A302 Algebraic Methods in GAGA

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Índice

Previous remarks: These notes correspond to an introduction to the chapter 2 of the module A_{30} (Foundations of GAGA) of the matter A_3 (Algebraic and Analytic Geometry). From the mathematical viewpoint, it is necessary to have some basic knowledge of Basic Algebra and Group Actions. It is advisable to have some basic knowledge of Afine Geometry.

As usual, in addition to this introduction and a final complementary section, materials are organized in four sections. They contain a list of exercises for self-verification of understanding of materials. Subsections or paragraphs marked with an asterisk ($*$) display a higher difficulty and can be skipped in a first lecture.

In the same way as for other chapters of this module, the introduction is written in englsh, whereas the sections are written in Spanish language.

0.1. Preface to A301

Algebraic Geometry studies algebraic varieties and algebraic maps between them. Initially, they are locally defined by a finite number of polynomials. More intrinsic presentations replace sets of polynomials by the ideal generated by them. It uses not only algebraic methods, but differential, analytical and topological methods, also. The predominance of algebraic methods is a consequence of the Weil Program (early 1950s), which includes the need to extend the results from other approximations to fields of arbitrary characteristic p.

Algebraic methods are present in Algebraic Geometry from the beginning of Mathematics. In a naive way. one can think of Algebraic Geometry in terms of solutions to simple ratios between parameters and indeterminates (Thales, Eudoxus), or linked to "geometric representation" for solutions of algebraic equations [1]. The search of numerical solutions for simple or double ratios between measurable quantities provide the first examples linking arithmetic and geometric aspects.

The introduction of coordinates by R.Descartes in the 17th century was the key to give an analytic expression to relations between "quantities", and ease the explicit computation of solutions in $\mathbb{R}^n$. The development of Linear Algebra on n dimensional vector spaces $V \simeq \urcorner^n$ over a field k was the next contribution to simplify the formalism. Its extension to Affine spaces $\mathbb{A}^n$ is the key for more flexible operations with "objects" in $\mathbb{A}^n_k$. The affine space was initially thought as $\{(x_1, \ldots, x_n) \mid x_i \in K\})$ up to translations. The extension of linear systems of equations (corresponding to linear subspaces) to non-linear systems of equations was the next challenge.

1. A first issue is linked to the study of solutions for non-linear systems of algebraic equations in the coordinate ring $k[x_1, \ldots, x_n]$. [2]

2. A second issue concerns to "counting" solutions "properly". i.e. having in account "improper solutions" (which require the use of more general fields), multiple solutions and/or "solutions at infinity" corresponding to the asymptotic behaviour of branches in graphical representations.

Both issues involve to the notion of "intersection multiplicity" for solutions, which has a long history [3] The analysis of multiple solutions appear already with the study of solutions for low order polynomials in one variable whcih can be solved by radicals (Cardano) for polynomials of degree ≤ 4. The incorporation of solutions at infinity was already known from the first perspective treatises in the Renaissance (L.B.Alberti around 1430, e.g.).

The first great result for the computation of solutions involving non-linear systems of equations is due to E.Bezout (1730-1783), who "proved" (1779) that

[1] See the precedent chapter A_{301} for details

[2] The recognition of complexity of this problem was acknowledged already in the Antiquity (crossing of vaults in Architecture), even if we restrict ourselves to the analysis of a low number of quadratic equations in the ordinary space.

[3] For a survey of modern approaches see the first chapter of A_{34} (Enumerative Geometry).

if two plane algebraic curves C_1 and C_2 of degrees d_1 and d_2 have no component in common, they have $d_1 d_2$ intersection points, counted with their multiplicity, and including "points at infinity" and points with complex coordinates. This result is naturally extended to hypersurfaces, and provides the starting point for the Intersection Theory [Ful84].

The incorporation of circular points "at infinity" shared by the intersection of pairs of conics was already known by I.Newton, who labelled as I (Isaiah) and J (Jacob). The first systematic approaches appear with the study of third and fourth degree equations, whose first (redundant) classification was performed by Newton, by using affine and metric properties. The introduction of the Projective Geometry (Poncelet, 1827), as a natural extension of the Euclidean and Affine Geometries, allows to reduce the 72 types of plane cubic curves to only 3 regular classes. A large advantage of Projective Geometry consists of the removal of "exceptional cases" and their integration in a common projective framework.

The *Golden Age of the Projective Geometry* (first half of the 19th century) gives structural nexus between synthetic and analytic methods for the study of higher degree algebraic varieties and their intersections in the complex projective space $\mathbb{CP}^r$ by the French school. A subset $S \subset \mathbb{A}^n_k$ is algebraic (resp analytical) if it is the zero locus of a finite set of polynomials f_i (resp. analytical functions).

In a simultaneous way, the development of algebraic methods allow an explicit computation of solutions for intersecting hypersurfaces in terms of the Resultant $R(f, g)$ of two polynomials. This method was initially developed by Sylvester (1853) and Cayley (1857) who called it the Bézoutian in honor of E.Bézout. From the second half of the 19th century, a subset $X \subset \mathbb{A}^n_k$ is algebraic (resp. analytic) iff $X = V(I)$ where I is an ideal of $k[x_1, \ldots, x_n]$.

The convergence with analytic methods is linked to the developments of Analytic Functions of One Complex Variable. In the complex domain, differential aspects are initially due to N.H.Abel (1802-1829) involving algebraic relations between integrals along cycles on a complex algebraic curve C (visualized as a real surface). These results are formulated even before the first (incomplete) proofs of Cauchy theorems about residues of integrals. Links between differential and integral aspects are initially developed by Jacobi (1804-1851), starting with properties of elliptic functions (see below). The study of fractional transformations linked to $SL(2; \mathbb{C})$ was developed by A.F.Moebius (1790-1868).

In the linear case, properties of vector spaces V over a field k are described in terms of sections and projections giving linear subspaces W and quotients $Q = V/W$ with the corresponding linear transformations. A natural issue consists of asking about the extension of these methods to eventually non-linear "objects". For projective varieties, sections and projections are projectively invariant, giving projective "extrinsic" invariants (i.e. dependent on the immersion(such as the degree d (number of intersection points with a generic line ℓ for plane curves, e.g.) and the class d^v (number of tangents from an exterior point).

In the non-linear case, one must replace vector spaces V over k by modules M over a ring R, and consider the corresponding maps $f : M_0 \to M_1$ between modules, and replace linear maps by rational morphisms. Unfortunately, nor even the simplest extrinsic invariants, such as the degree d and the class d^v of an algebraic plane curve, are intrinsic invariants, i.e. they are not preserved by arbitrary birational isomorphisms. The Pluecker formulae relate both projective invariants with the main birational invariant given by th *genus* $g(C)$. In these formulae there appear terms linked to the number of double points and cuspidal points, "properly" counted , e.. according to their multiplicity.

