On Matroid Chains [1]

Thomas Britz[2]

1. Introduction

What might be coined *combinatorial coding theory* aims to describe properties of linear codes over fields in combinatorial terms and, conversely, to extract and refine results from coding theory to set-combinatorial settings. The roots of this theory reach back to the seminal 1935 paper by Whitney [12] who explictly defined matroids. However, the theory did not blossom until Crapo and Rota [5] presented their Critical Theorem which describes how vector supports of a linear code over a field are determined by the associated vector matroid. The theory gained further attention when Greene [7] in 1976 showed how the weight enumerator of a linear code is determined by the Tutte polynomial of the associated vector matroid. Since then, these results have been generalised in many ways, and the main objectives of the theory have largely been met, namely to characterise the properties of linear codes that are determinable by the associated vector matroid - and those that are not; see [4, 1, 2, 6, 8], for instance. It is therefore time to consider new directions for research in combinatorial coding theory. Two new and promising directions are to investigate the combinatorial properties of linear codes over more general rings than fields (see [11] for instance), and to consider more general combinatorial objects than matroids, such as demi-matroids [3].

2. Matroid chains

The present research follows a third and potentially interesting direction, namely to consider chains of linear vectors, over a common field $\mathbb{F}$ and with a common ground set E,

$$C_1 \subseteq \cdots \subseteq C_k \subseteq \mathbb{F}^E$$

and the associated sequence of vector matroids $M_{C_1}, \ldots, M_{C_k}$. (For more information on matroids, see [9].) In the present paper, we will simply consider such two-matroid chains; many of our results however extend naturally to the general case:

Definition 1. *A pair of matroids* (M_1, M_2) *is a* (matroid) chain *over* $\mathbb{F}$ *if they share a common ground set* E *and* $M_i = M(C_i)$ *for some chain of linear codes* $C_1 \subseteq C_2$ *over* $\mathbb{F}$.

The purpose of this paper is to determine basic properties of the matroid chain and to partially characterise when a sequence of matroids (M_1, M_2) may arise from a given type of linear code chain.

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[2] University of New South Wales, Kensington, NSW 2052, Australia. E-mail: `britz@unsw.edu.au`

3. General properties of matroid chains

We begin by presenting general properties of matroid chains.

Proposition 2. *The relation $\preceq$ defined by $M_1 \preceq M_2$ exactly when (M_1, M_2) is a chain over $\mathbb{F}$ is a partial order on the family of uniquely $\mathbb{F}$-representable matroids.*

Proposition 3. *If (M_1, M_2) is a chain over $\mathbb{F}$, then*

(1) *(M_2^*, M_1^*) is a chain over $\mathbb{F}$;*
(2) *if $r(M_1) = r(M_2)$, then $M_1 = M_2$;*
(3) *if (M_2, M_1) is also a chain, then $M_1 = M_2$;*
(4) *$(M_1\backslash A, M_2\backslash A)$ is a chain over $\mathbb{F}$ for each $A \subseteq E$;*
(5) *$(M_1/A, M_2/A)$ is a chain over $\mathbb{F}$ for each $A \subseteq E$;*
(6) *each cocircuit of M_1 is a union of cocircuits of M_2; and each circuit of M_2 is a union of circuits of M_1.*

Lemma 4. *If (M_1, M_2) and (M_1', M_2') are chains over $\mathbb{F}$ on disjoint sets E and E', respectively, then $(M_1 \oplus M_1', M_2 \oplus M_2')$ is a chain over $\mathbb{F}$ on $E \cup E'$.*

If (M_1, M_2) are a pair of matroids on a common set E, then each pair of minors of the form $(M_1\backslash X/Y, M_2\backslash X/Y)$ for $X, Y \subseteq E$ are a *minor pair* of (M_1, M_2). By Lemma 3, each minor pair of a chain over $\mathbb{F}$ is also a chain over $\mathbb{F}$. A *quotient* is a map of the form $\mathcal{N} - A \mapsto \mathcal{N}\backslash A$ for some matroid $\mathcal{N}$ on some set E' and some subset $A \subseteq E'$.

Lemma 5. *If $M_2 \mapsto M_1$ is a quotient, then $M_2' \mapsto M_1'$ is a quotient for each minor pair (M_1', M_2') of (M_1, M_2).*

Quotients provide a necessary condition for matroid pairs to form chains:

Proposition 6. *If (M_1, M_2) is a chain over some field, then $M_2 \mapsto M_1$ is a quotient.*

The following result provides a general (but weak) necessary and sufficient condition for a matroid pair to be a chain over a given field.

Proposition 7. *The pair (M_1, M_2) is a chain over $\mathbb{F}$ if and only if there is a set E' disjoint from E and an $\mathbb{F}$-representable matroid N on $E \cup E'$ such that $M_1 = N\backslash E'$ and $M_2 = N/E'$.*

4. Some classes of matroid chains

In this section, we will consider the matroids over a ground set E belonging to some common class of matroids, namely the uniform, the binary, the graphic, and the transversal matroids, respectively. We show that any two uniform matroids over the same set will always form a chain over certain fields $\mathbb{F}$. Furthermore, various necessary and sufficient conditions are given for two binary matroids to form a chain over $\mathbb{Z}_2$, thus providing chain analogues of classical descriptions of uniform and binary matroids. For graphic and transversal matroids, we present natural sufficient conditions.

Uniform matroid chains.

This results in this subsection characterises the pairs of uniform matroids $(U_{k',n}, U_{k,n})$ that are chains over given fields $\mathbb{F}$.

