A400 Differential Topology. An introduction

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Índice

This chapter correspond to an introduction to the matter A_4 (Differential Topology) and some applications. It is necessary to have some basic knowledge of General Topology, Analysis of Several real and complex Variables, Differential Geometry A_1 and Algebraic Topology A_2. Some meaningful examples are taken from objects appearing in Algebraic and Analytic Geometry A_3, but most GAGA methods are not used in this module.

As usual, in addition of this introduction, materials of this chapter are organized in four sections. Subsections or paragraphs marked with an asterisk ($*$) display a higher difficulty and can be skipped in a first lecture.

0.1. Introduction to the chapter A400

Differential Topology can be considered as an extension of General Topology to the differentiable case or, more precisely, as the introduction of topological methods for the study of smooth manifolds M and smooth maps $f : N \to P$ between them. In the same way as Differential Geometry can be understood as an extension of Multivariate Analysis to the global case of manifolds, Differential Topology is also labelled as Global Analysis, for readers arising from Analysis.

Both approaches are naturally extended to eventually singular algebraic or analytic varieties X and non-necessarily C^r-maps $f : X \to Y$ between them. Source X and target space Y for f are initially considered as piecewise smooth (PS) manifolds. Systems of equations E_b which are C^r-dependent on the base point $b \in B$ are modelled in terms of different kinds of locally trivial superimposed structures (bundles, sheaves, fibrations). The local triviality is the key to extend the C^r-structre from the base space B to the whole total space $E = \cup_{b \in B} E_b$ of the superimposed structure.

The *fundamental goals of Differential Topology* are the characterization and classification of "objects" using differential methods. "Objects" of class C^r can be manifolds M or varieties X, C^r-maps $f : X \to Y$, superimposed structures $\xi = (E, \pi, B, F)$, locally trivial on a base space B with generic fibre F, and (not everywhere well defined) morphisms $\phi : \xi \to \eta$ between superimposed structures (representing systems of C^r-equations on the base space B). Characterization is a too hard problem, and by this reason we will be focused towards classification.

The first attempts to formulate a Differential Topology are due to PoincarÃ© at the beginning of the 20th century. However, an appropriate formalism was not yet available, since not even the notion of a smooth manifold was well defined. The first contributions for superimposed C^r-structures were developed throughout the first half of the 20th century by Poincaré, Cartan, Morse, Lefschetz, Hopf, Pontrjagine, Whitney, Chern, among others. They culminate in the first synthesis carried out by Steenrood in the early 1950s, and first reformulation in terms of sheaves A_{33}.

The confluence of formalism from Bundle Theory, and Local Algebra (Commutative and Homological Algebra), were the key to the introduction and explicit computation of C^r-invariants for equivalence classes of "objects". The most relevant results concern to the relation between local characters and global C^r invariants, involving the base space B and the superimposed structure E.

Next steps concern to the classification of C^r-maps and morphisms $\Phi : \xi - \to \eta$ between superimposed structures. They start with the regular case (submersions, immersions, embeddings), to be extended next to increasingly complex singularities, with a special regard to the (smooth vs analytic) C^r-classification of singular map-germs $f \in C^r(n, p)$. Most of the first great theorems are developed between 1950 and 1975 by Arnold, Atiyah, Chern, Mather, Milnor, Novikov, Smale, Thom and Wall (alphabetical order, not chronological) throughout the sixties and seventies, extending the work of Hopf, Lefschetz, Morse, and Whitney, between others.

The unification of different techniques is performed in algebraic terms as a by-product of the algebrization program proposed by A.Weil for the Algebraic Topollogy A_2. Instead of developing a purely algebraic approach as in GAGA A_{33}, most results of Local and Global Differential Topology are developed on the real $\mathbb{R}$ or the complex field $\mathbb{C}$, where one uses standard tools arising from the Analysis and Topology.

Invariants are given by cohomology classes of the base space $H^*(B; R)$, where R is usually $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Their relation with topological invariants $H^*(E; R)$ of the total space arise from the Kunneth formulae, having in account the "almost triviality" of the cohomology $H^*(B; F)$ of the fiber F (a Cartesian space $\mathbb{R}^n$, initially), for the ordinary or the compact support cohomology. Characteristic classes provide invariants for the real structure (Whitney), the complex structure (chern), relations between real and complex structures (Pontrjiagin) or the evolving volume form (Euler) for smooth manifolds.

Some elements for the unification of the basic types of interactions appearing in Quantum Mechanics are linked to

- Relations between the topology of base space B and superimposed structures E (bundles, fibrations, sheaves, e.g.), including their classification up to C^r-equivalence;
- the explicit computation of C^r-invariants linking the topology of B and E, and relations between them linked to C^r-maps or, more generally, morphisms to classify, also;
- the expression in terms of differential complexes (DeRham, Dolbeault, signature, Dirac);
- the interpretation in terms of optimal solutions for variational problems (Hodge)
- the introduction of integral operators of the curvature of an affine connection ∇_a on a Principal Bundle as an ideal support for the interaction.

All of them are key for the development of the Standard Model, which unifies Electromagnetic, Weak and Strong interactions along 1960s. As a pending big challenge, it remains the Great Unification Theory with a quantized model of gravitational interaction where supersymmetry (a model incorporating breaking symmetries for different interactions) plays a central role [1]

The development of superimposed structures from the mid-20th century was accompanied by the introduction of new notions that formalized previous ideas due to different schools of mathematicians. Some of the most important contributions are linked to the interrelationships between local and global aspects including the main objects corresponding to

[1] See the last chapters of the module A_{42} (Bundles) for more details.