A weak point of the naive approach based on "degeneration" principles and continuity arguments, was the lack of a "good" notion of algebraic multiplicity for the intersection of curves or more general varieties. As it is well known, the same set of solutions can be the support for initially different systems of equations. A central topic of Algebraic Geometry is the "right definition" of multiplicity. A historical survey of failed attempts from the 19th century till arriving to the definitive solution (Serre's Tor formula) can be read in the chapter 1 of the module A_{35} (Enumerative Geometry).

Classificaiton under birational isomorphism implies that the field $k(X)$ of rational functions on a variety X is the main object to be considered. This expalins the central locus played by the theory of fields extensions. The first formulation of connections with Group Theory appears already in the Galois work, initially developed to analyze the solvability of algebraic systems of equations. In particular, the group of automorphisms $Gal(L/K)$ of an extension L/K allows to identify if the extensions normal or separable, e.g.

In a complementary way, the need of relating different algebraic descriptions for the same support, opens the door for the notion of ideals I of a ring R with the corresponding notions of finite generation, and ascending vs descending conditioins for chains of ideals. These constructoins are naturally extended to modules M over a ring F in terms of "resolutions". The "apparent motion" along a variety X is formalized in terms of "change of base point" which is formulated in terms of a tensor product.

0.1.1. Algebraic methods and structures

Algebraic methods are ubiquitous in Algebraic and Analytic Geometry. They involve to objects (varieties, schemes, analytic spaces, e.g.), superimposed structures (fiber bundles, fibrations, sheaves, e.g.), and morphisms between them. They cover applications of all algebraic subareas, and have motivated the introduction of a lot of algebraic structures which have been extended to other knowledge areas. Several basic taxonomies to be commented along this paragraph involve to

- Discrete vs continuous approaches.
- Global vs Local frameworks

- Finite vs infinite-dimensional actions.
- Extrinsic vs intrinsic strategies.
- Superimposed C^r-structures for classificaiton issues.
- Relations with other knowledge areas.

Classical Algebraic Geometry was constructed on the complex field number $\mathbb{C}$. Deep relations with Number Theory, and the need of studying reductions modulo p (for any prime $p \in \mathbb{Z}$) motivated the extension of algebraic methods to the study of Algebraic Geometry on $\mathbb{F}_p$ or $\mathbb{F}_{p^n}$ in Algebraic Number Theory.

(*) Some specific features of characteristic p are linked to the ordinary differential calculus (where $dx^p = 0$, e.g.) or to the inseparable character of field extensions. Furthermore, they provide a more natural connection with discrete geometries with their corresponding actions of finite groups.

Discrete vs continuous approaches appear in regard to the resolution of Diophantine equations [4]. The introduction of Cartesian coordinates is the key for a continuous inteprretation of algebraic relations in terms of the continuous representations of variables and parameters.

Inversely, given a continuous relation between variables, a non-trivial (sometimes very difficult) problem consists of identifying how may solutions belong to $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, or some finite field extension. Perhaps the most known problem is the well-known Fermat's conjecture, who has attracted the attention of a lot of mathemticians from the 17th century, till its definitive proof by A.Wiles (1986). Relations with the Mordell's conjecture (proved by Faltings in 1983) and mainly with Elliptic curves, will be commented in the chapter 9 of the module A_{31} (Algebraic Curves).

Local vs global frameworks involve to the local character of Basic Algebra (with Commutative Algebra as paradigm) and their compatibility to generate global objects as they appear in Sheaves. Homological Algebra is the main paradigm extending Algebraic Topology methods. Both of them can be reformulated in terms of (pre-)sheaves which provide a common language for matching stacks in a similar way to fibres of a fibre bundle $\xi = (E_\xi, \pi_\xi, B_\xi, F_\xi)$ as a "toy model" for (pre-)sheaves. Local data can be understood as some kind of "localization" (some kind of"specialization") of a hypothetical global structure.

- The current *Commutative Algebra* emerges from the development of key notions along the 19th century linked to structures such as groups and fields (starting with Galois theory), rings (for internal properties of polynomials and their restrictions to subvarieties), ideals (to compare different "presentations" in terms of generators, e.g.), modules (as superimposed

[4] Diophantus lived around the third century a.C. and published the first compendium for solving discrete equations from the arithmetical viewoint, even if it is considered as the father of the "Diophantine Geometry" by the strong connections with geometric problems.

structures to rings extending the notion of vector space, e.g.), between other [5]. Their properties provide the core for basic GAGA.

- The current *Homological Algebra* emerges from the development of general methods for *solving* systems of equations involving all the precedent algebraic structures under finiteness conditions. They use some algebraic extensions of well-known techniques arising from Homology and Cohomology theories A_{22}. These strategies are formulated in terms of injective vs projective *resolutions*, which can be considered as an extension of nested collections of embedded algebraic structures of the same type.

The global reformulation of local and global aspects is performed in terms of sheaves corresponding to the same structure (groups, rings, modules, e.g.). Initially sheaves have a topological inspiration, and by this reason they can be applied to more general frameworks than those appearing in classical approaches to GAGA.

The Klein description of a Geometry as the set of invariant properties on a base space X by the action $\alpha : G \times X \to X$ of a group G, motivates different extensions to the GAGA framework. This characterization is adapted to GAGA by imposing the condition that G is an algebraic or an analytic group G on algebraic varieties (o more generally schemes). The action is algebraic (resp. analytic) locally defined by polynomials (analytic functions) [6]. The basic distinction between finite, finite-dimensional and infinite-dimensional actions is translated in terms of

- *Discrete groups* going from Galois extensions, symmetric vs alternating groups, or reflection groups (Euclidean vs hyperbolic geometries, e.g.). Soluble groups (in the Galois sense) play a specially important role.
- *Classical groups* linked to Klein geometries acting on a linear space, involving extensions to the Affine and Projective spaces for orthogonal, unimodular, unitary, symplectic or contact groups, and their mutual intersections, by taking in account the algebraic character of these Lie groups. They are commonly used in applications to Physics and Engineering.
- *Infinite-dimensional groups* as "subsets"' of the Homeomorphisms $Homeom(X)$ of a topological space X, including the restriction to the smooth (given by local diffeomorphisms), analytic (given by bianalytic transaformations), and algebraic (given by birational isomorphisms) contexts. Different kinds of birational transformations play a central role for classification issues in GAGA.

G-homogeneous spaces by the action of an algebraic group G are described in the same way as in Differential Geometry. In addition of the basic Cartesian,

[5] Modules replace vector bundles as fibres for generalizations of fibre bundles such as (pre-)sheaves, e.g.

[6] Equivalently, the composition $G \times g \to G$ and the inverse map $G \to G$ are morphisms between algebraic varieties.

Affine and Projective spaces (identify, as exercise, the corresponding algebraic group), spheres, Grassmann (1809-1877) and Flag manifolds are examples of G-homogeneous spaces by the corresponding algebraic group.

(*) Other more complicated "examples" are given by unipotent subgroups $U \subset GL(n)$ locally described by upper triangular matrices with 1 as entries of the principal diagonal. Linear representations of their corresponding nilpotent Lie algebras have 0 at the principal diagonal. They play a fundamental role for the classification of finitely determined map germs A_{44}, and their applications to models for dissipative phenomena (see the section 3 of B_{155} for details).

