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Basic Algebraic Geometry Codes

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Outline

Main references

Fulton, William *Algebraic curves. An introduction to algebraic geometry.* Mathematics Lecture Notes Series. W. A. Benjamin, Inc., New York-Amsterdam, 1969. xiii+226 pp.

Høholdt; van Lint; Pellikaan *Algebraic geometry of codes.* Handbook of coding theory, Vol. I, II, 871–961, North-Holland, Amsterdam, 1998.

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- **V.D. Goppa (1977)** Codes associated with divisors. *P. Information Transmission*, 13, 22-26.
- ...
- **Tsfasman, Vlăduț and Zink (1982)** Modular curves, Shimura curves and Goppa codes better that Gilbert-Varshamov Bound. *Math. Nachrichten*, 109, 21-28.
- ... algebraic curves ↔ algebraic function fields
- **Høholdt, van Lint, Pellikaan (1998)** Algebraic geometry codes, in Handbook of Coding Theory, Elsevier, 871-961. Weight functions...
- **Martínez, Munuera, Ruano Eds. (2008)** Advances in Algebraic Geometry Codes. World Scientific.

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Affine space, projective space

Algebraic Geometry codes are defined w.r.t. curves in the affine and projective space. Let $\mathbb{F}$ be a field, the n-dimensional affine space $\mathbb{A}^n(\mathbb{F})$ over $\mathbb{F}$ is just the vector space $\mathbb{F}^n$

$$\mathbb{A}^n(\mathbb{F}) = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{F}\}.$$

Let $\mathbf{x}, \mathbf{x}'$ be two elements in $\mathbb{F}^{n+1} \setminus \{\mathbf{0}\}$, they are equivalent $\mathbf{x} \equiv \mathbf{x}'$ if there is a $\lambda \in \mathbb{F}$ such that $\mathbf{x} = \lambda \mathbf{x}'$. The n-dimensional projective space $\mathbb{P}^n(\mathbb{F})$ over $\mathbb{F}$ is

$$\mathbb{P}^n(\mathbb{F}) = \mathbb{F}^{n+1} / \equiv .$$

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The equivalence class (projective point) containing $\mathbf{x} = (x_1, x_2, \ldots, x_{n+1})$ will be denoted by $\mathbf{x} = (x_1 : x_2 : \ldots : x_{n+1})$ (homogeneous coordinates). Thus projective points are 1 dimensional subespaces of $\mathbb{A}^{n+1}(\mathbb{F})$.

If $P \in \mathbb{P}^n(\mathbb{F})$ and $P = (x_1 : x_2 : \ldots : x_{n+1} = 0)$ then it is called point at infinity, those points not at infinity are called affine points and each of them can be uniquely represented as $P = (x_1 : x_2 : \ldots : x_{n+1} = 1)$.

Any point at infinity can be uniquely represented as with a 1 at its right-most non-zero position $P = (x_1 : x_2 : \ldots : x_i = 1 : 0 : \ldots : 0)$.

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The projective line and the projective plane

See the blackboard

Exercise

Let $\mathbb{F} = \mathbb{F}_q$, prove that

- $\mathbb{P}^n(\mathbb{F})$ contains $\sum_{i=0}^{n} q^i$ points.
- $\mathbb{P}^n(\mathbb{F})$ contains $\sum_{i=0}^{n-1} q^i$ points at infinity.

Note: Please, try even the trivial exercises.

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Let $\mathbb{F}[x_1, x_2, \ldots, x_n]$ be the set of polynomials with coefficients from $\mathbb{F}$. A polynomial $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ is homogeneous of degree d if every term of f is of degree d.

If $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ is not homogeneous and f has maximum degree d we can homogenizate it adding a variable as follows

$$f^H(x_1, x_2, \ldots, x_n, x_{n+1}) = x_{n+1}^d f\left(\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right).$$

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Clearly $f^H(x_1, x_2, \ldots, x_n, 1) = f(x_1, x_2, \ldots, x_n)$. Moreover, if we start with and homogeneous polynomial $g(x_1, x_2, \ldots, x_n, x_{n+1})$ of degree d,

$$g(x_1, x_2, \ldots, x_n, 1) = f(x_1, x_2, \ldots, x_n),$$

them f has degree $k \leq d$ and $g = x_{n+1}^{d-k} f^H$.

Thus there is a one-to-one correspondence between polynomials in n variables of degree d or less and homogeneous polynomials of degree d in $n+1$ variables.

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Theorem

Let $g(x_1, x_2, \ldots, x_n, x_{n+1})$ be an homogeneous polynomial of degree d over $\mathbb{F}$.

1. *If $\alpha \in \mathbb{F}$ then $g(\alpha x_1, \alpha x_2, \ldots, \alpha x_n, \alpha x_{n+1}) = \alpha^d g(x_1, x_2, \ldots, x_n, x_{n+1})$.*
2. *$f(x_1, \ldots, x_n) = 0$ if and only if $f^H(x_1, \ldots, x_n, 1) = 0$.*
3. *If $(x_1 : x_2 : \ldots : x_{n+1}) = (x'_1 : x'_2 : \ldots : x'_{n+1})$ then $g(x_1, \ldots, x_n, x_{n+1}) = 0$ iff $g(x'_1, \ldots, x'_n, x'_{n+1}) = 0$.*

Theorem above implies that the zeros of f in $\mathbb{A}^n(\mathbb{F})$ correspond precisely to affine points in $\mathbb{P}^n(\mathbb{F})$ that are zeros of f^H and the concept of a point of $\mathbb{P}^n(\mathbb{F})$ being a zero of a homogeneous polynomial is well defined.

For $k \geq 0$ let $\mathcal{P}_k \subset \mathbb{F}_q[x]$ be the set of all polynomials of degree less than k. Let α be a primitive n-th root of unity in $\mathbb{F}_q$ $(n = q - 1)$, then the code

$$\mathcal{C} = \left\{ \left(f(1), f(\alpha), \ldots, f(\alpha^{q-2}) \right) \mid f \in \mathcal{P}_k \right\} \tag{1}$$

is the narrow-sense $[n, k, n - k + 1]$ RS code over $\mathbb{F}_q$.

RS codes are BCH codes

See the blackboard

$\mathcal{C}$ can be extended to a $[n + 1, k, n - k + 2]$ code given by

$$\hat{\mathcal{C}} = \left\{ \left(f(1), f(\alpha), \ldots, f(\alpha^{q-2}), f(0) \right) \mid f \in \mathcal{P}_k \right\} \tag{2}$$

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Exercise

Show that if $f \in \mathcal{P}_k$ with $k < q$ then

$$\sum_{\beta \in \mathbb{F}_q} f(\beta) = 0.$$

Clue: $q > 2 \quad \sum_{\beta \in \mathbb{F}_q} \beta = 0.$

Thus $\hat{C}$ results from adding an overall parity check to a RS code and the minimum weight increases.

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Let n be an integer $1 \leq n \leq q$, and $\gamma = (\gamma_0, \ldots, \gamma_{n-1})$ a n-tuple of distinct elements of $\mathbb{F}_q$, and $\mathbf{v} = (v_0, \ldots, v_{n-1})$ a n-tuple of non-zero elements of $\mathbb{F}_q^*$. Let k be an integer $1 \leq k \leq n$, then

$$\mathcal{GRS}_k(\gamma, \mathbf{v}) = \{(v_0 f(\gamma_0), \ldots, v_{n-1} f(\gamma_{n-1})) \mid f \in \mathcal{P}_k\} \quad (3)$$

are the Generalized Reed-Solomon codes over $\mathbb{F}_q$.