- *bundles* (or, more generally, fibrations to include possible degenerations) for distributions $\mathcal{D}$ of vector fields or dual differential systems $\mathcal{S}$ of covectors;
- the explicit calculation of *invariants* corresponding to fields (index, e.g.) or differential forms (characteristic classes, e.g.) initially for smooth varieties M;
- G-*structures* including homogeneous and locally symmetrical spaces, with principal bundles as the central "example";
- *sheaves* as a global version of systems C^r-equations that allow representing the irregularities in fibrations corresponding to spaces of solutions for $\mathcal{D}$ or $\mathcal{S}$, e.g. item *Deformations* of any of the above C^r-structures for $r = \infty$, $r = \omega$ or $r = alg$;
- the *Global Variational Calculus* corresponding to the study of the extremal values of integral functionals (distance, area, volume, energy, curvature, e.g.).

Differential Topology develops tools for the C^r-*classification* for

- possible C^r-structures on the base space B corresponding to a manifold M or a variety X, and relations between different kinds of C^r-structures on the base space F;
- superimposed structures (bundles, sheaves, fibrations) up to isomorphism in the smooth, analytical and algebraic categories;
- the *global matching* of local data extending basic topological approaches to more advanced theories of cobordism;
- the introduction of *classifying spaces* B_G linked to classical groups G or more general algebraic groups, with a special regard to the semisimple, reductive and unipotent cases;
- the analysis and classification of *singularities* associated with each of the previous problems;

All of them are useful in regard to applications to Theoretical Physics (unification theories), Mechanics and other Engineering areas, including the specification of hierarchies associated with map-germ singularities.

The preceding list is not exhaustive, but it is illustrative of many of the problems developed in the second half of the 20th century by a large number of mathematicians, some of whom received the Field Medal for their contributions to the aforementioned problems. In this introduction we only comment on some "snapshots" that are intended to motivate reading the expanded version (currently only available in Spanish). Before showing them, we remember some basic ideas.

The most important areas correspond to the differentiable $r = \infty$, analytical $r = \omega$ or algebraic $r = rat$ categories. The geometric description in terms of groups of automorphisms of the fiber is insufficient, since the group of automorphisms of analytical spaces can be reduced to the neutral element. To do this, it is necessary to develop a topological approach that is a natural extension of the geometric approach.

- The C^r-classification is carried out in terms of C^r-equivalence that affects spaces as well as C^r-structures. Initially, the classification is carried out modulo the action C^r-subgroups of homeomorphisms, such as the diffeomorphisms for $r = \infty$, the bi-analytical transformations for $r = \omega$ or the birational transformations for $r = rat$.

- The C^r-equivalences acting on the source and target spaces of maps $f : X \to Y$, or on the graph $\Gamma(f)$ of f. The best-known case corresponds to the action of automorphisms on the fiber (transition functions) indicated by C^r-equivalences on the intersection $U_i \cap U_j$ of open a trivialization of the C^r-structure. The basic example is given by the transition functions of a vector bundle. This approach extends to the C^r-classification of morphisms between $\Phi : \xi \to \eta$ between overlapping structures.

Finding effective criteria for the C^r-classification of objects (manifolds or structures) and morphisms between objects is a highly non-trivial problem. This problem is similar to that described in Algebraic Topology A_2 where the classification module homeomorphism is “reflected” to the classification by the type of homotopy, where the homotooy A_{21} or (co)homology A_{22} techniques provide more computable criteria.

In this matter, the classification is performed modulo diffeomorphism, but even so the problem remains very difficult; one strategy is to “add more structure”. The analogue to closed paths are periodic integral curves of vector fields; the extent of the PL-decompositions (symplicial vs cuboidal) is given by the cellular complexes A_{23}. The analogue of the PL-cohomologies is given by the DeRham cohomologies for the real case or Dolbeault for the complex case A_{24}.

The smooth manifolds M, the differentiable maps $f : N \to P$, and the vector bundles provide the “initial basic objects” for Differential Topology. The C^r-transformations described above act on these objects. The unification is carried out using some basic algebraic notions. In particular, the graded algebras associated with the (co)tangent space are the basic building blocks.

An important difference w.r.t. the approach developed in the previous matters is that the tangent space refers not only to a variety, but to a space of functions or maps defined on them, as well. Even for simply connected varieties, the classification of differentiable structures is a highly non-trivial problem (Milnor). The extension to the non-simply connected case was one of the motivations for introducing topological K-theory (classification of bundles in terms of cleavage properties), but with limited results.

0.1.1. The interplay between local and global issues

Local issues are formulated in terms of open subsets U_i belonging to a covering $\mathcal{U}$ of the base space B which is initially given by a C^r-manifold or an algebraic vs analytic variety X. Two extreme cases corresponds to $r = 0$ (General Topology) and $r = \infty$ (Differential Geometry). Roughly speaking, $r = \infty$ is "dense" [2] in the C^r-topology for $r \geq 1$; by this reason, the "extreme" cases are the most meaningful ones to start with.

The clarification of relations between smooth, PL and topological structures was one of the central problems along the sixties and seventies. From the global viewpoint, characteristic classes provided criteria of "negative type", but not constructive. From an experimental viewpoint, data are initially unorganized and irregularly distributed. After introducing grouping criteria, it is necessary superimpose PL-structures (simplicial or cuboidal complexes, e.g.), jointly with their symbolic structures (lattices, networks, graphs) to ease their management. Regularization strategies provide PS models where Differential Topology can be applied.

