An overview on algebraic invariants and the main parameters of some parameterized codes

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Abstract

The main goal of this work is to describe the parameters of some evaluation codes known as parameterized codes by using the relationships with the algebraic invariants of a quotient ring. We use some tools of algebraic geometry and commutative algebra to do this description. Some results in the case of codes parameterized by the edges of a graph or a clutter are given.

Let $K = \mathbb{F}_q$ be a finite field with q elements. Let $L = K[Z_1, \ldots, Z_n]$ be a polynomial ring over the field K and let $Z^{a_1}, \ldots, Z^{a_m}$ be a finite set of monomials. As usual if $a_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{N}^n$, where $\mathbb{N}$ stands for the non–negative integers, then we set $Z^{a_i} = Z_1^{a_{i1}} \cdots Z_n^{a_{in}}$ for all $i = 1, \ldots, m$. In this situation we say that the following set X, which is a multiplicative group under componentwise multiplication, is the toric set parameterized by these monomials.

$$X = \{[(t_1^{a_{11}} \cdots t_n^{a_{1n}}, t_1^{a_{21}} \cdots t_n^{a_{2n}}, \ldots, t_1^{a_{m1}} \cdots t_n^{a_{mn}})] \in \mathbb{P}^{m-1} : t_i \in K^*\}, \tag{1}$$

where $K^* = K \setminus \{0\}$ and $\mathbb{P}^{m-1}$ is a projective space over the field K.

Let $S = K[X_1, \ldots, X_m] = \bigoplus_{d=0}^{\infty} S_d$ be a polynomial ring over the field K with the standard grading and let $X = \{[P_1], \ldots, [P_{|X|}]\}$. The evaluation map

$$\mathrm{ev}_d : S_d \to K^{|X|},$$
$$f \to \left(\frac{f(P_1)}{X_1^d(P_1)}, \ldots, \frac{f(P_{|X|})}{X_1^d(P_{|X|})} \right)$$

defines a linear map of K–linear spaces. The image of this map is denoted by $C_X(d)$ and it will be called a parameterized code of order d associated to the toric set (1). The vanishing ideal of X, denoted by I_X, is the ideal of S generated by the homogeneous polynomials of S that vanish on X. As we comment above the main objective is the description of the parameters of the code $C_X(d)$ by using the algebraic invariants of the ring S/I_X. The length of the code $C_X(d)$ is given by the degree of S/I_X and the dimension of the code $C_X(d)$ is given by the Hilbert funcion of $S_d/I_X(d)$, where $I_X(d)$ is the degree d piece of I_X.

Also the knowledge of the regularity index of S/I_X, r_X, Castelnuovo–Mumford regularity, allows to study the cases $0 \leq d < r_X$, which are the only interesting cases, because $C_X(d) = K^{|X|}$ if $d \geq r_X$.

Given any graph $\mathcal{G}$, or a clutter, we can associate a toric set to its edges and then we can define a parameterized code arising from these edges. Several particular, but important, cases are described: bipartite complete graphs, cycles, r–partite complete graphs, complete graphs and clutters. For many computations we use the software *Macaulay2*.

Keywords

Parameterized code, Finite field, Projective space, Clutter