GRS codes are $[n, k, n-k+1]$ MDS codes See the blackboard

Note that both, the narrow sense RS code and the extended RS code can be seen as Generalized Reed-Solomon codes.

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Since there is a one-to-one correspondence between $\mathcal{L}_{k-1}$ the homogeneous polynomials in two variables of degree $k-1$ and the non-zero polynomials of $\mathcal{P}_k$, let $P_i = (\gamma_i : 1) \in \mathbb{P}^1(\mathbb{F}_q)$, we can redefine the code $\mathcal{GRS}_k(\gamma, \mathbf{v})$ as follows

$$\{(v_0 g(P_0), \ldots, v_{n-1} g(P_{n-1})) \mid g \in \mathcal{L}_{k-1}\} . \qquad (4)$$

Let $t = \mathrm{ord}_q(n)$ and β a primitive n-th root of unity in $\mathbb{F}_{q^t}$. Choose $\delta > 1$ and let $\mathcal{C}$ the narrow sense BCH code of length n and designed distance δ, i.e.

$$c(x) \in \mathbb{F}[x]/(x^n - 1) \text{ is in } \mathcal{C} \Leftrightarrow c(\beta^j) = 0, \quad 1 \leq j \leq \delta - 1.$$

Note that

$$\begin{aligned}(x^n-1)\sum_{i=0}^{n-1}\frac{c_i}{x-\beta^{-i}} &= \sum_{i=0}^{n-1} c_i \sum_{l=0}^{n-1} x^l (\beta^{-i})^{n-1-l} \\ &= \sum_{l=0}^{n-1} x^l \sum_{i=0}^{n-1} c_i (\beta^{l+1})^i.\end{aligned} \tag{5}$$

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Because $c(\beta^l) = 0, \quad 1 \leq l \leq \delta - 2$, LHS in (5) is a polynomial with lowest degree term al least $\delta - 1$, thus RHS can be written as $p(x)x^{\delta-1}$ with $p(x) \in \mathbb{F}_{q^t}[x]$.

$$\begin{aligned} c(x) \in \mathcal{C} &\Leftrightarrow \sum_{i=0}^{n-1} \frac{c_i}{x - \beta^{-i}} = \frac{p(x)x^{\delta-1}}{x^n - 1} \\ &\Leftrightarrow \sum_{i=0}^{n-1} \frac{c_i}{x - \beta^{-i}} \equiv 0 \mod x^{\delta-1}. \end{aligned} \tag{6}$$

This equivalence means that if the LHS is written as a rational function $\frac{a(x)}{b(x)}$ then the numerator $a(x)$ will be a multiple of $x^{\delta-1}$ $(b(x) = x^n - 1)$.

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Following the discussion above, fix an extension $\mathbb{F}_{q^t}$ of $\mathbb{F}_q$ ($t = \mathrm{ord}_q(n)$ no longer needed). Let

$$L = \{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\} \subset \mathbb{F}_{q^t}$$

and let $G(x) \in \mathbb{F}_{q^t}[x]$ with $G(\gamma_i) \neq 0$ where $\gamma_i \in L$.

The Goppa code $\Gamma(L, G)$ is the set of vectors $(c_0, \ldots, c_{n-1}) \in \mathbb{F}_q^n$ such that

$$\sum_{i=0}^{n-1} \frac{c_i}{x - \gamma_i} \equiv 0 \mod G(x). \tag{7}$$

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The Goppa code $\Gamma(L, G)$ is the set of vectors $(c_0, \ldots, c_{n-1}) \in \mathbb{F}_q^n$ such that

$$\sum_{i=0}^{n-1} \frac{c_i}{x - \gamma_i} \equiv 0 \mod G(x). \tag{7}$$

This again means that if the LHS is written as a rational function then the numerator is a multiple of $G(x)$ the Goppa polynomial. Note that $G(\gamma_i) \neq 0$ guarantees that $x - \gamma_i$ is invertible in $\mathbb{F}_{q^t}[x]/(G(x))$.

Since

$$\frac{1}{x-\gamma_i} \equiv -\frac{1}{G(\gamma_i)} \frac{G(x)-G(\gamma_i)}{x-\gamma_i} \mod G(x) \qquad (8)$$

Substituting in eqn. (7) we have $(c_0, \ldots, c_{n-1}) \in \Gamma(L, G)$ iff

$$\sum_{i=0}^{n-1} c_i \frac{G(x)-G(\gamma_i)}{x-\gamma_i} G(\gamma_i)^{-1} \equiv 0 \mod G(x) \qquad (9)$$

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Suppose $\deg G(x) = w$ and

$$G(x) = \sum_{j=0}^{w} g_j x^j, \quad g_j \in \mathbb{F}_{q^t}.$$

$$\frac{G(x) - G(\gamma_i)}{x - \gamma_i} G(\gamma_i)^{-1} = G(\gamma_i)^{-1} \sum_{j=0}^{w} g_j \sum_{k=0}^{j-1} x^k \gamma_i^{j-1-k}$$
$$= G(\gamma_i)^{-1} \sum_{k=0}^{w} x^k \left(\sum_{j=k+1}^{w} g_j \gamma_i^{j-1-k} \right). \tag{10}$$

Some classical codes

Goppa codes. Parity check matrix.

Suppose $\deg G(x) = w$ and

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Some classical codes

Goppa codes. Parity check matrix.

From (9) setting the coefficients of x^k to 0 in the order $k = w-1, w-2, \dots, 0$ we have that $\mathbf{c} \in \Gamma(L,G)$ if $H\mathbf{c}^T = \mathbf{0}$, where H is

$$\begin{pmatrix} h_0 g_w & \dots & h_{n-1} g_w \\ h_0(g_{w-1} + g_w \gamma_0) & \dots & h_{n-1}(g_{w-1} + g_w \gamma_{n-1}) \\ \vdots & \vdots & \vdots \\ h_0 \sum_{j=1}^{w} g_j \gamma_0^{j-1} & \dots & h_{n-1} \sum_{j=1}^{w} g_j \gamma_{n-1}^{j-1} \end{pmatrix} \quad (11)$$

with $h_i = G(\gamma_i)^{-1}$.

H can be reduced to the matrix H'

$$\begin{pmatrix} G(\gamma_0)^{-1} & \dots & G(\gamma_{n-1})^{-1} \\ G(\gamma_0)^{-1}\gamma_0 & \dots & G(\gamma_{n-1})^{-1}\gamma_{n-1} \\ \vdots & \vdots & \vdots \\ G(\gamma_0)^{-1}\gamma_0^{w-1} & \dots & G(\gamma_{n-1})^{-1}\gamma_{n-1}^{w} \end{pmatrix} \qquad (12)$$

Note that the parity check matrix H' is the generator matrix of the $\mathcal{GRS}_w(\gamma, \mathbf{v})$ over $\mathbb{F}_{q^t}$ where $\mathbf{v} = (G(\gamma_0)^{-1}, \ldots, G(\gamma_{n-1})^{-1})$, i.e. we have that

$$\Gamma(L, G) = \mathcal{GRS}_w(\gamma, \mathbf{v})^{\perp}|_{\mathbb{F}_q}.$$

Since $\mathcal{GRS}_w(\gamma, \mathbf{v})^{\perp}$ is also a GRS code then classical Goppa codes are subfield subcodes of GRS codes.