In the classical case, one of the first important "examples" is linked to the use of fractional transformations performed by A.F.Moebius. (1790-1868). From the geometric viewpoint, fractional transformatios wer originally described as the composition of a stereographic projection of the plane onto the sphere, followed by a rotation or displacement of the sphere to a new location and finally a stereographic projection, this time from the sphere to the plane.

From a more modern viewpoint, it is more natural consider the Moebius transformations as transformations of the Riemann sphere $\mathbb{S}^2$ (i.e. of the complex plane increased with a point at infinity $\mathbb{C} \cup \{\infty\}$. However, the original Moebius viewpoint illustrates the reaching and the geometric meaning in regard to the Projective Geometry. Moebius transformations are the starting point for the *Conformal Geometry* with a lot of applications in Theoretical Physics and Engineering (including Animation in Computer Graphics, e.g.).

More generally, *Geometric invariant Theory* plays a central role to relate algebraic and analytic viewpoints. It has been mainly developed by the Britisch School (Cayley, Sylvester) in the second half of the 19th century by using relations between generators of ideals. This approach was reformulated and extended by the German school (Dedeking, Kronecker, Noether, Weber) by using the language of Basic Algebra at the late decades of the 19th century. Hilbert's contributions have allowed a reformulation in terms of Higher Algebra (Weber, Van der Waerden).

The reformulation of Algebraic Geometry that took place after 1960 was also transferred to the *Geometric Invariant Theory*, with the figure of D. Mumford as the greatest promoter of this subarea. It takes place in successive waves based on algebraic group actions on Commutative Algebra (Mumford), sheaf-theoretical based approach (Mumford and Fogarty) and G-equivariant approaches (Mumford, Fogarty, Kirwan) to moduli spaces $\mathcal{M}_g$ along the second half of the 20th century. In the chapter A_{309} we will make a short introduction to some applications to Natural Sciences (Physics and Chemistry, mainly) and Engineering.

(*) From a complementary viewpoint, along the last quarter of the 20th century, there appear strong interactions with the Differential Invariant Theory. The interplay can be thought in local or infinitesimal terms, by taking the k-jets bundles J^kE of a vector bundle which are generated by k-jets. The k-jet of a map germ $f \in C^r(n,p)$ is represented by the jets $j^k f_i$ (formal Taylor developments) truncated at order k of components f_i of $f = (f_1, \ldots, f_p)$.

In the algebraic framework, components f_i of a map-germ $f \in C^r(n,p)$ are interpreted as the generators of an ideal I. The reduction to finitely determined maps allows the introduction of Commutative and Homological Algebra on spaces of k-jets [7]

(*) So, for finitely k-determined map germs $f \in C^r(n,p)$ one can develop a Differential Invariant Theory (DIT) for the smooth case [8]. We denote by J^kE the space of k-jets of sections of a bundle with total space E. Morphisms $\xi \rightarrow$ are extended naturally to morphisms $J^kE_\xi \rightarrow J^k_\eta$ between their corresponding total spaces

(*) Jets spaces and Jet bundles provide a formal language for evolving systems of PDEs. DIT is far richer than the classical Geometric Invariant Theory (GIT), because it is extended to distributions $\mathcal{D}$ of vector fields, systems $\mathcal{S}$ of differential forms, systems of PDEs, and differential operators, including some aspects of Variational Calculus between others. A good reference for DIT is [Olv95] [9]

Intrinsic vs extrinsic strategies involve to the relation between the abstract description of the geometry of a variety X, and its description as a subvariety of a larger ambient variety Z. Initially, algebraic varieties were described as subvarieties of an affine $\mathbb{A}^n$ or a projective space $\mathbb{P}^n$ in terms of equations in the polynomial ring $k[\underline{x}]$ in n variables $\underline{x} = (x_1, \ldots, x_n)$ or their homogeneous completion.

The first reconversion is their expression by taking polynomial equations f_i as the generators of an ideal I_X of the coordinate ring $k[X]$.In this extension, projective equivalence is replaced by birational equivalence, where linear transformations are replaced by rational transformations of degree ≥ 2. The simplest example is given by Cremona transformations corresponding to degree 2. There appear a local problem in regard to the choice of the degree of birational transformations, and a harder problem in regard to the structure of the group of bitarional transformations.

Birational transformations play a central role for the explicit computation of a *non-singular model* of a variety X. Resolution of Singularities is solved (Hironaka, 1963) for varieties on an algebraically closed field k of characteristic zero. Hence, they are crucial to solve classification issues on $\mathbb{C}$, e.g.. As always, one follows an increasinlgy complex strategy by starting with the simplest case corresponding to curves B_{31}, where almost everything is already known.

(*) The algebraic classification of surfaces (1946) is essentially due to F.Enriques (1871-1946) in terms of powers of the canonical divisor K_X (including intersections with cycles). Its algebraic reformulation by O.Zarsiki (along 195Os). Additional contributions linked to deformations due to Shafarevich school, were completed for the analytic case by K.Kodaira along 1960s.

[7] A more formal approach in terms of sheaves of Principal Parts is developed in the module A_{33}.

[8] A map-germ f is k-determined if it is equivalent to its k-jet j^kf.

[9] P.J.Olver. *Equivalences, Invariants and Symmetry*, Cambridge Univ. Press, 1995.

(*) For algebraic surfaces, the structure of the group of birational transformations for surfaces is affordable because birational maps of surfaces factor as sequences of blow-ups at points, and by the existence of a unique minimal model for non-ruled surfaces, which can not be reduced to a simpler model by blow-downs [10]. Neither of these results is true for algebraic threefolds, which requires a more sophisticated strategy called the Mori's program which is developed in the module A_{36} (Algebraic and Analytic Threefolds).

The next big package for taxonomies concerns to *superimposed structures* on varieties and morphisms between them. The simplest ones have been described as an extension by using sheaves of vector bundles A_{12} and Tensor Bundles A_{13} of Differential Geometry of Manifolds A_1. Similarly, one can define the analogue of Principal Bundles whose structural groups are given by algebraic group actions.

Solvable and Unipotent groups play now an important role, which introduces a first important difference w.r.t. the smooth case. The same structures can be defined on algebraic and analytical varieties by using "enough" local sections $s \in \Gamma(U, \mathcal{F})$ of sheaves $\mathcal{F}$ as a tool for "tracking" the behaviour along the total space. The geometry of the Unipotent Variety plays a central role for the classification of finitely determined singular map-germs A_{44}.

(*) The existence of increasingly complex singularities introduces additional troubles for the corresponding allgebraic groups (and their correspond Lie algebras) giving behaviour is not "so good" as in the smooth case. Furthermore, the kernel and the cokernel of a morphism between vector bundles is not, in general, a vector bundle. The corresponding structural groups display "bifurcations" to be described in terms of adjacencies between algebraic groups. So, we have additional troubles to compute invariants (dimensions of spaces of sections, e.g.) in terms of the cohomology of sheaves.