Matching of local data to generate a base space is performed by using local coordinate charts $\phi : U_i \to \mathbb{K}^m$, fulfilling "good constraitns" on interections $U_i \cap U_j$ of open sets of an open covering $\mathcal{U}$. good constraint conditions are formulated by using C^r-equivalences $\phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ between open sets of $\mathbb{K}^m$ whose properties are well-known from Multivariate Analysis. The basic model has been described in Differential Geometry A_1.

Compatibility between coordinate charts are "lifted" to superimposed structures (E, π, B, F) by means locally trivial conditions, i.e. $\pi^{-1}U \simeq_{C^r} U \times F$ (in a non-canonical way). Typical examples are given by vector bundles or principal bundles in Differential Geometry A_1, and their extensions to presheaves and sheaves in GAGA A_{33}. In this way, locally trivial conditions on the base space are naturally extended to the superimposed structure where one has a simple algebraic structure on the fibre.

Global issues involve to non-trivial topological structure for the base space B or, alternately, non-integrability of the superimposed structure (non-unique, discontinuous solutions, or bifurcations phenomena, e.g.). All of them are ubiquitous in all scientific or technological areas. Last chapters of each module of this matter are devoted to sketch some applications in Natural Sciences, Engineering and Health Sciences.

Differential Topology provides a general framework for a unified treatment in terms of models arising from almost all mathematical areas. Some of the most difficult theoretical issues concerns to the classification of non-regular maps. Stratifications A_{45} allow to consider as nested matching of regular maps such as embeddings, immersions and submersions. classification issues are related with the recognition of shapes and behaviours.

[2] A more precise statement will be given in the chapter 2 of the module A_{41} (Basic Differentiall Topology) in terms of Baire sets.

(*) A non-trivial problem consists of the semi-automatic identification of local regularity. Recent advances in AI linked to learning manifolds (resp. maps between manifolds) representing objects (resp. behaviours). A basic strategy consists of "reducing information" which is formulated in terms of composition of submersions (going from the SOM framework of nineties to more recently in OpenAI solutions, e.g.).

(*) Inversely, the simplest topological models to generate new multimedia contents are formulated in terms of a composition of immersions. Some solutions have been developed in the DALL-E framework by using transformers and Stable diffisuion, as a neural model or diffusion-reaction equations. Currently, one tries of developing a more consistent mathematical formulation for these computational developments [3].

Some more classical challenges correspond to try of extending local to global models of non-simply connected manifolds M. In addition of the above matching strategies for local data, one can use global relations between characteristic classes and Hermitian forms on the fundamental group $\pi_1(X)$ of the manifold and the homology $H_*(X;R)$ These relations play a fundamental role for linking local and global aspects.

In particular, the use of functional methods (on Sobolev spaces), topological methnods (homotopy invariance of characteristic classes) and algebraic properties (involving Hermitian forms on co-chains with an involution) illustrate the interplay between different matthematical areas to solve hard classificaiton issues.

An " external" motivation for the growth of the interplay between local and global issues is motivated by a "convergence" between algebraic (group representation theory), topological (Differential Topology) and Analytical areas (Global Analysis). This "convergence" paves the way for the unification between the different types of interaction that appear in Theoretical Physics [4] . The unification also affects the strategies for solving the systems of equations specific to each theoretical framework. For this reason, this topic appears recurrently in several sections and, obviously, throughout the entire module.

In particular, the resolution of differential or integral equations is a central topic in Engineering. This observation motivates the introduction of some more recent applications to different areas of Engineering. In this more technological framework, some topics related to the different subjects B_j that are presented in block II of these notes have been selected. These topics include a geometric approach to Continuous Media Mechanics B_1, Computer Vision B_2, Robotics B_3 and Computer Graphics B_4 which are presented below (section 3).

[3] See the last paragraph of this subsection for some additional details

[4] A first approach to the problems of unification is presented in section 3 of this chapter

0.1.2. Extending group actions to Functional Operators

Vector bundles play a fundamental role to unify differential aspects involving distributions $\mathcal{D}$ of vector fields and systems $\mathcal{S}$ of differential forms. Integrability results (Frobnius theorems) ease the geometric interpretation in terms of foliations $\mathcal{F}$ whose maximal dimension solutions (leaves $\mathcal{L}$) can be interpreted in terms of solutions parametrizing "multipahts" or "multiconstraints".

The multilinear combination of weighted products of s vector fields and r covector fieldds gives tensors of type (r, s), which are interpreted as the simultaneous fulfilment of paths and constraints for simplified evolving system. When transformations between local data follow standard multilinear rules A_{13}, tensor are interpreted as sections of a tensor bundle $T^{r,s}M$ on M. This interpretation explains the ubiquity of tensors in almost all scientific or technological areas, including AI, recently.

Often, the support is not a manifold M, nor even an algebraic or analytic variety X. Thus, we have not a vector space on the fiber corresponding to an evolving system of equations E on the base space B. typical examples appear in discrete models, which can be managed in terms of "particle systems" (as it occurs in Quantum Mechanics, e.g.). To solve classification issues, one must lokk at some kind of local homogeneity on the base space B to be described in terms of local symmetries.

In presence of a group action $\alpha : G \times \mathcal{C} \to \mathcal{C}$ acting on configurations $C \in \mathcal{C}$ with "goood properties", one can replace vector bundles by principal bundles $\mathcal{P} = (P, \pi, B, G)$, with similar locally trivial conditions $\pi^{-1}(U) \simeq_{C^r} U \times G$ for opoen sets U of a covering $\mathcal{U}$ of the base space B. In this case, the structural group G plays a similar role to the generic fibre F of a vector bundle, i.e. the above C^r-equivalence is restricted to an isomorphism $\pi^{-1}(b) \simeq_b \{b\} \times G$ between groups.