Theorem

The Goppa code $\Gamma(L, G)$ *with* $\deg(G(x)) = w$ *is an* $[n, k, d]$ *code where* $k \geq n - wt$ *and* $d \geq w + 1$.

Proof.

The entries of H' are in $\mathbb{F}_{q^t}$. By choosing a base of $\mathbb{F}_{q^t}|\mathbb{F}_q$ each element of $\mathbb{F}_{q^t}$ can be represented by a $t \times 1$ column vector, and if we replace each entry in H' by the corresponding vector we get a matrix H'' with entries in $\mathbb{F}_q$ such that $H''\mathbf{c}^T = \mathbf{0}$, $\mathbf{c} \in \Gamma(L, G)$.

The rows of H'' may be independent thus $k \geq n - wt$. If $\mathbf{0} \neq \mathbf{c} \in \Gamma(L, G)$ has weight $\leq w$ then when the LHS of (7) is written as a rational function the numerator has degree $\leq w-1$, but it has to be a multiple of $G(x)$, which contradicts the fact $\deg(G(x)) = w$.

Up to here the first session (28/11)

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Up to here the first session (28/11)

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Up to here the first session (28/11)

Let $\mathcal{R}$ be the vector space of all the rational functions f with coefficients in $\mathbb{F}_{q^t}$ such that

1. $f = \frac{a(x)}{b(x)}$ where a, b are relatively prime.
2. The zeros of $a(x)$ include the zeros of $G(x)$ with at least the same multiplicity.
3. The only possible poles of f (i.e. the zeros of $b(x)$) are $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ with multiplicity at most one.

$f \in \mathcal{R}$ has a Laurent series expansion about γ_i

$$f = \sum_{j=-1}^{\infty} f_i (x - \gamma_i)^j \tag{13}$$

where $f_{-1} \neq 0$ if f has a pole at γ_i or $f_{-1} = 0$ otherwise.

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The residue of $f(x)$ at γ_i denoted as $\mathrm{Res}_{\gamma_i} f$ is the coefficient f_{-1} above. Let

$$\mathcal{C}_{Res}(G,\gamma) = \left\{ \left(\mathrm{Res}_{\gamma_0} f, \ldots, \mathrm{Res}_{\gamma_{n-1}} f \right) \mid f \in \mathcal{R} \right\} \tag{14}$$

Exercise

Show that $\mathcal{C}_{Res}(G,\gamma)|_{\mathbb{F}_q} = \Gamma(L,G)$.

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Exercise

Show that $\mathcal{C}_{Res}(G,\gamma)|_{\mathbb{F}_q} = \Gamma(L, G)$.

Let $m > 0$, $n = q^m$ and $\{P_1, \ldots, P_n\} = \mathbb{A}^m(\mathbb{F}_q)$. Let $0 \leq r \leq m(q-1)$ and $\mathbb{F}_q[x_1, \ldots, x_m]_r$ the set of polynomials of total degree r or less.

The *r-th order generalized Reed-Muller code* of length $n = q^m$ is

$$\mathcal{R}_q(r, m) = \{(f(P_1), \ldots, f(P_n)) \mid f \in \mathbb{F}_q[x_1, \ldots, x_m]_r\} \tag{15}$$

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Note that since $\beta^q = \beta$ for all $\beta \in \mathbb{F}_q$ if we note $\mathbb{F}_q[x_1, \ldots, x_m]_r^*$ the set of polynomials of total degree r or less with no variable with exponent q or higher we have

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$F_q[x_1, \ldots, x_m]_r^*$ is a vector space with a basis

$$\mathfrak{B} = \left\{ x_1^{e_1} x_2^{e_2} \ldots x_m^{e_m} \mid 0 \leq e_i < q, \sum_{i=0}^{m} e_i \leq r \right\}$$

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Clearlythe words $\{(f(P_1), \ldots, f(P_n)) \mid f \in \mathfrak{B}\}$ spand the code $\mathcal{R}_q(r, m)$.

Exercise

Prove that $\{(f(P_1), \ldots, f(P_n)) \mid f \in \mathfrak{B}\}$ are independent.

Algebraic curves

An affine plane curve $\mathcal{X}$ is the set of affine points $(x, y) \in \mathbb{A}^2(\mathbb{F})$ denoted as $\mathcal{X}_f(\mathbb{F})$ such that $f(x, y) = 0$, $f \in \mathbb{F}[x, y]$.

A projective plane curve $\mathcal{X}$ is the set of projective points $(x : y : z) \in \mathbb{P}^2(\mathbb{F})$ denoted (also) as $\mathcal{X}_f(\mathbb{F})$ such that $f(x, y, z) = 0$, $f \in \mathbb{F}[x, y, z]$ an homogeneous polynomial.

If $f \in \mathbb{F}[x, y]$ then $\mathcal{X}_{f^H}(\mathbb{F})$ is called the projective closure of $\mathcal{X}_f(\mathbb{F})$ (i.e. we add the possible points at infinity).

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If $f = \sum_{i,j} a_{ij} x^i y^j \in \mathbb{F}[x,y]$ the partial derivative f_x of f w.r.t. x is

$$f_x = \frac{\partial f}{\partial x} = \sum_{i,j} i a_{ij} x^{i-1} y^j.$$

The partial derivative f_y of f w.r.t. y is defined analogously.

A point (x_0, y_0) of $\mathcal{X}_f(\mathbb{F})$ is singular if $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. A point of $\mathcal{X}_f(\mathbb{F})$ is nonsingular or simple if it is not singular.

A curve that has no singular point is called nonsingular, regular or smooth. Analogous definitions hold for projective curves.

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The **Fermat curve** $\mathcal{F}_m(\mathbb{F}_q)$ is a projective plane curve defined by

$$f(x,y,z) = x^m + y^m + z^m = 0.$$

$f_x = mx^{m-1}$, $f_y = my^{m-1}$, $f_z = mz^{m-1}$, thus it has no singular points if $\gcd(m,q) = 1$.

Exercise

- Find the three projective points of $\mathcal{F}_3(\mathbb{F}_2)$.
- Find the nine projective points of $\mathcal{F}_3(\mathbb{F}_4)$.

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Let $q = r^2$ where r is a prime power. The Hermitian curve $\mathcal{H}_r(\mathbb{F}_q)$ is a projective plane curve defined by

$$f(x, y, z) = x^{r+1} - y^r z - yz^r = 0.$$

Since r is a multiple of the characteristic then $\mathcal{H}_r(\mathbb{F}_q)$ is non singular.

Exercise

- Show that $(0 : 1 : 0)$ is the only point at infinity of $\mathcal{H}_r(\mathbb{F}_q)$.
- Find the eight affine points $(x : y : 1)$ points of $\mathcal{H}_2(\mathbb{F}_4)$.

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Theorem

There are r^3 affine $(x : y : 1)$ points in $\mathcal{H}_r(\mathbb{F}_q)$.

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Proof.