Theorem 8. *Let $n \geq k \geq k' \geq 0$ be given. If a field $\mathbb{F}$ contains at least $k + n$ elements, then $(U_{k',n}, U_{k,n})$ is a chain over $\mathbb{F}$.*

Proposition 9. *The non-trivial uniform two-matroid chains over $\mathbb{Z}_2$ are all the pairs $(U_{1,n}, U_{n-1,n})$, where $n > 2$ is even.*

Proposition 10. *For all $n \geq 2$, the pair $(U_{1,n}, U_{n-1,n})$ is a chain over all fields $\mathbb{F} \neq \mathbb{Z}_2$. The dual pairs $(U_{1,n}, U_{2,n})$ and $(U_{n-2,n}, U_{n-1,n})$ form chains over any field $\mathbb{F}$ with $|\mathbb{F}| > n$. The non-trivial uniform chains over $\mathbb{Z}_3$ all assume one of the forms above. Apart from these and from trivial chains, the uniform chains over $\mathbb{F}_4$ are of the following forms:*

$$(U_{1,5}, U_{3,5})\,,\ (U_{2,5}, U_{4,5})\,,\ (U_{1,6}, U_{3,6})\,,\ (U_{3,6}, U_{5,6})\,,\ (U_{1,6}, U_{4,6})\,.$$

Binary matroid chains.

The following result characterises the binary matroid pairs (M_1, M_2) that are chains over $\mathbb{Z}_2$.

Theorem 11. *If M_1 and M_2 are binary matroids on a common set E, then (M_1, M_2) is a chain over $\mathbb{Z}_2$ if and only if each cocircuit of M_1 is a disjoint union of cocircuits of M_2.*

Indeed, if (M_1, M_2) is a chain over $\mathbb{Z}_2$, then cocircuit of M_1 is the symmetric difference of cocircuits of M_2; this extends Whitney's characterisation of binary matroids [12]. The next result is the two-matroid chain analogue of the forbidden $U_{2,4}$-minor characterisation of binary matroids due to Tutte [10].

Theorem 12. *If M_1 and M_2 are binary matroids on a common set E, then (M_1, M_2) is a chain over $\mathbb{Z}_2$ if and only if $M_2 \mapsto M_1$ is a quotient and $(U_{1,3}, U_{2,3})$ is not a minor pair of (M_1, M_2).*

Graphic matroid chains.

The following proposition provides a sufficient condition for a pair of graphic matroids to form a chain over any field.

Proposition 13. *Let G_2 be a graph with edge set E, and suppose that G_1 may be obtained from G_2 by identifying vertices. Then $(M(G_1), M(G_2))$ is a chain over all fields.*

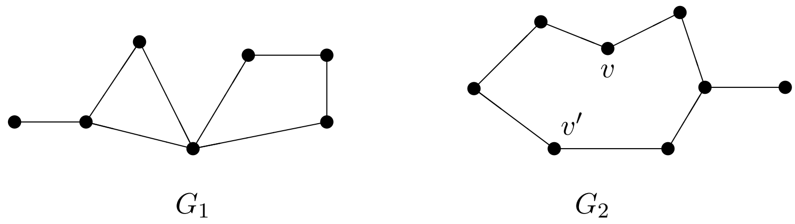

FIGURE 1. Graphic chains

Example 14. *Consider the graphs G_1 and G_2 in Figure 1. The graph G_1 is 2-isomorphic to the graph obtained from G_2 by identifying the vertices v and v'. By Proposition 13, the matroids $M(G_1)$ and $M(G_2)$ then form a chain over any field. Indeed,*

$$M(G_1) = M\begin{bmatrix} 1&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&1&0 \\ 0&0&1&0&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix} \qquad \textit{and} \qquad M(G_2) = M\begin{bmatrix} 1&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&1&0 \\ 0&0&1&0&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&1&0 \end{bmatrix}.$$

Problem 15. *Is the converse of Proposition 13 true? That is, is G_1 2-isomorphic to a graph that is obtained from G_2 by identifying vertices, whenever $(M(G_1), M(G_2))$ is a chain over all fields? Is this indeed true whenever $(M(G_1), M(G_2))$ is a binary chain?*

Example 16. *Consider the graphs G_1 and G_2 in Figure 2. The matroids $M(G_1)$ and $M(G_2)$ are represented over $\mathbb{Z}_2$ by matrices row equivalent to the matrices $\begin{bmatrix} 1&1&1 \end{bmatrix}$ and $\begin{bmatrix} 1&0&1 \\ 0&1&1 \end{bmatrix}$, respectively, and do therefore not form a chain over $\mathbb{Z}_2$. However, $(M(G_1), M(G_2))$ is a chain over all other fields.*


G_1 G_2

FIGURE 2. A graphic chain over all fields except $\mathbb{Z}_2$

Problem 17. *Are all binary graphic chains also chains over all other fields? In general, is it true that binary chains of matroids representable over all fields are chains over all fields? Are binary chains of matroids representable over some other field also chains over that other field? Certainly, the converse is not true, as Example 18 demonstrates.*

Example 18. *Suppose that a chain (M_1, M_2) is represented over $\mathbb{Z}_3$ as follows:*

$$M_1 = M\left[\begin{smallmatrix} 1 & 1 & 0 & 1 \end{smallmatrix}\right] \qquad M_2 = M\left[\begin{smallmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right]$$

Over $\mathbb{Z}_2$, the matroids are represented as follows.

$$M_1 = M\left[\begin{smallmatrix} 1 & 1 & 0 & 1 \end{smallmatrix}\right] \qquad M_2 = M\left[\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right]$$

The pair (M_1, M_2) does therefore not form a pair over $\mathbb{Z}_2$.

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