In practice, the structural group G does not remain the same, and there can appear "breaking symmetries" phenomena. They can correspond to changes of state in the base space B, phase transitions in the space of the first order variation rates for data which is called the Phase space (Poincaré), or even the second order variation rates which we call the Euler space E. In all cases, breaking vs capturing symmetries corresponds ideally to de- vs re-compositions are formualted in terms of the description of an initial structural group in products of subgroups. Thus, the classification of groups plays a central role.

The precedent description is formulated in regard to the basic hierarchy for Theoretical Mechanics given by geometric, kinematic and dynamic aspects corresponding to the "localization", their space-time evolution (velocities, accelerations) and the analysis of interactions (forces and moment) between configurations. A naive representation for this hiearchy is formulated in terms of a manifold M , its Phase space P_M given by the total space of the tangent bundle τ_M (or its dual), and the Phase space P_M^2 of P_M. So, local symmetries at the base space can be lifted to "succesive extensions" to Phase space P or τ_P.

The description based on successive "extensions" has not good functorial properties, and it can not incorporate the high complexity of shape or behaviour mutations. By this reason, we introduce the space of k-jets $J^k(n,p)$ of C^r-map germs $[f] \in C^r(n,p)$. From a local viewpoint, a k-jet $j^k f$ of a function $f \in C^r(n,1)$ is given by the formal Taylor development truncated at order k. This reduction to a formal polynomial is the key to apply algebraic techniques for the classification of finite determined k-jets, i.e. those which are C^r-equivalent to its k-jet by the action of C^r-equivalences on the source and target space.

The k-jet of a map $finC^r(n,p)$ is defined as the k-jets of components $f_i \in C^r()n, 1$ for $1 \leq i \leq p$. Furthermore, they can be extende to superimposed structures $E \to B$ where $J^k E$ denotes the "set" of k-jets $j^k s$ of local sections $s : U \to E$. In this way, one can define morphisms $J^k E_0 \to J^k E_1$ for superimposed structures on the same base space B. The k-jets formalism allows not only the extension of local ODEs and PDEs, but a reformulation of variational principles which are the key to connect local and global aspects in the functional framework.

0.1.3. Weakening regularity conditions

Increasingly complex strategies start start with smooth manifolds M and regular maps $f : N \to P$ maps (immersions and submersions) between smooth manifolds. Lastones provide general strategies to construct manifolds. These are ideal conditions to be replaced to ease their adaptability to more realistic morphological and functional aspects. The next step corresponds to the analysis of functions $f : M \to \mathbb{R}$ of real valued functions de3fined on a compact m -dimensional manifold M.

The singular locus is given by the vanishing of the gradient field ∇f; the simplest non-trivial case corresponds to quadratic singularities characterized by a non degenerate Hessian matrix, where the index of the resulting quadratic form (Taylor development) determines the local topology of cells $(e^k, \partial e^k) \simeq (\mathbb{D}^k), \mathbb{S}^{k-1}$ to be adjointed by each non-degenerate critical points. Typicla examples of Morse functions are given by scalar linear (height, depth, e.g.) or quadratic functions (different types of distance, maps, e.g.).

in this way, it is possible to recoonstruct the global topology of the manifold M, and inverssely. First versions of this study was performed along the third decade of the 20th centiury by M–Morse for the real case, and by S.Lefschtez for the complex case. Global invariants (as the euler-PoincarÃ© characteristic, e.g.) can be described in terms of local characterisitics (index of the gradient vector field). Functions having non-degenerate critical points are "dense" (any other functions with arbitrary singularities can be "approached" by Morse functions). Hence, "generic deformations" of more complex functions display only Morse singularities.

This viewpoint is naturally extended to the Phase space P, which is modeled in terms of the total space of the (co-)tangent bundle τ_M giving the ambient space for most issues in Classical Kinematics. The energy functional provides a Morse function on the Phase space P. In this case, by replacing the ordinary gradient ∇ by the symplecti gradient $\nabla_J := \mathbf{J}\nabla$ one obtains the Hamilton-Jacobi equations to describe the space-time evolution of particles. Similarly to the static case, level surfaces of ∇_J provide the support for a lot of issues regarding staiblity. This "example" is naturally extended to higher degree Hamiltonian functions $H : P \to \mathbb{R}$, but the topological analysis displays higher trobules due to the apparition of more complex singularities than quadratic ones.

The *moment map* provides a natural G-equivariant extension of conservative mechanical systems linked to Hamiltonians. The added value consists ot re-introducing a G-equivariant approach (solutions as unions of orbits). which is compatiblle with G-equivariant bifurcations. The specification of this program requires an identification of adjacency orbits appearing in the adherence of regular strata. An advantage of this viewpoint consists of the availability of tools for extending local propagation models to the whole decomposition of the space as a union of orbits. So, hierarchies between group actions (llinked to trees of groups, e.g.) can be translated in a natural way to hierarchies between solutions of differential systems or their dual distributions.

The support for this hierarchical G-equivariant approach is given by locally symmetric spaces, whose basic examples have been already introduced in the module A_{24} (Geometric Topology). Here, we put the accent on differential aspeccts, involving eventually nested subgroups. Basic techniques can be considered as an extension of the structure of flag manifolds, which reappear in different ways in the module A_{45} (Stratifications).

Some recent developments involve to the discretization of smooth models classically developed in the second half of the 20th century. Discrete versions can be motivated by the very large diversity of phenomena appearing in solid structures in regard to gases al liquids. Last ones are modelled in terms of particles systems with a random structure, where stochastic processes (initially modelled as random perturbations of ODEs) play a central role.