$z = 1$ implies $x^{r+1} = y^r + y = \mathrm{Tr}_2(y)$ where Tr_2 is the trace map from $\mathbb{F}_{r^2}$ to $\mathbb{F}_r$.

$\mathrm{Tr}_2(y)$ is $\mathbb{F}_r$-linear and surjective, so its kernel is a 1-dim. $\mathbb{F}_r$-subspace of $\mathbb{F}_{r^2}$, thus has r values with $\mathrm{Tr}_2(y)$ that leads to r affine points on $\mathcal{H}_r(\mathbb{F}_q)$ of type $(0 : y : 1)$.

(Cont. ...) □

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(Cont. ...)

If $x \in \mathbb{F}_{r^2}$ then $x^{r+1} \in \mathbb{F}_r$, as $r^2 - 1 = (r+1)(r-1)$ and the non zero elements of $\mathbb{F}_r$ in $\mathbb{F}_{r^2}$ are those satisfying $\beta^{r-1} = 1$. When y is one of the $r^2 - r$ elements in $\mathbb{F}_{r^2}$ with $\mathrm{Tr}_2(y) \neq 0$, there are $r+1$ solutions $x \in \mathbb{F}_{r^2}$ of $\mathrm{Tr}_2(y) = x^{r+1}$. Thus there are $(r^2 - r)(r+1) = r^3 - r$ more affine points on $\mathcal{H}_r(\mathbb{F}_q)$, and the theorem follows. $\square$

The Klein quartic $\mathcal{K}_4(\mathbb{F}_q)$ is a projective plane curve defined by

$$f(x,y,z) = x^3y + y^3z + z^3x = 0.$$

Exercise

- Find the three partial derivatives of f and show that if $\mathrm{char}(\mathbb{F}_q) = 3$ then $\mathcal{K}_4(\mathbb{F}_q)$ is non singular.
- If $(x:y:z)$ is a singular point in $\mathcal{K}_4(\mathbb{F}_q)$ show that $x^3y = -3y^3z$, $z^3x = 9y^3z$ and $7y^3z = 0$.
- Show that if $\mathrm{char}(\mathbb{F}_q) \neq 7$ then $\mathcal{K}_4(\mathbb{F}_q)$ is non singular.

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The degree of a point in a curve depends on the field under consideration. Let $q = p^r$ (p prime) and $m \geq 1$, the map $\sigma_q : \mathbb{F}_{q^m} \longrightarrow \mathbb{F}_{q^m}$ given by $\sigma_q(\alpha) = \alpha^q$ is an automorphism of $\mathbb{F}_{q^m}$ that fixes $\mathbb{F}_q$ ($\sigma_q = \sigma_p^r$ where σ_p is the Frobenius map).

If $P = (x, y)$ or $P = (x : y : z)$ in $\mathbb{A}^2(\mathbb{F}_{q^m})$ or $\mathbb{P}^2(\mathbb{F}_{q^m})$ denote by $\sigma_q(P) = (\sigma_q(x), \sigma_q(y))$ and $\sigma_q(P) = (\sigma_q(x) : \sigma_q(y) : \sigma_q(z))$ respectively.

Exercise

- Show that that $\sigma_q(P)$ is well defined if $P \in \mathbb{P}^2(\mathbb{F}_q)$.
- Show that if $f \in \mathbb{F}_q[x, y]$ (or homogeneous in $\mathbb{F}_q[x, y, z]$) then $f(P) = 0$ implies $f(\sigma_q(P)) = 0$ ($P \in \mathbb{A}^2(\mathbb{F}_{q^m})$ or $\mathbb{P}^2(\mathbb{F}_{q^m})$ respectively).

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If $P = (x, y)$ or $P = (x : y : z)$ in $\mathbb{A}^2(\mathbb{F}_{q^m})$ or $\mathbb{P}^2(\mathbb{F}_{q^m})$ denote by $\sigma_q(P) = (\sigma_q(x), \sigma_q(y))$ and $\sigma_q(P) = (\sigma_q(x) : \sigma_q(y) : \sigma_q(z))$ respectively.

Exercise

- Show that that $\sigma_q(P)$ is well defined if $P \in \mathbb{P}^2(\mathbb{F}_q)$.
- Show that if $f \in \mathbb{F}_q[x, y]$ (or homogeneous in $\mathbb{F}_q[x, y, z]$) then $f(P) = 0$ implies $f(\sigma_q(P)) = 0$ ($P \in \mathbb{A}^2(\mathbb{F}_{q^m})$ or $\mathbb{P}^2(\mathbb{F}_{q^m})$ respectively).

From exercise above if $P \in \mathcal{X}_f(\mathbb{F}_{q^m})$ then $\left\{\sigma_q^i(P) \mid i \geq 0\right\} \subseteq \mathcal{X}_f(\mathbb{F}_{q^m})$, and there are at most m distint points in the set since $\sigma_q^m = \mathrm{Id}$.

A point P on $\mathcal{X}_f(\mathbb{F}_q)$ of degree m over $\mathbb{F}_q$ is a set of m distinct points $P = \{P_0, \ldots, P_{m-1}\}$ with $P_i \in \mathcal{X}_f(\mathbb{F}_{q^m})$ and $P_i = \sigma_q^i(P_0)$. We will denote the degree of P over $\mathbb{F}_q$ as $\deg(P)$. Notice that points of degree m over $\mathbb{F}_q$ are fixed by σ_q just as the elements of $\mathbb{F}_q$, that's why they are cosidered to be on $\mathcal{X}_f(\mathbb{F}_q)$.

The points of degree one on $\mathcal{X}_f(\mathbb{F}_q)$ are called rational points or $\mathbb{F}_q$-rational points.

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$$f(x,y,z) = x^3 + xz^2 + z^3 + y^2z + yz^2 \in \mathbb{F}_2[x,y,z].$$

A point at infinity satisfies $z = 0$, thus $x^3 = 0$, therefore there is only one point at infinity $P_\infty = (0 : 1 : 0)$ and is $\mathbb{F}_2$-rational.

When considering the affine points we can assume $z = 1$, thus $x^3 + x + 1 = y^2 + y$. If $x, y \in \mathbb{F}_2$ then

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Example: Degree of points in a elliptic curve (Cont.)

Consider now $x, y \in \mathbb{F}_4$. If $y = 0, 1$ then $0 = y^2 + y$, but $x^3 + x + 1$ has no solution in $\mathbb{F}_4$.

If $y = \omega, \bar{\omega}$ are the roots of $y^2 + y = 1$, thus $x^3 + x = x(x+1)^2 = 0$. Therefore the points of degree 2 are

$$P_1 = \{(0 : \omega : 1), (0 : \bar{\omega} : 1)\},\ P_2 = \{(1 : \omega : 1), (1 : \bar{\omega} : 1)\}$$

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When defining AG codes we will need to compute the points in the intersection of two curves and the multiplicity at the point of intersection.

We will not define it because the definition is quite technical. Instead of it we will show with the following example how can we compute multiplicity similarly to the way multiplicity of zeros is computed for one variable polynomials.

