Hamming and Simplex Codes for the Sum-Rank Metric

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Abstract. Sum-rank Hamming codes, with minimum sum-rank distance 3, are introduced, together with their duals, called sum-rank simplex codes. It is shown that sum-rank isometric classes of sum-rank Hamming codes are in bijective correspondence with maximum-size partial spreads. It is also shown that sum-rank Hamming codes are perfect codes for the sum-rank metric. This is in contrast with the rank-metric case, where no non-trivial perfect codes exist. Finally, bounds on the minimum sum-rank distance of sum-rank simplex codes are given based on known upper bounds on the size of partial spreads.

Let $\mathbb{F}_q \subseteq \mathbb{F}_{q^m}$ be an extension of finite fields. For numbers ℓ, N and $n = \ell N$, we may define the sum-rank weight of $\mathbf{c} = (\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(\ell)}) \in \mathbb{F}_{q^m}^n$ by $\mathrm{wt}_{SR}(\mathbf{c}) = \sum_{i=1}^{\ell} \mathrm{wt}_R(\mathbf{c}^{(i)})$, where wt_R denotes rank weight. The sum-rank metric is then defined by $\mathrm{d}_{SR}(\mathbf{c}, \mathbf{d}) = \mathrm{wt}_{SR}(\mathbf{c} - \mathbf{d})$. This metric measures the error-correction capability of codes in multishot network coding, and gives an estimate on the global erasure correction capability of locally repairable codes. Furthermore, it recovers the Hamming metric when $N = 1$ and it recovers the rank metric when $\ell = 1$.

For $m = 1$ and fixed N, we define *sum-rank Hamming codes* as linear codes $\mathcal{C} \subseteq \mathbb{F}_q^n$ with minimum sum-rank distance $\mathrm{d}_{SR}(\mathcal{C}) = 3$ and maximum possible length $n = \ell N$. Such codes are given by parity check matrices of the form

$$H = (H_1, H_2, \ldots, H_\ell) \in \mathbb{F}_q^{r \times n},$$

where $\mathcal{H}_i \cap \mathcal{H}_j = \{\mathbf{0}\}$ if $i \neq j$, being $\mathcal{H}_i \subseteq \mathbb{F}_q^r$ the column space of $H_i \in \mathbb{F}_q^{r \times N}$. Thus $\{\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_\ell\}$ forms a maximum-size partial N-spread in $\mathbb{F}_q^r$. Note that classical Hamming codes are recovered by choosing $N = 1$, in which case $\{\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_\ell\}$ forms the $(r-1)$-dimensional projective space.

If N divides r, it was shown by Beutelspacher that a maximum-size partial spread has size

$$\ell = \frac{q^r - 1}{q^N - 1}.$$

In such a case, sum-rank Hamming codes have length $n = N\frac{q^r-1}{q^N-1}$, dimension $k = N\frac{q^r-1}{q^N-1} - r$ and minimum sum-rank distance 3. By computing the size of a ball of sum-rank radius 1, we may check that these sum-rank Hamming codes are *perfect codes* for the sum-rank metric.

Finally, define s*um-rank simplex codes* as duals of sum-rank Hamming codes. It can be shown that the non-zero components of their codewords correspond to certain (possibly not maximum-size) partial spreads. By applying known bounds on the maximum size of a partial spread, the minimum sum-rank distance of sum-rank simplex codes can be lower bounded.

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