However, solid materials require different clustering strategies at different levels (molecular vs atomic), with different physical-chemistry laws to each level. A basic distinction consists of crystalline vs amorphous solids. Roughly speaking they correspond to regular arrangements of crystalline micro-structures vs non-regular structures. Their detection is performed by using operators for thermodynamical modelling.

(*) In particular, the presence of "defects" (very rare in Nature) can be modelled in terms of singularities linked to propagation models. Crystallization phenomena appearing at critical ambient conditions (pressure,l temperature, e.g.) are some of the most interesting issues to be modelled in the interplay between smooth and discrete structures. For more details see [Fah23] [5].

[5] B.D. Fahlman: *Materials Chemistry (4th ed)*, Springer, 2023.

0.1.4. Controlling the evolution of living systems

Complex living systems evvle from birth till death. The large diversity of patterns requires a very flexible approach able of identifying structures, their evolution, bifurcations, decompositions and regrouping. The simplest mathematical quasi-static models have been described in the precedent matters. Differential Topology provides models and tools for their static grouping-decomposition, internal kinematic evolution along the space-time, dynamical patterns for the interplay with the environment. These aspects can be considered as a natural extension of Static, Kinematics and Dynamics appearing in Mechanics.

Differential Geometry provides a mathematical framework for Classical Mechanics, Optics and Electromagnetism. Differential Topology extends this framework to some more complex phenomena appearing in Quantum Mechanics. Simplest models for successive levels (Statics, Kinematics, Dynamics) of the hierarchical approach to Mechanics have been developed in the precedent matters. They have been initially described in terms of "regular" maps, i.e. maps whose differential has maximal rank (immersions and submersions provide typical examples), and their extensions to superimposed structures (fiber bundles, fibrations, e.g.). *Irregular models* are developed from A_{33}.

In the first paragraph of this subsection, we have commented the role played by immersions and submersions in artificial life, or more precisely, in Artificial Intelligence. In particular, learning in the SOM (Self-Organizing Maps) framework [Koh97] [6] can be understood as a composition of submersions holding on discrete structures with a similar behavior to a discrete vector bundle.

Reduction dimensionality preserving the regularity is a goal which can be reformulated in terms of submersions. Typical clustering self-adaptive techniques use clustering in subspaces labeled as ASSOM (Adaptive Subspaces in SOM). Thus, SOM-based learning strategies can be immediately translated to learning on a (discrete version of a) Grassmann manifold. If we admit the existence of reinforcement learning by following successive steps, is clear that (a discrete version of) Flag manifolds provide the right framework.

In a complementary way, the generation of new multimedia contents can be understood as a composition of immersions in extensions of the DALL-E framework by using transformers and Stable diffisuion) illustrate possible applications of this viewpoint which lack of a riguour foundation, still. An intuitive idea consists of thinking of the generation as a composition of encoders (given by successive Gaussian blurring) and decoders (given by successive Gaussian filtering) from initial multimedia data.

A "toy model" is suuported by $y = ax^2$ where encoders (resp. decoders) located along the left (resp. right) branch. Obviously, the parabola can be replaced by a paraboloid or a discrete veerson, which is the support not only for good metrical properties [7], but by diffusion properties (Heat Equation, e.g.).

[6] T. Kohonen: *Self-Organizing Maps* (2nd ed), Springer-Verlag, 1997.

[7] See the chapter 6 (Voronoi diagrams) of the module B_{11} (Computational Geometry) of

0.2. An overview of the matter

In accordance with the previous observations, the subject is organized around issues related to differential classification and basic properties of

1. *Smooth manifolds and basic operations* A_{41} with special attention to the regular case (immersions and submersions), the subregular case (Morse Theory) and gluing or segmentation techniques (bordism vs surgery).
2. *Simpler bundle structures* A_{42} with special attention to vector bundles and principal bundles, associated with systems of equations or linear operators, as well as linear actions with their applications to Theoretical Physics.
3. *Function germs* A_{43} to classify the singularities that can appear, with special attention to the case of simple singularities (finite number of types of orbits), with Thom catastrophes as basic types.
4. *Application germs* A_{44} where the methods developed in the previous module are extended (using Local Algebra techniques) to the "vector" case (finite number of components). Some applications to Continuous Media Mechanism are outlined.
5. *Analytical stratifications* A_{45} where the methods of the previous modules are extended to (semi-)analytical varieties, superimposed structures and morphisms between structures. A geometric approach is adopted to facilitate adaptation to engineering applications and interpretation in terms of deformations.
6. *Dynamic systems* A_{46} where basic models of interaction with the environment are developed in local terms (ODE or PDE systems and their extension to the global case.

0.2.1. Some methodological issues

A first basic taxonomy involves to morphological and functional aspects. In global terms it is initially re-formulated in terms of the support (a PS manifold, a cellular complex, a variety, a functional space) and the superimposed structures (systems of equations, vector bundles, principal bundles, sheaves, fibratioins, functional operators). Morphological and functional aspects display a dual behaviour which has been developed in the matter A_1 for the smooth frameworks and in A_2 in the PL-framework.

We follow an increasingly complex methodology starting with spaces, next maps or more generally fields, then superimposed structures, and finally morphisms between them. A novelty (to be visualized as flows) of this matter consists of

the matter B_1 (Computational Mechanics of Continuous Media) incorporating “irregular” spaces and behaviors in terms of singularities involving the support and generalized systems of equations corresponding to the superimposed structures. In this way, one can incorporate well-known “features” (changes of state, phase transitions) to structural models appearing in different kinds of evolving flows.