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Algebraic curves
Intersection multiplicity example

Consider the elliptic curve defined by

$$f(x,y,z) = x^3 + xz^2 + z^3 + y^2z + yz^2 \in \mathbb{F}_{2^r}[x,y,z].$$

Intersection with $x=0$:

We have either $z=0$ or $z=1$. In the first case we get P_∞ and in the latter $(0:\omega:1), (0:\bar{\omega}:1) \in \mathbb{P}^2(\mathbb{F}_4)$.
We can see this in two ways:

- The curve and $x=0$ intersect at three degree 1 points in $\mathbb{P}^2(\mathbb{F}_4)$ with intersection multiplicity 1.
- The curve and $x=0$ intersect at two points in $\mathbb{P}^2(\mathbb{F}_2)$, one with degree 1 and the second with degree 2, both with with intersection multiplicity 1. (Notice that there are more points of higher degress)

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Intersection with $x^2 = 0$:

Notice that $x^2 = 0$ is the union of the line $x = 0$ with itself. Thus any point at $x^2 = 0$ and the elliptic curve occurs twice as frequently as it did at $x = 0$. Thus

- The curve and $x^2 = 0$ intersect at three degree 1 points in $\mathbb{P}^2(\mathbb{F}_4)$ with intersection multiplicity 2.
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We have seen that there is only one point P_∞ of the elliptic curve with $z = 0$. P_∞ has degree 1 over any extension field of $\mathbb{F}_2$. When we plug $z = 0$ in the equation of the elliptic curve we get $x^3 = 0$, thus P_∞ occurs with multiplicity 3.

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Intersection with $y = 0$:
$z = 0$ is not possible ($\Rightarrow x = 0$), so $z = 1$and we have

$$x^3 + x + 1 = 0.$$

The solutions to this equation occur in $\mathbb{F}_8$ and give us the points

$$(\alpha : 0 : 1), \quad (\alpha^2 : 0 : 1), \quad (\alpha^4 : 0 : 1) \quad \in \mathbb{P}(\mathbb{F}_8)$$

Threfore over $\mathbb{F}_8$ there are 3 points in the intersection, each of them of degree 1 and multiplicity 1. Over $\mathbb{F}_2$ they combine in a single degree 3 point P_3 with intersection multiplicity 1.

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We have seen that there is a "type" of uniformity when counting properly the number of points in the intersection of two curves, where properly means take into account both degree and multiplicity. This was stated in the following theorem

Theorem (Bézout)

Let f, g be homogeneous polynomials in $\mathbb{F}[x, y, z]$ of degrees d_f, d_g respectively. Suppose that f and g have no common nonconstant polynomial factors. Then $\mathcal{X}_f$ and $\mathcal{X}_g$ intersect at $d_f d_g$ points counted with multiplicity and degree.

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A divisor D on $\mathcal{X}$ over $\mathbb{F}$ is a formal sum

$$D = \sum n_P P, \tag{17}$$

where n_P is an integer and P is a point of arbitrary degree on the curve $\mathcal{X}$, with only a finite number of n_P being nonzero.

The divisor D is efective if $n_P \geq 0$ for all P. The support $\mathrm{supp}(D)$ of the divisor D is the set $\{P \mid n_P \neq 0\}$. The degree of the divisor is

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If $\mathcal{X}_1$ and $\mathcal{X}_2$ are two projective curves then their intersection divisor over $\mathbb{F}$ denoted by $\mathcal{X}_1 \cap \mathcal{X}_2 = \sum n_P P$ where the sumation runs over all the poins both in $\mathcal{X}_1$ and $\mathcal{X}_2$ and n_P is the multiplicity of the point in the intersection of the curves.

If $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined by homogeneous polynomials of degrees d_f, d_g respectively with no common nonconstant polynomial factors then

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- Intersection with $x = 0$: $P_\infty + P_1$.
- Intersection with $x^2 = 0$: $2P_\infty + 2P_1$.
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When finding the minimum distance of an AG code it will be connected to the genus of a curve. This is related to a topological concept of the same name but quite offtopic in this course. We will just show Plüker's formula that will serve in our case as a definiton for the genus.

Theorem (Plüker's formula)

The genus of a nonsingular projective plane curve determined by an homogeneous polynomial of degree $d \geq 1$ is

$$g = \frac{(d-1)(d-2)}{2}. \tag{20}$$

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Let $p(x, y, z)$ an homogeneous polynomial that defines a projective curve $\mathcal{X}$ over $\mathbb{F}$. We define the field of rational functions on $\mathcal{X}$ over $\mathbb{F}$ as

$$\mathbb{F}(\mathcal{X}) = \left(\left\{ \frac{g}{h} \middle| \begin{array}{l} g, h \text{ homogeneous,} \\ \text{same degree, } p \nmid h \end{array} \right\} \cup \{0\} \right) / \approx_{\mathcal{X}} . \quad (21)$$

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Exercise

Show that $\mathbb{F}(\mathcal{X})$ is a field containing $\mathbb{F}$ as a subfield. Notice that the class of 0 is precisely when g is a multiple of $p(x, y, z)$.

Let $f = \frac{g}{h} \in \mathbb{F}(\mathcal{X})$ such that $f \not\approx_{\mathcal{X}} 0$. Then the divisor of f is

$$\operatorname{div}(f) = (\mathcal{X} \cap \mathcal{X}_g) - (\mathcal{X} \cap \mathcal{X}_h) \tag{22}$$

By Bézout theorem $\deg(\operatorname{div}(f)) = d_p d_g - d_p d_h = 0$. Since f is an equivalence class remains to proof that $\operatorname{div}(f)$ is well defined. This is true but we will not prove it.

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By Bézout theorem $\deg(\operatorname{div}(f)) = d_p d_g - d_p d_h = 0$. Since f is an equivalence class remains to proof that $\operatorname{div}(f)$ is well defined. This is true but we will not prove it.

Exercise

Let $\mathcal{X}$ the elliptic curve

$$f(x,y,z) = x^3 + xz^2 + z^3 + y^2 z + yz^2 \in \mathbb{F}[x,y,z].$$

where $\mathrm{char}(\mathbb{F}) = 2$. Let $f = \frac{g}{h}$ and $f' = \frac{g'}{h'}$ where $g = x^2 + z^2$, $h = z^2$, $g' = z^2 + y^2 + yz$ and $h' = xz$. Let $P_\infty = (0:1:0)$ and $P_2 = \{(1:\omega:1), (1:\bar{\omega}:1)\}$.

- Show that $f \approx_{\mathcal{X}} f'$.
- Show that $\mathrm{div}(f) = 2P_2 - P_\infty$.
- Show that $\mathrm{div}(f') = 2P_2 - P_\infty$.

Given two divisors on a curve we will say

$$D = \sum n_P P \succeq D' = \sum n'_P P$$

provided that $n_P \geq n'_P$ for all the points. (I.e. D is effective if $D \succeq 0$).

Given a divisor D on a projective curve $\mathcal{X}$ over $\mathbb{F}$ let

$$L(D) = \{f \in \mathbb{F}(\mathcal{X}) \mid f \not\equiv_{\mathcal{X}} 0, \operatorname{div}(f) + D \succeq 0\} \cup \{0\}. \quad (23)$$

Exercise

Prove that $L(D)$ is a $\mathbb{F}$-vector space.

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Prove that $L(D)$ is a $\mathbb{F}$-vector space.

