined in more advanced symmetric spaces where an extension of differential and integral calculus are developed.

The topology of homogeneous spaces and Riemannian symmetric spaces can be described in terms of (tensor products of) the topology of Lie groups, where “group reduction” plays a fundamental role. The simplest example corresponds to the “polar decomposition” of a regular real (resp. complex) matrix as a product of an orthogonal (resp. unitary) matrix and a symmetric matrix; as the set of symmetric matrices is a vector space, it is contractible from the topological viewpoint and therefore, one can reduce the action of the general linear group to the action of the Orthogonal (resp. Unitary) group, which is essentially the Gram-Schmidt orthogonalization procedure.

More generally, by virtue of the Cartan-Maltsev-Iwasawa theorem any connected Lie group is homeomorphic to the product of one of its compact subgroups and an Euclidean (hence contractible) space. In other words, one can perform a description of the topology of connected Lie groups in terms of the topology of its compact Lie subgroups. This remark justifies the study of representations of Compact Lie Groups which was developed along seventies of the 20th century.

The description and development of effective computations for cohomology classes of C^r -structures on varieties (vector and principal bundles on manifolds. along this module) has been carried out (initially manually) since the late 1940s. The most important landmarks correspond to real (Stiefel and Whitney, independently), quasi-complex (Pontrjagin) and complex (Chern) structures.

All of them use symmetric functions for explicit computations. Their application to algebras of differential operators poses additional challenges. Global issues are developed in the $\mathcal{D}$-modules framework A_{33}. Local aspects involving classification of function- and map-germs are developed in the modules A_{43} and A_{44}, respectively. The introduction of computational tools at the end of the eighties makes it possible to automate these calculations (in some cases very tedious and prone to errors) [25].

The unification of explicit computations is performed by hand for cohomology of vector bundles along this module. It allows the discrmination between different structures (real, complex, quasi-complex, e.g.) is carried out by using graded differential complexes associated with DeRham (real differentiable manifolds), Hodge-Dolbeault (complex manifolds), Pontrjagin (symplectic manifolds), signature (Hirzebruch, Dirac). The great synthesis was carried out by Atiyah and Singer in a geometric-functional framework of elliptical differential operators (Fredholm operators) on compact manifolds.

A *Fredholm operator* is a bounded linear operator $T : X \to Y$ between two Banach spaces with finite-dimensional kernel $\ker T$ and finite-dimensional (algebraic) $Coker(T)$, and with closed range for T. The *index of a Fredholm operator* is the integer

$$T := \dim \ker T - \operatorname{codim} \operatorname{rank}$$

or in other words,

$$\operatorname{ind} T \;=\; \dim \ker T - \dim \operatorname{coker} T.$$

Let us remark that the $Coker$ is the ker of the adjoint operator T^*. Hence, we are measuring the difference between the dimensions of solutions sets for differential operators In the real case, they correspond to spaces of real-valued differential l-forms. The expression of the Euler-Poincaré characteristic $\chi(E)$ as an alternating sum, can be written as a sum of even degree cohomology, minus the sum of the odd degree cohomology; in the smooth case cohomology classes are represented (Hodge's theorem) by harmonic differential forms, i.e solutions of the operator $D^2 = (d + d^*)^2 = dd^* + d^*d$.

From the functional viewpoint, the expression of the Laplace-Hodge operator as $D^2 = dd^* + dd^*$, and the representation of cohomology forms by harmonic forms (solutions of the Laplace operator), one has that the odd degree terms are nothing else than $Ker(D^*) = Coker(D)$. In the global complex case, they correspond to the dimension of spaces of sections of holomorphic vector bundles. In this way, one sees that the Atiyah-Singer Index Theorems are "quite natural" for the real (DeRham) cohomology and complex (Dolbeault) cohomology. At the end of this module we give some additional details, including signature (Pontrjagin) and Dirac differential complexes.

[25] The computational approach is presented in the B_{12} (Computational Algebraic Topology) module of the matter B_1 (Computational Mechanics of Continuous Media)

0.2.5. Some problems to solve

As in any Geometry or Topology, the *general problems to solve* are the classification and characterization, both of varieties and of maps. The characterization problem is, up to the $1D$ case, very difficult to solve in general. Therefore, we focus on the *classification problem* that is solved by calculating C^r*-invariants* associated with the C^r - structure of the total space E; the local topological triviality of the bundle allows to express the invariants of E in terms of the invariants of the base space B and of the fiber F by virtue of Kunneth-type theorems A_{22}; they express the (co)homology of a local trivialization in terms of the tensor product of the (co)homologies of the base and the fiber.

The *most relevant invariants* are initially of a co-homological type, that is, they INVOLVE the classes of co-cycles (linear operators on cycles) defined locally on the base manifold. Therefore, they are invariants of the homotopy class of the base space B or of the total space E of the bundle; in the differential framework they are analyzed in terms of equivalence classes for fields defined on the variety. In general, they can be interpreted as invariants modulo C^r -deformations of the base or of the C^r-overlapping structure [26].

The non-vanishing of *characteristic classes* "measures the deviation" of a local (real, complex, quaternionic) structure from a trivial product structure. They were initially introduced by E.Stiefel and H.Whitney (1935) to study the topology of vector fields over a smooth real manifold M, and extened by S.S.Chern to the complex case along the 1940s.

In the general case, characteristic classes are associated to graded (complex) structures built from differential operators in the smooth framework. Similar to what happens for any invariant of the homotopy class, the classification based on these invariants only provides criteria of " negative type "; that is, if two bundles have different invariants they cannot be C r-equivalents, but if they have the same invariants an analysis using other types of criteria is necessary.