Furthermore, algebraic or analytic fibrations are not bundles (there can appear singular fibers, e.g.). It is necessary to extend the original superimposed structure from bundles to sheaves, and even to schemes to include embedded components and even "jumps" in the dimensionality of fibers (such it appears with the Hilbert scheme, e.g.). In addition, the stalk structure can display different kinds of hierarchies represented by ascending or descending chains of algebraic structures, e.g. (typical "examples" appear in Group Schemes).

0.1.2. Commutative Algebra for GAGA

Commutative algebra is initially described in terms of polynomial rings $k[x_1, \dots x_n]$ and their quotients by ideals I. First dictionaries were established by D.Hilbert (Nullstellensatz), giving a correspondence between $V(I)$ and the set of prime ideals (called spectrum) of $k[\underline{x}]/I$ in the modern terminology.

This correspondence provides a a common language for Algebraic Geometry and Algebraic Number Theory, which is the key for the proof of some classical

[10] See details in the module A_{35} (Algebraic and Analytic Surfaces)

conjectures. Ideal primes $\mathfrak{p}$ define the basic closed sets for the Zariski's topology (coarser than the usual topology of coefficients).

In a nutshell, the local ring $\mathcal{O}_{X,x}$ of a variety at a regular point encodes information about the variety's behaviour near the regular point $x \in X$. In presence of several "branches,", ona has a semilocal ring. These constructions are naturally extended to localization w.r.t. an ideal I. In particular, if $\mathfrak{p} \subset A$ is a prime ideal, one has

$$A_{\mathfrak{p}} \ := \ \{\frac{f}{g} \in K \ \mid \ f, g \in A \ teextwith \ g \notin \mathfrak{p}\}$$

which is erestricted to $\mathfrak{m}_{\mathfrak{p}}$ where $f \in \mathfrak{p}$, or extended to more general ideals $I \subset A$, by replacing $\mathfrak{p}$ by I.

More generally, there exists a dictionary between rings A and Affine varieties (extended to affine schemes in A_{33}). Following this dictionary, prime ideals $\mathfrak{p}$ correspond to orindary points (no embedded components) or irreducible subvarieties [11]. The set $Spec(A)$ of prime ideals plays a central role to define maps (or more generally, morphisms) between affine subvarieties. In particular, smooth points are replaced by regular local rings.

The study of particular properties around a point $x \in X$ is performed in terms of the localization $A_{\mathfrak{p}}$ of the ring A w.r.t. the prime ideal $\mathfrak{p}$. A first relation with Analytic Geometry is performed in terms of the *completion* ^{A} which is topologically interpreted as a formal neighborhood. The study of the module of differentials (or more general modules) is carried out in terms of modules over the base ring A (to be extended to Modules, i.e. sheaves over $Spec(A)$ in A_{33}-

Finiteness conditions about embedded collections of subspaces os successive quotient spaces in Linear Algebra are naturally extended to ascending vs descending chains of embedded ideals in $k[\underline{x}]$. So, one obtains a characterization of Noetherian and Artininan rings, where finiteness conditions are fulfilled. Finite free resolutions extend this idea in the Homological Algebra context. Simlar constructions are developed in the Analytic framework.

The algebraic formalism is naturally extended to other analytic, differential and topological frameworks, which provides basic keys for the unification developed from the 1950s (as part of the Weil's program). So, one extends traditional approaches for the smooth case, to include the study of singular loci involving "objects" and morphisms between objects appearing in GAGA. In particular, the polynomial ring $k[x_1, \ldots, x_n]$ must be replaced by the ring $k\{x_1, \ldots, x_n\}$ of *convergent series* in n variables, which allows to manage several branches in the neighborhood of a s singular point. Analytic topology plays a similar role to the Zariski topology in the analytic framework.

A partial smoothing is performed in terms of normalization (integral closure of the fractions field at each point), where singularities disappear in codimension

[11] They are the "basic objects" appearing in OOP-based computational apparoces developed in the part II.

one. In particular, for a curve or a hypersurface with an isolated singularity, normalization gives a non-singular model. Key notions such as different notions of geometric dimension are extended to Krull dimension and multiplicity in terms of ascending vs descending nested chains (Noether vs Artin) of ideals or, more generally, modules. These extensions play a fundamental role to extend the analysis of regular to different kinds of singularities where torsion phenomena appear in a natural way.

All the above aspects have a local character, but they can be extended to a global framework by using "sheaves". The introduction of sheaves is key to match together (modules on) local rings in a new structure as "stalks" by generalizing the notion of "germ", almost ubiquitous in Differential Geometry and Topology (Global Anlaysis). It is the key to unify any kind of (affine, projective, quasi-projective varieties) in the notion of scheme. In particular, finiteness conditions are key for the notion of *coherence* (which plays a similar role to vector bundles in Differential Geometry):

(*) A *coherent sheaf* on a ringed space $(X, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules satisfying the following two properties:

- $\mathcal{F}$ is of finite type over $\mathcal{O}_X$, that is, every point in X has an open neighborhood U in X such that there is a surjective morphism $\mathcal{O}_X^n|_U \to \mathcal{F}_U$ for some natural number n;
- for any open set $U \subseteq X$, any natural number n, and any morphism $\varphi : \mathcal{O}_X^n|_U \to \mathcal{F}|_U$ of $\mathcal{O}_X$-modules, the kernel of φ is of finite type.

(*) Coherence hypothesis provides the key for matching local data in global objects. In particular, there exists an equivalences between coherent algebraic sheaves and analytic sheaves on projective varieties over $\mathbb{C}$ (Serre, 1956). This ewaulr explains the role played by coherence properties as a tool to unify Algebraic and Analytic Geometries in the GAGA framework A_{33}. From a ,local viewpoint, the key is the systematic use of Commutative (completions, e.g.) and Homological Algebra (resolutions, e.g.) in a formal framework.

(*) The notion of scheme (initially proposed by C.Chevalley and developed by A.Grothendieck) includes the possibility of embedded components with higher multiplicity, and components of dimension lesser than the generic one. In this way, the reaching of classification issues is extended not only for "geometric objects", but to number theory, topology, and representation theory, also.

Sheaves were originally introduced by J.P.Serre (1956) by imposing "coherence" conditions about the variation of (modules on) local rings. Their topological invariants are computed in a combinatorial way, by using the Čech cohomology. The introduction of other topologies (ℓ-adic, cruystalline, e.g.) by A.Grothendieck in the late 1950s, became the key for the extension of these constructions to other mathematical areas, contributing to the "Algebrization program" of Topology proposed by A.Weil (around 1950).

The notion of *scheme* plays a central role for these extensions. Their local objects are affine schemes or prime spectra $Spec(A)$, which curiously play a similar role to the notion ‘f “object” in the OOP computational framework. These “objects” are locally ringed spaces, which form a category that is antiequivalent (dual) to the category of commutative unit rings, extending the duality between the category of affine algebraic varieties over a field k, and the category of finitely generated reduced k-algebras.