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Theorem

Let D be a divisor on a projective curve $\mathcal{X}$. The following statements hold:

- *If $\deg(D) < 0$, then $L(D) = \{0\}$.*
- *The constant functions are in $L(D)$ if and only if $D \succeq 0$.*
- *If P is a point in $\mathcal{X}$ with $P \notin \operatorname{supp}(D)$, then P is not a pole in any $f \in L(D)$.*

Proof.

- If $f \in L(D)$ with $f \not\approx_{\mathcal{X}} 0$ then $\operatorname{div}(f) + D \succeq 0$, i.e. $\deg(\operatorname{div}(f) + D) \geq 0$, but $\deg(\operatorname{div}(f) + D) = \deg(D)$, which is a contradiction.
- Let $f \not\approx_{\mathcal{X}} 0$ a constant function. If $f \in L(D)$ then $\operatorname{div}(f) + D \succeq 0$. But $\operatorname{div}(f) = 0$ (is constant), thus $D \succeq 0$. Conversely, if $D \succeq 0$ then $\operatorname{div}(f) + D = D \succeq 0$.
- If P is a pole in $f \in L(D)$ with $P \notin \operatorname{supp}(D)$ then the coefficient of P in $\operatorname{div}(f) + D$ of $\mathcal{X}$ is negative, contradicting $f \in L(D)$.

$\square$

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$\square$

Let $p(x, y, z)$ an homogeneous polynomial that defines a projective curve $\mathcal{X}$ over $\mathbb{F}_q$. Let D be a divisor on $\mathcal{X}$ and choose a set $\mathcal{P} = \{P_1, \ldots, P_n\}$ of n distinct $\mathbb{F}_q$-rational points on $\mathcal{X}$ such that $\mathrm{supp}(D) \cap \mathcal{P} = \emptyset$. If we order the points in $\mathcal{P}$ consider the evaluation map

$$\begin{array}{rccc} \mathrm{ev}_{\mathcal{P}}: & L(D) & \longrightarrow & \mathbb{F}_q^n \\ & f & \longmapsto & \mathrm{ev}_{\mathcal{P}}(f) = (f(P_1), \ldots, f(P_n)) \end{array} \tag{24}$$

Exercise

Is $\mathrm{ev}_{\mathcal{P}}$ well defined?

Let $p(x, y, z)$ an homogeneous polynomial that defines a projective curve $\mathcal{X}$ over $\mathbb{F}_q$. Let D be a divisor on $\mathcal{X}$ and choose a set $\mathcal{P} = \{P_1, \ldots, P_n\}$ of n distinct $\mathbb{F}_q$-rational points on $\mathcal{X}$ such that $\mathrm{supp}(D) \cap \mathcal{P} = \emptyset$. If we order the points in $\mathcal{P}$ consider the evaluation map

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Is $\mathrm{ev}_{\mathcal{P}}$ well defined?

If $f \in L(D)$ then P_i is not a pole of f, however if f is represented by $\frac{g}{h}$ then h may have P_i as a zero occurring in $\mathcal{X} \cap \mathcal{X}_h$ and it will occur at least so many times in $\mathcal{X} \cap \mathcal{X}_g$. If we choose $\frac{g}{h}$ to represent f then $f(P_i) = \frac{0}{0}$, we must avoid this situation. It can be shown that for any $f \in L(D)$ we can choose a representative $\frac{g}{h}$ with $h(P_i) \neq 0$.

Suppose now that f has two such representatives $\frac{g}{h} \approx_{\mathcal{X}} \frac{g'}{h'}$ where $h(P_i) \neq 0 \neq h'(P_i)$. Then $gh' - g'h$ is a polynomial multiple of p and $p(P_i) = 0$. Thus $g(P_i)h'(P_i) = g'(P_i)h(P_i)$, i.e. $\frac{g}{h}(P_i) = \frac{g'}{h'}(P_i)$.

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Exercise

Prove that $\mathrm{ev}_{\mathcal{P}}$ is a $\mathbb{F}_q$-linear mapping.

With the notation above we define the algebraic geometry code associated to $\mathcal{X}$, $\mathcal{P}$ and D to be

$$\mathcal{C}(\mathcal{X}, \mathcal{P}, D) = \{\mathrm{ev}_{\mathcal{P}}(f) \mid f \in L(D)\}. \tag{25}$$

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In order to get some information on the dimension and minimum distance we will use the following version of the Riemann-Roch's Theorem.

Theorem (Riemann-Roch)

Let D a divisor in a nonsingular projective plane curve $\mathcal{X}$ over $\mathbb{F}_q$ of genus g. Then

- $\dim(L(D)) \geq \deg(D) + 1 - g$.
- *Furthermore, if* $\deg(D) > 2g - 2$ *then*

$$\dim(L(D)) = \deg(D) + 1 - g.$$

Theorem

Let D a divisor in a nonsingular projective plane curve $\mathcal{X}$ over $\mathbb{F}_q$ of genus g. Let $\mathcal{P}$ a set of n distinct $\mathbb{F}_q$-rational points on $\mathcal{X}$ such that $\operatorname{supp}(D) \cap \mathcal{P} = \emptyset$. Assume that

$$2g - 2 < \deg(D) < n.$$

Then $\mathcal{C}(\mathcal{X}, \mathcal{P}, D)$ is an $[n, k, d]$ code over $\mathbb{F}_q$ where

$$k = \deg(D) + 1 - g.$$

Proof.

In order to check $k = \deg(D) + 1 - g$ by Riemann-Roch theorem we just need to show that $\mathrm{ev}_{\mathcal{P}}$ has trivial kernel. Suppose that $\mathrm{ev}_{\mathcal{P}}(f) = \mathbf{0}$, then $f(P_i) = 0$ for all i, i.e. is a zero of f, since $P_i \notin \mathrm{supp}(D)$ we have $\mathrm{div}(f) + D - (\sum_{i=1}^{n} P_i) \succeq 0$. Therefore $f \in L(D - (\sum_{i=1}^{n} P_i))$, but $\deg(D) < n$, thus $\deg(D - (\sum_{i=1}^{n} P_i)) < 0$ and we have $L(D - (\sum_{i=1}^{n} P_i)) = \{0\}$ and $f = 0$.

Suppose that $\mathrm{ev}_{\mathcal{P}}(f)$ has minimum weight d. Thus $f(P_i) = 0$ for $n - d$ indices $\{i_j \mid 1 \leq j \leq n - d\}$. Thus $f \in L(D - (\sum_{j=1}^{n-d} P_{i_j}))$ and therefore $\deg(D - (\sum_{j=1}^{n-d} P_{i_j})) \geq 0$. Hence $\deg(D) - (n - d) \geq 0$.

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As a corollary of previous theorem we have that if $\{f_1, \ldots, f_k\}$ is a basis of $L(D)$ then a generator matrix of the code $\mathcal{C}(\mathcal{X}, \mathcal{P}, D)$ is

$$\begin{pmatrix} f_1(P_1) & f_1(P_2) & \ldots & f_1(P_n) \\ f_2(P_1) & f_2(P_2) & \ldots & f_2(P_n) \\ & & \vdots & \\ f_k(P_1) & f_k(P_2) & \ldots & f_k(P_n) \end{pmatrix}. \tag{26}$$

Consider the projective curve $\mathcal{X}$ over $\mathbb{F}_q$ given by $z = 0$. The points in the curve are $(x : y : 0)$. Let $P_\infty = (1 : 0 : 0)$, $P_0 = (0 : 1 : 0)$ and $P_1, \dots P_{q-1}$ the remaining rational points. For narrow sense RS codes we will let $n = q - 1$ and $\mathcal{P} = \{P_1, \dots P_{q-1}\}$ and for the extended narrow-sense RS codes $n = q$ and $\mathcal{P} = \{P_0, \dots P_{q-1}\}$.