In this introduction a historical approach is adopted, starting with some relationships with other areas of Mathematics, where Vector Bundles and Fibrations have already shown their usefulness. This approach is complemented with applications to Theoretical Physics as a consequence of the advances in the standard model that unifies the electromagnetic, weak and strong interactions. The classical equivalence (in absence of external forces) between differential (Hamilton-Jacobi) and integral (Euler-Lagrange) approaches to Analytical Mechanics, is reformulated now as to the minimization of the curvature functional (a variational problem) in the space of connections on a Principal Bundle (homogeneous differential approach).

First unified formulations are proposed by Yang and Mills (1954), giving the starting point for an explosive growth of this topic from the 1960s. From this time, there appears an overlapping and mutual fertilization between different

[26] For details and references on basic results and examples, see module A_{22} (Homology and Cohomology) of the matter A_2 (Algebraic Topology)

areas (geometric, topological, analytical) of Mathematics and Theoretical Physics. Roughly speaking, bundles can be said to provide a *global framework* to assess whether the *local resolution* of any kind of C^r -equation systems can be extended or not to a whole C^r-manifold or a space of functions. This argument justifies the ubiquity of bundles (or their generalization to bundles) in all areas of Mathematics.

The simplest example for a vector bundle is given by the trivial vector bundle $\varepsilon^r := (M\ times \mathbb{R}^r, \pi, M, \mathbb{R}^r)$ of range r over a base variety M. In this case, the system of equations to be solved is always the same; that is, the resolution for a point extends to all points of the base variety M; therefore, it is reduced to a Linear Algebra problem. If the manifold is not topologically trivial (the torus $\mathbb{T}^2$, for example), integration obstructions may appear from both the base manifold and the fiber.

Over a smoothly trivial manifold like the circumference $\mathbb{S}^1$, e.g. you can have a trivial bundle $\varepsilon^1 = \mathbb{S}^1\ times \mathbb{R}^1 \simeq \tau_{\mathbb{S}^1}$ corresponding to the tangent bundle or a non-trivial bundle $\gamma_{\mathbb{S}^1}$ (extended Moebius band) both of rank 1. This example shows the importance of comparing the fibers corresponding to closed paths over a variety. The distribution can always be locally embeddable, but not globally.

Therefore, it is necessary not only to solve different systems of equations, but to "match" the local data each time closed paths are completed. When the rank r of the system of equations or the dimension of the solution space (kernel of an operator) is *constant*, a local product structure is obtained for the set of solutions. The set of solutions is then said to be a locally free module on the local ring of regular functions $\mathcal{O}_{X,x}$ at $x \in X$.

If the (co-)dimension or (co-)rank $f : X \to Y$ is locally constant up to homotopy, we only have a structure of C^r-fibration, (pre-)sheaf or analytical space. Fibrations and analytical spaces are covered in module A_{45} (Stratifications) of this matter A_4. The (pre) beam structure is studied in more detail in module A_{33} of the matter A_3 (Algebraic Geometry). The key in all these cases is the study of local sections $s : U \to E$ verifying $\pi \circ s = id_U$ of the C^r-fibration on an open U of the base space.

The study of sections allows addressing issues such as

- the prolongation of local structures and their gluing in C^r -global structures of different types;
- the existence or not of non-trivial deformations in C^r-categories
- the " obstructions " to the existence of a C^r -Non-trivial deformation for a prolongation of solutions.

Obstructions are expressed in terms of non-vanishing of C^r-invariants called characteristic classes. The most relevant ones for PS-structures are labeled as Stiefel-Whitney, Chern, Pontrjagin and Dirac. As any cohomological class the are invariants of the homotopy class. The operations defined on these groups or the exact sequences e.g. they play a fundamental role for classification issues of superimposed structures (vector vs principal bundles, fibrations).

These motivations explain the ubiquity of Vector Bundles and their generalizations in all scientific areas, Engineering or Economic Theory. The most spectacular applications correspond to Theoretical Physics developed in a systematic way from the sixties in relation to the standard model.

The Grand Unification of the Standard Model with the gravitational interaction (the weakest of all) continues to be the most difficult challenge to solve, as there is still no satisfactory model for the quantization of this interaction. More recently, an exponential growth of applications to Engineering is taking place, some of which are presented in the modules B_i of these notes.

0.3. References for this introduction

References are not the most recent ones, nor exhaustive. They are included only to give the reader some more complete or alternative insights, in order to construct his/her own vision of this subject.

0.3.1. Basic bibliography

Only some textbooks are included. For more enlarged bibliography, see the subsection §5,4. References for meaningful research articles are included as footnotes.

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0.3.2. Software resources

In the module B_{13} (Computational Differential Topology)of the matter B_1 (computational Mechanics of Continuous Media) we develop a computational approach to basic aspects of Differential Topology. Their foundations can be found in

[Her13] M.Herlihy, D.Kozlov, S.Rajsbaum: *Distributed Computing Through Combinatorial Topology*, 2013.

[Zom05] A.J. Zomorodian: *Topology for Computing*. Cambridge Univ. Press, 2005.

It is necessary to develop more specific software for a computational treatment of most aspects developed in this module. To my knowledge, the only reference is Singular. Additional information about other software packages is welcome.

Final remark: Readers which are interested in a more complete presentation of this chapter or some chapter of the module A_{42} (Fiber bundles), must write a message to franciscojavier.finat@uva.es or to javier.finat@gmail.com