(*) The gluing of objects in the category of locally ringed spaces was initially performed by using the Zariski topology. Next, the Zariski topology in the set-theoretic sense is replaced by a Zariski topology in the sense of Grothendieck topology. The last one and his school developed Grothendieck topologies having in mind more “exotic” but geometrically finer and more sensitive examples than the coarse Zariski topology, namely the étale topology, and the two “flat Grothendieck topologies” [12].

(*) However, the notion of sheaf is sometimes “too strict” in regard to recent developments, and applications to other knowledge areas such those appearing in the part II of these notes. It is convenient to use a weaker notion, where pre-sheaves provide the first candidate to match together “algebraic stacks” (Deligne-Mumford) where sharper discontinuities are allowed [13].

0.1.3. Homological Algebra for GAGA (*)

The basic idea is linked to the expression of successive systems of equations to solve any kind of algebraic problem in terms of finite collections of maps between A-modules. In the literature they are called ‘resolutions with two basic types labelled as injective and surjective resolutions. In a naive approach, they would correspond to an algebraic extension of successive immersions and submersions (as the smooth version of linear sections and projections of Linear Geoemtry).

They are commonly used in Algebraic Topology, where they involve to incidence relations between elements of increasing or decreasing dimension in graded (simplicial vs cuboidal) complexes. Combinatorial properties of graded compelxes play a fundamental role in Algebraic Topology. Intrinsic properties are expressed in terms of the homology and cohomology of the graded complex of (co)chains corresponding to descending and ascending degree(linked to boundary ∂ and coboundary d operators).

A more general functional approach is described in terms of the Čech cohomology, which is commonly used in GAGA from the early 1950s. In this case, combinatorial properties are expressed in terms of successive intersections $U_{ij} = U_i \cap U_j$, $U_{ijk}, \dots$ between open subsets U_i of a covering $\mathcal{U}$ of avariety X. Initially one imposes “good properties” for successive intersections (a typical constraint is$U_{ijk} = \emptyset$).

An elementary introduction ot the Čech Cohomology on smooth manifolds (including the comact support version) can be read in the module A_{42} (Bund-

[12] See the module B_{33} (Sheaves, Cohomology, Schemes) for more details

[13] Typical “examples” are linked to “events”, which modify the topology of the “scene” or maps representing their space-time evolution.

les and Cohomology) of the matter A_4 (Differential Topology), where we have followed [Bot83]. The adaptation of PL-complex to the smooth framework is the key for a drastic reduction of the number of cells $(e_i^k, \partial e_i^k) \simeq (\mathbb{D}^k), \mathbb{S}^{k-1}$, and consequently for explicit computations in (co(homology. By naturality and better functorial properties, usually we develop a cohomological approach.

Another fundamental ingredient is linked to the ubiquitous tensor operations appearing in Mathematics and tehir applicaitons to other scientific or technological areas. A precedent in Algebraic Topology is linked to like Kunneth theorems appearing inHomology and Cohomology theories A_{22}. They express the relation between the q(-th co)homology $H_q(X \times (X \times Y))$ of $X \times Y$ as a direct sum of $H_i(X) \times H_j(Y)$ for $i + j = q$.

In this way, Kunneth formulae allow to relate cohomological invariants of total spaces space E and the base space B of a bundle by using the cohomology $H^*(F)$ of the fiber,which is symbolically exopressed as $H^*(E; R) \simeq H^*(B; R) \otimes H^*(F; R)$ for a domain R. Luckily, this formalism remains valid when one replaces R by a sheaf $\mathcal{F}$, which allows to extend it to fibrations, deformations or base changes operations, between other.

A third ingredientfor the application of (co)homology methods is linked to different kinds of duality extending the original Poincaré duality between cycles and cocycles or between cocycles of complementary dimension (in terms of integral operators). In the module A_{22} we have described several types of duality which are commonly used in Algebraic Topology A_2, and their applications to low-dimensional topological manifolds (Alexander), Complex Manifolds (Hodge) or Porjective Algebraic Manifolds (Lefschetz).

All the above approaches are reformulated in the context of sheaves cohomology. They start wioth the Serre's duality (the first adaptation of Poincaré duality to regular algebraic varieties).and continue with the Grothendieck's cohomologies for schemes, which was developed along 1960s by Grothendieck and his school. They require additional information of more advanced Homological Algebra involving functors, bigraded complexes and sheaf cohomology. By this reason, we postpone a more formal presentation to the module A_{33} (Sheaves, Cohomology, Schemes).

(*) The interplay between Algebraic Geometry and Differential Topology plays a central role along the following decades. It can be initially motivated by the development of the Standard Model unifying the electromagnetic, weak and strong interactions in Quantum mechanics along 1960s. Classical approaches suppose a smooth support given by a manifold M, as the support for principal bundles with affine connections ∇. The minimization of the curvature functional (Yang-Mills) on the space of affine connections is the key-stone for the Standard Model.

(*) The unification with a quantized version of Relativistic Gravitation is an open problem, which is the goal of GUT (Great Unificaiton Theory). It is necessary to solve alot of problems including the extension of classical cohomology theories to the singular case. The naturaol framework to minimize integral functionals is given by Morse theory, where geodesics providethe first “exampole” linked to the quadratic distance functional. Extensions of Classical Morse theory to singular varieties has been performed along 1980s by developing a Singular Homolgy Theory on stratifications with “good properties” A_{45} (Stratifications).

Resolutions

Ext and Tor functors

Stratified Morse Theory

singular Homology Theory

and Spectral Sequences

Classical homology and cohomology theories provide combinatorial methods to extract information and construct invariants linked to PL-configurations of basic geometric units (simplices, cuboids) and linear operators on them (involving algebraic, analytic, differentiable or continuous maps, e.g.). This simple remark explains the unbiquity of (co)homological methods in almost all mathematical areas and their applications to other knowledge areas. The precedent description is algebraically formalized in terms of graded complexes involving (co)simplicial or (co)cuboidal structures with the corresponding (co)homological invariants. The last ones are invariants of the homotopy type (roughly nspeaking, “deformations” with “good” topological properties), but not w.r.t. homeomorphisms. The largest advantage is their computability by using combinatorial tools.

Alternately one can think of graded complexes linked to systems of equations or, more specifically, to the generators of an ideal I. First order relations between generators give the kernel of a map

Alternately, to avoid the dependence w.r.t. coordinate representations, one can use “pre-sheaves” given by functionals $U_i \mapsto \mathcal{F}(U_i)$ defined on open sets of a covering $\mathcal{U} = \{U_i\}_{i \in I}$ fulfilling “good incidence” conditions. Initially, one requests “natural” compatibility conditions linked to regular C^r-structures, and the existence of “enough” local sections for superimpoosed structures. Later, both conditions are “relaxed”; a typical example is given by presheaves). So, it is possible to adapt initial definitions to broader contexts, specially in regard to applications to less ideal contexts, where there can appear non-regular behaviours for the support X of the functionals defined on X (a $\mathcal{O}_X$-module, e.g.).