Fix k $(1 \leq k \leq n)$ and let $D = (k-1)P_\infty$ $(D = 0$ when $k = 1)$. We have that $\mathrm{supp}(D) \cap \mathcal{P} = \emptyset$ and $\mathcal{X}$ is non singular of genus $g = 0$. Also $k - 1 = \deg(D) > 2g - 2$ thus $\dim(L(D)) = \deg(D) + 1 - g = k$.

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$$\mathfrak{B} = \left\{1, \frac{x}{y}, \frac{x^2}{y^2}, \ldots \frac{x^{k-1}}{y^{k-1}}\right\}$$

is a basis of $L(D)$.
First $\mathrm{div}(x^j/y^j) = jP_0 - jP_\infty$, thus $\mathrm{div}(x^j/y^j) + D = jP_0 - (k-1-j)P_\infty$ which is effective since $0 \leq j \leq k-1$.

Consider a linear combination of elements of $\mathfrak{B}$

$$f = \sum_{j=0}^{k-1} a_j \frac{x^j}{y^j} \approx_{\mathcal{X}} 0.$$

$f = g/h$ and by definition of $\approx_{\mathcal{X}}$ g must be a multiple of z, clearly this multiple should be 0 since z does not appear on f, therefore $a_i = 0$ for all i.

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Geometric Reed Solomon codes

Reed-Solomon codes are AG codes

Using $\mathfrak{B}$, any nonzero element $f \in L(D)$ can be written as

$$f(x, y, z) = \frac{g(x, y, z)}{y^d}, \quad g(x, y, z) = \sum_{j=0}^{d} g_j x^j y^{d-j}$$

with $g_d \neq 0$ and $d \leq k - 1$.

Notice that $g(x, y, z)$ is the homogenization in $\mathbb{F}_q[x, y]$ of $m(x) = \sum_{j=0}^{d} g_j x^j$ thus there is a 1-1 relation between the elements of $L(D)$ and those of $\mathcal{P}_k \subseteq \mathbb{F}_q[x]$.

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Moreover, if $\beta \in \mathbb{F}_q$ then $m(\beta) = f(\beta, 1, 0)$ and additionally $f(\beta, 1, 0) = f(x_0, y_0, z_0)$ where $(\beta : 1 : 0) = (x_0 : y_0 : z_0)$.

Let α a primitive element of $\mathbb{F}_q$ and order the points $P_i = (\alpha^i : 1 : 0)$ for $1 \leq i \leq n$. The discussion shows that the following sets are the same

$$\{(m(1), m(\alpha), \ldots, m(\alpha^{n-1})) \mid m(x) \in \mathcal{P}_k\}$$

$$\{(f(P_1), f(P_2), \ldots, f(P_n)) \mid f \in L(D)\}$$

and by R-R theorem $\deg(D) + 1 - g = k - 1 + 1 + 0 = k$, $d \geq n - \deg(G) = n - k + 1$, hence by Singleton Bound $d = n - k + 1$ and they are MDS. $\square$

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As an exercise show that Generalized Reed-Solomon codes are AG codes using the discussion above and using the following steps:

- Let $\gamma = (\gamma_0, \ldots, \gamma_{n-1})$ a n-tuple of distinct elements of $\mathbb{F}_q$, and $\mathbf{v} = (v_0, \ldots, v_{n-1}) \in \mathbb{F}_q^n$. Compute the polynomial given by the Lagrange Interpolation Formula

$$p(x) = \sum_{i=0}^{n-1} v_i \prod_{j \neq i} \frac{x - \gamma_j}{\gamma_i - \gamma_j}.$$

- Let $\mathcal{X}$ be the curve defined by $z = 0$ and $h(x, y)$ the homogenization of polynomial $p(x)$ of degree $d \leq n - 1$. We will assume that the v_i's are noncero, thus $h \neq 0$.

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- Let $u(x,y,z) = \frac{h(x,y)}{y^d} \in \mathbb{F}_q(\mathcal{X})$ and $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ such that $P_i = (\gamma_{i-1} : 1 : 0)$. $P_\infty = (1 : 0 : 0)$ and $D = (k-1)P_\infty - \mathrm{div}(u)$.
- Prove that $u(P_i) = v_{i-1}$.
- Prove that $\mathrm{supp}(D) \cap \mathcal{P} = \emptyset$.
- Since the divisor of any element in $\mathbb{F}_q(\mathcal{X})$ is cero then $\deg(D) = k-1$.
- Prove that a basis of $L(D)$ is

$$\mathfrak{B} = \left\{ u, u\frac{x}{y}, u\frac{x^2}{y^2}, \ldots u\frac{x^{k-1}}{y^{k-1}} \right\}$$

- Prove that $\mathcal{GRS}_k(\gamma, \mathbf{v}) = \mathcal{C}(\mathcal{X}, \mathcal{P}, D)$

Geometric Reed Solomon codes

A worked example

Let $\mathcal{X} = \mathcal{F}_3(\mathbb{F}_4)$ the Fermat curve over $\mathbb{F}_4$ given by the eqn.

$$x^3 + y^3 + z^3 = 0$$

It has nine projective points given by

Q	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
0	0	0	1	α	$\bar{\alpha}$	1	α	$\bar{\alpha}$
1	α	$\bar{\alpha}$	0	0	0	1	1	1
1	1	1	1	1	1	0	0	0

where $\bar{\alpha} = \alpha^2 = 1 + \alpha$.

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where $\bar{\alpha} = \alpha^2 = 1 + \alpha$.

By R-R's theorem $\dim(L(3Q)) = 3$. The functions

$$1, \frac{x}{x+y}, \frac{y}{y+z}$$

are regular outside Q and have a pole of order 2 and 3 respectively. They are a basis of $L(D)$.

A generator matrix of $\mathcal{C}(\mathcal{X}, \mathcal{P}, D)$ is

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & \alpha & \bar{\alpha} & 1 & \alpha & \bar{\alpha} \\ \bar{\alpha} & \alpha & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

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Up to here the third session (30/11)

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Let $\mathcal{V}$ be a vector space over $\mathbb{F}(\mathcal{X})$. An $\mathbb{F}$-linear map $D : \mathbb{F}(\mathcal{X}) \to \mathcal{V}$ is called a derivation if it satifies the product rule

$$D(fg) = fD(g) + gD(f).$$

Example

Let $\mathcal{X}$ be the projective line with funtion field $\mathbb{F}(x)$. Define $D(F) = \sum ia_i x^{i-1}$ for a polynomial $F = \sum a_i x^i \in \mathbb{F}[x]$ and extend this to quotients by

$$D\left(\frac{F}{G}\right) = \frac{GD(F) - FD(G)}{G^2}$$

Then $D : \mathbb{F}(x) \to \mathbb{F}(x)$ is a derivation.