The most relevant ideas are a natural extension of other ones appearing in the precedent matters. They involvin to

- Replace the linearization of manifolds in terms of (co)tangent bundles by graded structures but involving now to spaces of function. Jets spaces provide methods and tools to develop this idea. The underlying topological structure is more intrincate that those of cell or CW-complexes A_{23}, but it can be managed as an extension of orbifolds to infinite-dimensional spaces.
- Replace the action of Lie groups and algebras on manifolds or superimposed structures (bundles, coverings, sheaves, fibrations) by infinite-dimensional groups of diffeomorphisms acting acting on vector maps or more general operators. This gives some kind of euivariant stratifications.
- Develop the interplay between an evolving topology of the base space B and the fiber F by introducing stratifications for the base space, and relations between different kinds of eventally infinite-dimensional Lie algebras for controlling the evolution of systems, including possible “degenerations” linked to dissipative phenomena (by using nilpotent operators, e.g.)

0.2.2. A basic hierarchy

The basic hierarchy of Mechanics affects Geometry on a manifold, Kinematics on the phase space P (initially the total space TM of the tangent bundle τ_M of a PS-manifold M), and Dynamics on $E = TP$ (labelled here as the “Euler” space) which includes the interaction with itself or with the environment. This approach gives rise to a “natural hierarchy” that is ubiquitous in all applications developed in part II of these notes.

The hierarchy that appears in Mechanics is translated into the action of discrete groups, finite-dimensional continua and infinitesimals of infinite dimension, also. The characterization of possible “raising” or “lowering” of the different types of symmetries that can appear at a discrete, continuous or infinitesimal level is formulated in terms of bifurcations (very relevant for dynamical systems A_{46}). The characterization also affects the types of structural equations that can appear for the simplest cases. The behaviour of models and systems of equations (or, more generally, of C^r-fibrations) is analyzed in terms of C^r-equivalences (diffeomorphisms, bi-analytical or birational morphisms).

A typical example is given by the Newton-like approach for motion in M, the Hamilton-Jacobi equations in $P = TM$ (natural extension of the gradient field), or the Euler-Lagrange equations in TP (extending computation of extrema for functions). Let us reiterate that, in the absence of external forces, the Euler-Lagrange forces are equivalent to the Hamilton-Jacobi forces (the fiber of $TP \to P$ behaves globally in a trivial way).

The discrete symmetries of the rigid solid, the Euclidean group $SE(n)$ and its Lie algebra $\mathfrak{se}(n)$ provide the basic patterns for actions to consider on M, $P = TM$ and TP. This well-known example extends to any other classical group and its discrete and infinitesimal versions.

Most developments appearing in the first chapters of each module A_{4k} have a theoretical character, where top-down or models-based approach is the most relevant one. along the last chapters of each module, one introduces some applications to other scientific or technological areas, where bottom-up (based on data or "features", e.g.) is the most relevant ones.

Connections between them are formulated in symbolic terms, by using extensions of analytical graphs $\mathcal{G} = (G, \mathcal{O}_G) \in \mathfrak{G}$ to represent basic elements and relations between them. The introduction of morphisms $\mathcal{E}_0 - \to \mathcal{E}_1$ between superimposed structures $\mathcal{E}_t \in \mathfrak{E}$ is the key for a simultaneouss management of complex phenomena appearing in applications.

0.2.3. Comparing classification criteria

The absolute and relative modelling of "objects" (manifolds, varieties), superimposed structures (fiber bundles, sheaves, fibrations), maps $f : X \to Y$ between objects and morphisms $\Phi : \xi - \to \eta$ between superimposed structures are highly non-trivial problems. Classification is performed w.r.t. equivalence relations which can be done in terms of algebraic actions (given by groups or algebras, e.g.) or operations (K-theory, e.g.).

The topologiccal classification is a very hard problem, because there are no enough general effective criteria for classification up to homeomorphisms. Thus, one uses weaker criteria (homotopy type, e.g.) or, alternately, one restricts to diffeomorphisms preserving "some geometric quantity" linked to the (Riemannian, Symplectic, Contact) structure linked to a Classical Group. So, relations between groups are translated to hierarchies between classifications. A similar approach can be performed for algebras of operators.

If we start with "objects", PL- or PS-manifolds (eventually singular) are used as well as their C^r-transformations. Often, there are an uncountable infinity of types, which motivates the study of the topology of moduli spaces, which can have a C^r-structure, also. Typical"examples" are given in the modules A_{31} (Algebraic Curves) and A_{35} (Algebraic Surfaces); both show the difficulty of the problem.

The most favourable conditions from the mathematical point of view correspond to the smooth objects and their differentiable transformations, which are the key for the Topology of Differential Invariants to be developed in k-jets spaces $J^k(N, P)$, and the corresponding k-jet bundles $J^k(E, F)$. Up to some exceptions [Olv84], they have received less attention than the Geometric Invariant Theory. The best reference is [Olv95].

Objects of class C^r are locally described in implicit terms (as a place of function override, e.g.) or explicit terms (including parametric representations) on spaces or functions. An independent formal presentation of the system of generators (using prime ideals for irreducible varieties) has been developed in Algebraic Geometry A_3. Their adaptation to the smooth case can be read in [Tou72]. The C^r-equivalences are defined locally in terms of regular maps with regular inverses that act on the space or pairs of C^r-equivalences that act on the starting and arrival spaces or on the graph.