The computation of (co)homologies as graded complexes for the absolute and the relative case is extended to the superimposed structures in terms of morphisms $\varphi : \mathcal{E} - - \to \mathcal{F}$ between (pre)sheaves on the same base source or target space of a rational map $f : X - - \to Y$. Categories provide the natural language for the simultaneous management of sets of functions and morphisms between sets or other algebraic structures (groups, vector spaces, rings, modules on a ring, e.g.) to solve systems of equations $\rceil$ defined on the support X.

2. Derived Categories

The notion of a derived category is crucial for working with complexes of sheaves. Derived categories allow one to handle not just sheaves but also their higher-order interactions (via derived functors). In algebraic geometry, derived categories help classify varieties, study their morphisms, and investigate their singularities.

3. Cohomology

Cohomology groups measure the global sections of sheaves or their higher cohomologies, providing important topological information about the underlying variety. For example, the singular cohomology of a variety gives information about its topology, while sheaf cohomology informs about algebraic structure, like the solution space to certain algebraic equations on the variety.

4. Grothendieck's Theorem

Grothendieck developed a foundational framework for algebraic geometry that combined sheaf theory, category theory, and homological methods. This framework leads to important tools like the Grothendieck spectral sequence, which connects the cohomology of various sheaves and provides a powerful computational method.

5. Functoriality and Derived Functors

Homological methods are heavily based on derived functors like Ext and Tor, which measure extensions and torsion, respectively. These functors help to understand the derived categories of sheaves and their relations. For example, the Ext functor gives a measure of how one sheaf can be extended by another, while Tor measures how torsion occurs in certain sheaves.

6. Localization and Finiteness

Homological methods often leverage localization techniques in algebraic geometry to break down complicated global problems into simpler local ones. This is especially useful in the study of singularities and the behavior of sheaves on non-smooth varieties. Finiteness conditions, such as finiteness of cohomology, help classify varieties by controlling the possible algebraic structures of their sheaves.

7. Algebraic K-Theory

K-theory (especially algebraic K-theory) is another important homological tool. It provides a way of understanding vector bundles and other algebraic objects on a variety, often in relation to its topology. K-theory is concerned with classes of vector bundles and provides a bridge between algebraic geometry and topological invariants. 8. Homological Algebra in Moduli Problems

Moduli spaces, which classify families of algebraic varieties or objects, are often studied using homological methods. These include techniques like perverse sheaves and the study of the monodromy action on cohomology, which helps in understanding the structure of moduli spaces, particularly in the study of families of varieties with singularities or degenerations.

Key Examples:

The Grothendieck-Riemann-Roch theorem: This connects the Euler characteristic of a sheaf with its cohomology and topological data, often providing a bridge between geometry and topological invariants. Hodge theory: In the context of algebraic varieties, particularly projective varieties, Hodge theory uses cohomology groups to study the structure of the variety, especially for smooth varieties.

Applications of Homological Methods in Algebraic Geometry: Singularity Theory: Studying the singularities of varieties using the tools of sheaf cohomology and homological algebra. Moduli Spaces: Understanding the geometry of moduli spaces of varieties or vector bundles via the derived category. Intersection Theory: Using homological methods to study intersections of divisors on varieties, leading to results like the Hirzebruch-Riemann-Roch theorem.

These methods are often complex and technical but are fundamental to much of modern algebraic geometry. They enable the study of varieties in a much more flexible and detailed way than traditional geometric methods alone would allow.

0.1.4. Some applications of GAGA to Physics (*)

Sheaves, and the broader tools of algebraic geometry, have found a number of important applications in physics, especially in areas that involve geometric or topological aspects, such as string theory, quantum field theory (QFT), and topological field theories. Below are some of the key ways in which sheaf theory and related concepts have been applied to physics:

1. String Theory and D-branes

In string theory, sheaves appear in the study of D-branes, which are objects on which open strings can end. These branes often live in a certain geometrical space, and the structure of D-branes can be described using sheaf theory.

More specifically, the category of sheaves is used to model the space of states of a D-brane, including both the physical (quantum) and geometric aspects of the branes.

The concept of sheaf cohomology is used in string theory to classify different possible states of these D-branes, including their interactions. The cohomology of sheaves also helps describe the possible quantum states of these branes.

2. Topological Field Theory (TFT)

Topological field theories are quantum field theories in which the physical observables depend only on the topology of the underlying space-time and not on its specific geometry. These theories are important in understanding phenomena that are insensitive to local geometric details (such as certain types of anomalies in quantum fields). Sheaf theory plays a role in topological quantum field theories (TQFTs) by helping to describe how physical quantities (like correlation functions) change when the space-time topology changes. In this setting, sheaves encode information about topological defects, singularities, and the global structure of the space-time. The relationship between sheaves and categories of topological defects is important in understanding the interplay between geometry and physics in topological quantum field theory.

3. Gauge Theories and Bundle Theory

In gauge theories, which are the foundation of the standard model of particle physics, the mathematical objects used to describe the fields are fiber bundles, and the sections of these bundles are related to the physical fields. Sheaves help model the sections of these fiber bundles, particularly in the study of gauge fields (connections on bundles). The sheaf cohomology of such bundles can describe the global structure of gauge fields, which are critical in understanding the dynamics of force fields like electromagnetism and gravity. Additionally, sheaf theory helps describe instanton solutions and other solutions to gauge theories, which have important applications in understanding non-perturbative phenomena in quantum field theory.

4. Quantum Field Theory (QFT) and Sheaf Theory

In quantum field theory, sheaves provide a powerful tool for describing fields and their interactions. Specifically, sheaves allow the study of local field theories and the structure of operator algebras. One of the key applications of sheaves in QFT is in the study of distributional fields and their singularities. Sheaves offer a framework for generalizing classical fields to incorporate quantum effects, where classical fields (like scalar fields or gauge fields) are generalized as sheaves or complexes of sheaves. Sheaf theory also plays a role in the study of quantum gravity. Since space-time itself can have a non-trivial topology or geometry at the quantum level, sheaf-theoretic techniques can be used to model such structures in a way that incorporates quantum fluctuations.

5. Holography and the AdS/CFT Correspondence

The AdS/CFT correspondence is a conjecture in string theory that relates a type of string theory formulated in Anti-de Sitter (AdS) space to a conformal field theory (CFT) on its boundary. This duality has led to a deep understanding of the relationship between quantum gravity and quantum field theory.

Sheaf theory can be used to understand certain aspects of the boundary conditions and quantum states on the boundary of AdS spaces. More generally, sheaves help describe the types of localized excitations or "defects"that may exist in the boundary field theory, which are crucial for understanding the correspondences between bulk and boundary physics.

6. Topological Insulators and Quantum Matter

Topological insulators are materials that have insulating bulk properties but conductive edge states, and these edge states have been extensively studied using tools from topology and geometry. In this context, sheaf theory has been applied to study the global structure of the field configurations in such systems. Sheaf theory helps describe the flow of currents on the boundary of these materials and provides a framework for understanding the topological invariants that characterize these systems, such as the Chern number.