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The set of all derivations $D : \mathbb{F}(\mathcal{X}) \to \mathcal{V}$ will be denoted by $\mathrm{Der}(\mathcal{X}, \mathcal{V})$ (or $\mathrm{Der}(\mathcal{X})$ if $\mathcal{V} = \mathbb{F}(\mathcal{X})$). Notice that $\mathrm{Der}(\mathcal{X}, \mathcal{V})$ is a $\mathbb{F}(\mathcal{X})$-vector space.

Let $\mathcal{X}$ be a projective variety and P be a point on $\mathcal{X}$. Then a rational function f is called regular in the point P if one can find homogeneous polynomials F and G same degree, such that $G(P) \neq 0$ and f is the coset of F/G.

The set of all the regular rational functions at P will be denoted by $\mathcal{O}_P(\mathcal{X})$, the local ring at P and indeed it is a local ring, i.e. it has a unique maximal ideal, given by

$$\mathcal{M}_P = \{f \in \mathcal{O}_P(\mathcal{X}) \mid f(P) = 0\} \tag{27}$$

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Geometric Goppa Codes

Local ring at a point

Example

In $\mathbb{P}^2(\mathbb{F})$ consider the parabola $\mathcal{X}$ defined by $XZ - Y^2 = 0$. now with It has one point at infinity $P_\infty = (1:0:0)$. The function x/y is equal to y/z on the curve, hence it is regular in the point $P = (0:0:1)$.

$\frac{(2xz+z2)}{(y2+z2)}$ is regular in P and this function is equal to $\frac{(2x+z)}{(x+z)}$ and therefore also regular in P_∞.

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Let see that $\mathcal{M}_P$ is generated by a single element (i.e. is a principal ideal). Let $\mathcal{X}$ be a smooth curve in $\mathbb{A}^2(\mathbb{F})$ defined by the equation $f = 0$, and let $P = (a, b)$ be a point on it.

$$\mathcal{M}_P = \langle x - a, y - b \rangle \text{ and}$$
$$f_x(P)(x - a) + f_y(P)(y - b) \equiv 0 \mod \mathcal{M}_P^2$$

The $\mathbb{F}$-vector space $\mathcal{M}_P/\mathcal{M}_P^2$ has dimension 1 and therefore $\mathcal{M}_P$ has one generator. Let $g \in \mathbb{F}[x]$ be the coset of a polynomial G. Then g is a generator of $\mathcal{M}_P$ if and only if $d_P G$ is not a constant multiple of $d_P f$, where

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Let $< t >= \mathcal{M}_P$, and $z \in \mathcal{O}_P(\mathcal{X})$, then it can be written in a unique way as

$$z = ut^m,$$

where u is a unit and $m \in \mathbb{N}_0$. The function t is called a local parameter or uniformizing parameter in P.

If $m > 0$, then P is a zero of multiplicity m of z. We write $m = \mathrm{ord}_P(z) = \upsilon_P(z)$. ($\upsilon_P(0) = \infty$).

Theorem

Let $< t >= \mathcal{M}_P$ a local paramenter for t, then there exist a unique derivation

$$D_t : \mathbb{F}(\mathcal{X}) \to \mathbb{F}(\mathcal{X}) \text{ s.t. } D_t(t) = 1, \tag{28}$$

Moreover, $\mathrm{Der}(\mathcal{X})$ is one dimensional over $\mathbb{F}(\mathcal{X})$ and D_t is a basis element.

A rational differential form or differential on $\mathcal{X}$ is an $\mathbb{F}(\mathcal{X})$ linear map from $\mathrm{Der}(\mathcal{X})$ to $\mathbb{F}(\mathcal{X})$. The set of all rational differential forms on $\mathcal{X}$ is denoted by $\Omega(\mathcal{X})$.

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Consider the map $d : \mathbb{F}(\mathcal{X}) \to \Omega(\mathcal{X})$ given by for each $f \in \mathbb{F}(\mathcal{X})$ the image $df : \mathrm{Der}(\mathcal{X}) \to \mathbb{F}(\mathcal{X})$ is defined by $df(D) = D(f)$ for all $D \in \mathrm{Der}(\mathcal{X})$. Then d is a derivation. and provides to $\Omega(\mathcal{X})$ a vector space structure over $\mathbb{F}(\mathcal{X})$.

Theorem

The space $\Omega(\mathcal{X})$ has dimension 1 over $\mathbb{F}(\mathcal{X})$ and d_t is a basis for every point $P \in \mathcal{X}$ with local parameter t.

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The space $\Omega(\mathcal{X})$ has dimension 1 over $\mathbb{F}(\mathcal{X})$ and d_t is a basis for every point $P \in \mathcal{X}$ with local parameter t.

That is, for each differential we have a unique representation $\omega = f_P dt_P$, where f_P is a rational function at point P. We can not evaluate P at ω as by $\omega(P) = f_P(P)$ since it depends on the choice of t_P.

Let $\omega \in \Omega(\mathcal{X})$. The order or valuation of ω in P is defined by $ord_P(\omega) = \upsilon_P(\omega) := \upsilon_P(f_P)$. It is called regular if it has no poles. This definition does not depend on the choices made.

The canonical divisor (ω) of the differential ω is defined by

$$W = (\omega) = \sum_{P \in \mathcal{X}} \upsilon_P(\omega) P. \tag{29}$$

If D is a divisor, $\Omega(D) = \{\omega \in \Omega(\mathcal{X}) \mid (\omega) - D \succeq 0\}$.

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Let $P \in \mathcal{X}$ and t is a local parameter, let $\omega = fd_t$ a differential form , $f = \sum a_i t^i$, then the residue at point P is defined as

$$\mathrm{Res}_P(\omega) = a_{-1}.$$

As usual, Let D be a divisor on $\mathcal{X}$ and choose a set $\mathcal{P} = \{P_1, \ldots, P_n\}$ of n distinct $\mathbb{F}_q$-rational points on $\mathcal{X}$ such that $\mathrm{supp}(D) \cap \mathcal{P} = \emptyset$.

The linear code $\mathcal{C}^\star(\mathcal{P}, D)$ of length n over $\mathbb{F}_q$ is the image of the linear map $\alpha^\star : \omega(\sum P_i - D) \to \mathbb{F}_q^n$ defined by

$$\alpha^\star(\eta) = (\mathrm{Res}_{P_1}(\eta), \mathrm{Res}_{P_2}(\eta), \ldots, \mathrm{Res}_{P_n}(\eta)). \quad (30)$$

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Theorem

The code $\mathcal{C}^\star(\mathcal{P}, D)$ has dimension $k^\star = n - \deg(D) + g - 1$ and minimum distance $d^\star \geq \deg(D) - 2g + 2$.

The proof follows from Riemann-Roch's theorem and the isomorphims between $L(W - D)$ and $\Omega(D)$.

Theorem

The codes $\mathcal{C}^\star(\mathcal{P}, D)$ and $\mathcal{C}(\mathcal{P}, D)$ are dual codes.

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Theorem

Let $\mathcal{X}$ be a curve defined over $\mathbb{F}_q$. Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ rational points on $\mathcal{X}$. Then there exists a differential form ω with simple poles at the P_i such that $\mathrm{Res}_{P_i}(\omega) = 1$ for all i. Furthermore

$$C^\star(\mathcal{P}, D) = C(\mathcal{P}, W + \sum P_i - D)$$

for all divisors D that have a support disjoint from $\mathcal{P}$, where W is the divisor of ω.

Further topics and reading