- The homeomorphisms are the C^0-equivalences for the topological case. General Topology deals with the classification modulo homeomorphisms. Algebraic Topology introduces more effective but weaker criteria for classification, such as homotopy or (co-)homology groups on PL-structures (simplicial vs. cuboidal) in A_{22} or, more generally, cellular complexes A_{23}.
- The diffeomorphisms are the C^∞-equivalences for the differentiable case. The basic notions have been developed in the module A_{11} (Differentiable Varieties) and A_{12} (Linearization). Differential Topology deals with the classification of varieties and applications module diffeomorphisms.
- The bianalytic transformations are the C^ω-equivalences for the analytical case (convergent series development at each point). Analytical Geometry deals with classification module analytical equivalence. When the base body is $\mathbb{C}$ the analyticity condition is equivalent to holomorphy.
- The birational transformations that characterize Algebraic Geometry can be considered as a truncation of differentiable and analytical strategies. The truncation strategy adapts to the differentiable case in terms of the k-jet spaces that can be interpreted as (finite collections of) truncated formal Taylor polynomials.

In Differential Topology the classification of map-germs $f \in C^r(n,p)$ is performed up to diffeomorphisms giving two main types corresponding to double conjugacy or $\mathcal{A}$-equivalence, and the contact of $\mathcal{K}$-equivalence. They correspond to decoupled vs coupled actions. They can be applied to map-germs between smooth M, algebraic, or analytical varieties X. The existence of C^r-structures on a topological manifold, the characterization and the relationships between classifications for different values ââof r are some of the central problems.

To study objects and transformations of class C^r it is appropriate to introduce additional structures (bundles or bundles, for example) that provide support for the linearization of objects or their transformations, facilitating the adaptation of methods from Linear Algebra and elementary Differential Analysis. To fix ideas, in this matter the attention is initially focused on the “smooth” case or class C^∞ whose objects and morphisms (including regular transformations) form the differentiable category. In later phases, different C^r-structures on the same topological variety for $r = \infty$ are considered.

From the nineties, inverse problems play an increasing role in Applied Differential Topology. roughly speaking, instead of performing an analysis, they try of making a "synthesis" of shapes and behaviours.

- *Shape synthesis* involves to different scales and structures going from the atomic or molecular till industrial prototyping or multimedia graphic contents B_4. All of them use symbolic representations (graphs, e.g.) to represent relations between components, at different depth levels. An efficient design and computational implementation of Learning Manifolds are the key for these issues-
- *Behaviour synthesis* involves to different kinds of functionals defined at each level, with the corresponding optimization strategies, where one must identify critical levels corresponding to changes of state, phase or interactions between components. Tensor flows provide a first general framework in Deep Learning for their management. The most diffiult issues are linked to the simulation control of chemical reactions at molecular vs atomic levels.

The most immediate applications are related to Physical-Chemistry and Pharmacological Design [8]. In both knowledge subareas the interplay between discrete and continuous representations of the support and the corresponding dynamics plays a fundamental role.

0.2.4. An overview of the chapter

This chapter has four sections labelled as follows:

1. *Goals and methods* including a description of the smooth category, resoluton strategies, a discussion about methodological issues and effective strategies for the resolution.
2. *Some applications to Theoretical Physics* starting from a reformulatioin of basic aspects of Classical Analytical Mechanics (following the French tradition of Lagrange, Legendre, Liouville, PoincarÃ©) and their syntgeitc reformulation by Hamilton and Jacobi (differential formulation) or E.Noether (variational approaches). Their unification is performed in terms of group actions extending to diffeomorphism groups, giving the first steps for Gauge theories according to H.Weyl [9]. A short presentation of applications to other areas is sketched in the subsection §2,4.

[8] Some additional details are introduced in A_{406} (differential Topology in Physics, Chemistry and Engineering) and A_{408} (Differential Topology in Life Sciences).

[9] A gauge field is initially undertood as a connection on a fiber bundle on the space-time as the base space.

3. The third section is devoted to introduce some *applications* to different *applications to IST areas*, with a special emphasis on Computational Mechanics B_1, Computer Vision B_2, Robotics B_3 and Computer Graphics B_4, whcih will be developed in the part II of these notes. The introduction of symbolic representations given by graphs provides a nexus supporting tools and methods for usual query, extraction, recognition and classification issues to be applied in the part II.
4. The last section is devoted to explain the contents of the six modules of this matter, where we adopt a circular scheme , including connections between global, local and infinitesimal problems involving objects (varieties vs superimposed structures) and morphisms between them. One follows an increasingly complex strategy starting with regular, non-degenerate vs degenerate singularities, their package in stratified spaces and maps, and finally, the description of evolving systems in terms of the Topology of Dynamical Systems.

The descirption of smooth structures includes some comments about the superposition of the nearet ones topological and PL-categories, levels of detail for their analysis, and some relations between geometric and topological aspectss. They provide demarcation criteria helping to develop "approximations" which are commonly ued in other scientific or technological areas. According to the precedent remarks, we have organized this matter around the following six modules:

1. *Basic Differential Topology* A_{41} with Intersection Theory,, Morse Theory and basic Cobordism as the main goals.
2. *Fiber bundles and cohomology* A_{42} with structural classification results (including universal maps) and Index theorems as the main goal. Non-vanishing cohomology classes provide a support for a topological management of non-integrable systems wwhich have been described in the module A_{14} (Differential Forms).
3. *Singular function germs* where one extends the Morse theory to more complicated singularitiess of smooth function germs with Catastrophye Theory as the main paradigm. We follow a coarse-to-fine strategy going fromgeneric to "increasingly specialized" phenomena by introducing models for changes of state, phase transitions or sudden changes in interaction dynamical models.
4. *Singular map germs*, extending the precedent construction, where deformations, stability and classificaitons play a functamente role with a view to classification issues, and explicit constructions for versal deformations helping to stabilize complex non-linear phenomena.
5. *Stratifications* invovling spaces and maps, where one recovers again some connections between local and global issues, specially for the (semi-)analytic case which will reappeear along the part II. Qualitative theory

of folications provide some deep insight about the geometry, kinematics and dynamics.