7. Mirror Symmetry in String Theory

Mirror symmetry is a duality between pairs of Calabi-Yau manifolds that plays a central role in string theory, particularly in the context of compactifications of string theory on Calabi-Yau spaces. In mirror symmetry, the moduli spaces of complex structures and symplectic structures of these manifolds are related in a way that can be studied using sheaf cohomology. The relationship between the geometry of these spaces and the behavior of physical observables in string theory often relies on deep results from sheaf theory, especially in the study of moduli spaces of these geometries.

8. Quantum Cohomology and Sheaf Counting

In the study of quantum cohomology, the space of cohomology classes on a variety is augmented by quantum corrections. These corrections encode the effect of "quantum.effects in geometry, such as the contribution of curves or other subvarieties to the space's geometry.

Sheaf theory can be used to compute Gromov-Witten invariants, which count the number of curves of given classes that can be embedded in a variety. This has applications in string theory, particularly in understanding how different types of branes interact with the underlying geometry.

9. Localization Techniques in Path Integrals

Localization is a powerful method used in quantum field theory and statistical mechanics to simplify the computation of path integrals. This approach uses mathematical tools like sheaves and sheaf cohomology to reduce complex integrals to more tractable forms.

The method of localization has applications in computing the partition function of certain quantum field theories, such as in the study of topological quantum field theories and supersymmetric gauge theories.

10. Singularity Theory in Physics

Singularities are points where the usual description of a space breaks down. Sheaf theory can be used to describe the structure of singularities in both classical and quantum systems. In physics, understanding the behavior of fields near singularities (such as black holes or other exotic objects) often involves sheaf-theoretic methods.

Conclusion

Sheaves have become an indispensable tool in modern theoretical physics, particularly in areas involving complex geometrical structures, quantum field theory, string theory, and topological phenomena. They provide a rigorous and flexible framework for understanding the intricate relations between geometry, topology, and physics.

0.2. Outline of the chapter A301

In addition of this Introduction and a fifts section about Complementes (Conclusions, Practices, Chalenges, References). this chapter has the following four sections:

1. Algebraic structures
2. Commutative Algebra.
3. Intersection and invariants
4. Some connections with basic statistics

0.2.1. Methodological issues

In this chapter we privilege a structural approach based on Commutative Algebra for the study of systems of equations and their solutions. This approach uses notions of Basic Algebra which are linked to algebraic groups, rings, modules and fields. A motivation is given by the Galois Theory linked to the general resolution of equatons. In despite of the strong parallelism with general properties of Algebraic Number theory, these properties are only sketched in regard to some basic properties of divisors.

The intuitive notion of divisor is formalized in terms of principal ideals (f). They are reinterpreted in terms of "linear combinations of hypersurfaces" $[f]$ (defined up to non-null multiplication) with positive $[p] = [f]_0$ and negative part $[q] = [f]_\infty$, which provide the symbolic representation $[f] = [f]_0 - [f]_\infty$ of rational functions $f = p/q$ (meromorphic functions in the complex case).

The intuitive version of a divisor as a formal sum of codimension one subvarities, was exploited in an intensive way by the Italian school of Algebraic Geometry to understand the geometry of curves in terms of linear series, and the geometry of surfaces in terms of linear subsytems. Both of them provide the foundations for the "adjunction theory", where one describes the geometry of a variety from "incidence" conditions fulfilled by sections verifying some prescribed conditions (relative zeroes and poles, e.g.)

A more systematic approach to these ideas was performed by Zariski (initially in Italy, later in USA) and Van der Waerden (in Germany) along the 1930s. By using algebraic tools of the German school they were able of reformulating classical results of the Italian school, and reorganize a theoretical corpus in a much more systematic way. The first global compendium including a systematic treatment from the algebraic viewpoint are [Wae49][14], and [Zar58][15]

A geometric interpretation of the resolution of systems of equations is linked to the computation of their solutions. This interpretation involves to the Intersection Theory, which has several parts:

- *Existence:* corresponding to the number or solutions on the base field k. The first result is the Bézout theorem for plane curves, and its generalizations to higher dimension.
- *Properties of solutions sets* which are formulated in terms of "mobile cycles" and their equivalence classes in terms of numerical, homological or algebraic equivalence clsses.
- *Explciit computation* involving the "loclaization" of solutions, identification of "branches" and their overlapping linked to multiple solutions.

[14] B.L.Van der Waerden: *Moderne algebra*, 1s ed in 1930, 2nd ed 1949.

[15] O.Zariski and P.Samuel: *Commutative Algebra (2 vols)*, 1st ed by Van Nostrand , 1958-60; 2nd ed corrected by Springer. GTM, 1975.

All these topics, make part of the Enumerative Geometry A_{34} which uses a mixture of algebraic, topological and differential criteria. Thus, it can be considered as a transversal subarea with multiple connections with different approaches to GAGA and their applications. The request for a rigorous foundation was the 15th Hilbert's problme (talk in the ICM Paris Conference of 1900). Their algebrization is part of the Weil's Program (1949) The most general recent formulation appears in [Ful84] [16]

0.2.2. A categorical approach

In addiition of the intrinsic approaches to GAGA, it is necessary to consider extrinsic approaches, which are linked to maps $f : X \to Y$ or morphisms $X - - \to Y$ (not everywhere well defined). This distinction involves to superimposed structures (bundles, fibrations, sheaves) as "adiditional objects" defined on the source X and target spaces Y for f. The simples "examples" in the smooth case correspond to embeddings and submersions between smooth manifolds A_{11}, which are locally equivalent to the classical linear sections and projections of the Projective Geometry. Both of them provide criteria to identify regular submanifolds or fibrations giving classification criteria.

In the same way as in the smooth case, we are interested in maps or morphisms preserving the algebraic structure. Embeddings of the smooth case are replaced by immersions, whereas submersions are replaced by proper maps. So, immersions and proper maps allow a characterization of subvarieties in the GAGA framework. When one has "enough sections" fulfilling regular conditions, one says that the sheaf of sections is "ample". For each equivalence relation one has an ampleness criterium. So, formal properties for sets of regular maps are replaced by ampleness ceriteria.

By definition, if f is a regular map, then it has maximal rank, i.e. the kernel $Ker(f)$ or the cokernel $Coker(f)$ vanishes, corresponding to immersions and submersions, respectively, in the smooth case. In practice, a map or a morphism is not regular, and one must work with non-vanishing kernels and cokernels. This argument is naturally extended to morphisms $\phi : E - - \to F$ between total spaces of superimposed structures (bundles, fibrations, sheaves). They are linked between them in terms of long sequences which are not exact. Exactness is important because it allows to "predict" the behaviour of a system, but mos sequences of Modules (sheaves of modules) are not exact ones.

To solve this problem, one uses injective vs surjective "resolutions" in terms of locally free sheaves, extending the notion of free local rings. The general construction is realized in terms of "Derived Functors" which are introduced in the next chapter A_{303}. Intuitively, they are linked to "expanding" (Ext functors) or restricting (Tor funcgtors) the original information.

Before developing these extensions, it is convenient to acquire some familiarity with the "objects" where one applies this viewpoint. They concern to and

[16] W.Fulton: *Intersection Theory*, Springer-Verlag, 1984.

adaptation of classical categories involving sets, groups, vector spaces, rings or modules, e.g. So, in the GAGA framework, we follow an increasing difficult:

- *Categories of Varieties* with three *basic subcategories* corresponding to Affine Varieties (defined by polynomial equations), Projective (homogeneous polynomials), and Quasi-Projective Varieties (non-necessarily closed subsets of the Projective space), depending on the ambient space.
- *Categories of Schemes* with three *basic subcategories* corresponding to Affine, rojective, and Quasi-Projective Schemes, non-necessarily purely-dimensional and admiting embedded components, also, corresponding locally to non-reduced polynomials or analytical functions.

Each one of the above "objects" can display different kinds of mpas or, more generally morphisms. The most relevant ones are the folloing ones

- *Regular Maps:* A morphism between varieties or schemes that corresponds to a regular function between them.
- *Projective Morphisms:* Morphisms that map projective varieties into other projective varieties.
- *Finite Morphisms:* Morphisms where the preimage of any point consists of a finite number of points.
- *Flat Morphisms:* A morphism with a certain "uniformity" property in the fibers.

Finally, in the same way as the categories $\mathfrak{V}$ for Vector Bundles ξ and $\mathfrak{P}$ for Principal Bundles $\mathcal{P}$ on smooth manifolds M, one can consider categories for

- Different kinds of Sheaves $\mathcal{F}$, including ample or coherent sheaves, e.g.
- Morphisms $\mathcal{F}_0 - - \to \mathcal{F}_1$ between sheaves to compara or infer structures, e.tg.
- Different kinds of cohomologies (Čech, ℓ-adic, Crystalline, e.g.) for the computation of invariants.
- *Derived Categoriess* in terms of the cohimology of injective vs projective resolutions as a general method for managing the lack of regularity (injectivity or surjectivity) for maps or morphisms.

The precedent remarks show the power of categories to unify and compare properties invvling objects, maps, superimposed structures and their invariants. In despite of its abastract character can be considered as some kind "universal language" which simplifies the analysis of properties, and provides a unification between related areas with "similar" characteristics.

In the chapter A_{308} we will show some applications of categories to some topics of Theoretical Physics and some Engineering areas developed in the part

II of these notes, by showeing their power to relate knowledge domains for each matter. Some *advanced topics* for the applicaitons of categories to Moduli theory will be develop edin the chapteer A_{304} (analytical metods in GAGA). concern to:

0.2.3. Commutative Algebra and Affine Algebraic Geometry

(section 3) Commutative algebra is essentially the study of the rings occurring in algebraic number theory and algebraic geometry.Several concepts of commutative algebras have been developed in relation with algebraic number theory, such as Dedekind rings (the main class of commutative rings occurring in algebraic number theory), integral extensions, and valuation rings.

Polynomial rings in several indeterminates over a field are examples of commutative rings. Since algebraic geometry is fundamentally the study of the common zeros of these rings, many results and concepts of algebraic geometry have counterparts in commutative algebra, and their names recall often their geometric origin; for example “Krull dimensio”, “localization of a ring”, “local ring”, “regular ring”.

An affine algebraic variety X corresponds to a prime ideal $\mathfrak{p}$ in a polynomial ring $k[\underline{x}]$, and the points $x \in X$ of such an affine variety correspond to the maximal ideals $\mathfrak{m}$ that contain this prime ideal $\mathfrak{p}$. The Zariski topology, originally defined on an algebraic variety, has been extended to the sets of the prime ideals of any commutative ring A; for this topology, the closed sets are the sets of prime ideals that contain a given ideal.

The spectrum $Spec(A)$ of a ring A is a *ringed space* formed by the prime ideals equipped with the Zariski topology, and the localizations of the ring at the open sets of a basis of this topology. This is the starting point of scheme theory, a generalization of algebraic geometry introduced by Grothendieck, which is strongly based on commutative algebra, and has induced, in turns, many developments of commutative algebra.

0.2.4. Algebraic methods in Discrete Mathematics (*)

Some of the most relevant algebraic techniques for Discrete Mathematics are the following ones::

- Rank argument
- Restricted intersection theorems
- Multilinear linear algebra (tensor products and exterior powers)
- Linear algebra modulo composite number
- Combinatorial Nullstellensatz

- Extrapolation arguments
- Tensor rank

A more amenable approach can be read in https://mastermath.datanose.nl/Summary/294 including nearer issues:

The old combinatorial geometry (Whitney, Tutte, e.g.) based on geometric configurations, and their symbolic representations (graphs, lattices, posets) provides an initial support for a large number of combinatorial issues (Rota, e.g.). In the "simplest" cases, they have been reformulated in (co)homological terms or more recently in terms of Commutative Algebra (Stanley)

In particular, graphs G play a central role involving operations (contraction vs expansion), maps between graphs, superimposed structures expressed in analytic terms as $\mathcal{G} = (G, \mathcal{O}_G)$, and superimposed structures $\mathcal{E} = (E, \mathcal{O}_E)$ in a similar way to ringed spaces. All of them help to understand the role of symbolic representations. Analytical graphs $\mathcal{G}$ and superimposed structures $\mathcal{E}$ support the applications of standard tools arising from Linear Algebra (SVD and PCA, e.g.), Basic Algebra (groups, polunomial rings, number fields, ideals), and Diufferential Geometry (scalar, vector, covector, tensor sifelds)

The application of algebraic methods in combinatorics has been remarkably successful in recent years. Examining combinatorial problems from an algebraic perspective also yields connections to combinatorial geometry, probability theory and theoretical computer science. In this chapter we give basic notions and simple results. In the chapter $A306$ (Discrete and Combinatorial methods), we develop more some more advanced methods (Spectral Graph Theory) and their applications.

The *polynomial method* is a relatively recent innovation in combinatorics borrowing some of the philosophy of algebraic geometry. The starting point is given by the Hilbert's Nullstellensatz and Alon's combinatorial Nullstellensatz. These results can be applied to classical problems in additive number theory and graph colouring. We treat some very recent applications of the polynomial method to cap set problems. They can be apllliedd to stable polynomials to give a construction of Ramanujan graphs (an important family of expanded graphs).

0.3. References for this introduction

References appearing below are not exhaustive. They are included to encourage the reader to search more complete information by him/herself. Any suggestion is welcome.

0.3.1. Basic bibliography

Only basic textbooks are included. At the last subsection §5,4 (References) one includes some other paragraphs including references and some applications

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0.3.2. Software resources

Only those based in symbolic programming are included. It is necessary to develop interpreters in tersm of OOP paradigms and the corresponding Programming Languages (C, ++, C#) or their extensions to Functional Programming under the common Python framework.

- Macaulay 2
- Singular
- CoCoA
- Normaliz
- Sagemath (under Python).

Final remark: Readers which are interested in a more complete presentation of the above materials (in Spanish language, still) or some another chapter of the module A_{30} (Foundaitons of GAGA), please write a message to javier.finat@gmail.com