6. *Topology of Dynamical Systems* where one extends integrability issues appearing in the module A_{14}, but extending them now to the global case, to try of understanding complex interaction phenomena appearing in Natural Sciences or different Engineering areas. Hyperbolic systems play a central role which is linked to the characterization of stable vs unstable behaviour.An extension of characteristic classes A_{42} provides some connections between local and global issues. Codimension one phenomena are well known, but our knowledge for higher codimension is much more limited.

The most important motivations for the two first modules are linked to Theoretical Physics. The other modules are being applied to Edifferent technological areas with a speial regard to the four areas Computational Mechancis of Continuous media B_1, Computer Vision B_2, Robotics B_3 and Computer Graphics B_4 of the part II (by following an increasingly complex difficulty).

Some basic applications to Natural Sciences or Economic Theory will be sketched in soms paragraphs from the section 2. Other more advanced applications related to the topology of solutions on crystal liquids, and the corresponding phase transitions (including liquid helium at low temperatures, e.g.) will be ignored in this chapter.

0.3. References for this introduction

References are not exhaustive, nor the most recent ones. They are included as an invitation to the reeader to acqquire a more complete insight of this knowledge area, according to his/her own interests.

0.3.1. Basic bibliography

Only some textbooks are included. For more enlarged bibliography, see the subsection §5,4. References for meaningful research articles are included as footnotes.

[Bot82] R.Bott and L.W.Tu: *Differential Forms in Algebraic Topology*, GTM, Springer, 1982.

[Bro72] W.Browder: *Surgery on Simply-Connected Manifolds.* Ergeb. Math. 65, Springer-Verlag 1972.

[Che79] S.S.Chern: *Complex manifolds without potential Thoery (2nd ed)*, Springer-Verlag, 1979.

[Gil84] P.B.Gilkey: *Invariance theory, the Heat Equation and the Aityah-Singer Index thoorem*, Publish or Perish, 1984.

[Gui74] V. Guillemin and Pollack: Differential Topology, Prentice Hall 1974.

[Hir76] M.W.Hirsch: *Differential Topology*, GTM, Springer-Verlag, 1976.

[Hir66] F.Hirzebruch: *Topological Methods in Algebraic Geometry*, Springer-Verlag, 1966.

[Hus73] D.E.Husemoller: *Fibre bundles*, GTM, Springer-Verlag, 1973.

[Kir77] R.Kirby, and L. Siebenmann: *Foundational essays on topological manifolds, smoothings and triangulations.* Ann. Math. Stud. 88, Princeton Univ. Press, 1977.

[Mad79] I.Madsen and R.J.Milgram: *Classifying Spaces in Surgery and Cobordism of manifolds*, Annals of mathematics studies no.92, Princeton University Press, 1979.

[Mil62] J. Milnor: *Morse theory*, Ann. of Maths Study 51, Princeton Univ. Press, Princeton, NJ, 1962.

[Mil65] J. Milnor: *Topology from a differentiable viewpoint*, Virginia Univ. Press 1965.

[Mil74] J.W.Milnor and J.D.Stasheff: *Characterisstic classes*, Princeton Univ. Press, 1974.

[Mil71] J.W. Milnor: *Morse Theory*, Princeton Univ. Press, 1971.

[Nak90] M.Nakahara: *Geometry, Topology and Physics*, Adam Hilger, IOP, 1990.

[Nov97] S.P.Novikov: *Topology I*, Springer-Verlag, 1997.

[Olv95] P.J.Olver: *Equivalence, Invariants and Symmetry*, Cambridge Univ. Press, 1995.

[Ste51] N. Steenrood: *The Topology of Fibre Bundles*, Princeton Unive. Press, 1951.

[Sto68] R.E.Stong: *Notes on Cobordism Theory.* Princeton, N.J.: Princeton Univ. Press, 1968.

[Tou72] J.C.Tougeron: *Idéaux de fonctions différentiables*, Springer-Verlag, 1972.

[Van90] J.Van Mill and G.M.Reed (eds): *Open Problems in Topology*, Elsevier, 1990.

[Wal70] C.T.C.Wall: *Surgery on compact manifolds. Lond. Math. Soc. Monogr.* 1. Acad. Press 1970.

[Wel80] R.O.Wells: *Differential Analysis on Complex Manifolds (reprt)*,Springer-Verlag, 1980.

0.3.2. Software resources

To my knowledge, there is no a freely available Open Source library for Differential Topology, but some functionalities can be extracted from the entry "List of open source software for mathematics" in Wikipedia. Some general references including specific applicatioins to Differential topology are

- Mapple V: https://www.maplesoft.com/products/Maple/
- SINGULAR: https://www.singular.uni-kl.de/Manual/4-4/

- ... any suggestion is welcome

In the module B_{13} (computational Differential Topology) we will develop a more computational approach to the above issues. Their foundations can be found in

[Ama16] S.T.Amari: *Information Theory and Applications*, Springer-Verlag 2016.

[Her13] M.Herlihy, D.Kozlov, S.Rajsbaum: *Distributed Computing Through Combinatorial Topology*, 2013.

[Zom05] A.J. Zomorodian: *Topology for Computing*. Cambridge Univ. Press, 2005.

Final remark: Readers which are interested in a more complete presentation of this chapter (currently available in Spanish language, only), please write a message to javier.finat@gmail.com