Exploring the order domain conditions

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Abstract. Ruud Pellikaan in *On the Existence of Order Functions* (and independently Shinji Miura) showed how to represent any union $\cup_{m=0}^{\infty}\mathcal{L}(mQ)$ as a quotient ring $\mathbb{F}[X_1,\ldots,X_m]/I$, where m is the number of generators of the corresponding Weierstrass semigroup, and where I satisfies some simple conditions. Here, Q is a rational place. Using this description we conduct computer searches for curves with given Weierstrass semigroups and with many rational places.

Consider an algebraic function field over K of transcendence degree 1. Let Q be a rational place and denote by m the minimal number of generators of the corresponding Weierstrass semigroup $\Gamma = \langle w_1,\ldots,w_m\rangle$. Then there exists an ideal $I \subseteq K[X_1,\ldots,X_m]$ such that $\cup_{m=0}^{\infty}\mathcal{L}(mQ) \simeq K[X_1,\ldots,X_m]/I$ and such that I satisfies

1. There exists a Gröbner basis $\{G_1,\ldots,G_s\}$ of I such that each polynomial has exactly two monomials of highest weight in its support. Here, the weight is defined by $w(X_1^{i_1}\cdots X_m^{i_m}) = w_1 i_1 + \cdots + w_m i_m$.
2. The set $\{M \mid M \text{ is a monomial}, M \notin \mathrm{lm}(I)\}$ does not possess two monomials of the same weight.

In the other direction any such description defines a union of $\mathcal{L}$-spaces as above or a subalgebra thereof. If (and only if) the curve is non-singular (which is checked by considering the derivatives of the generators) then equality holds. In such a description there is a one-to-one correspondence between the rational places different from Q and the affine roots of I.

Using a computer program we investigate for different semigroups with not too many gaps (i.e. small genus) what is the maximal number of rational places.

It is interesting to note that the Goppa bound works for the corresponding evaluation codes also if the generating set $\{G_1,\ldots,G_s\}$ is not a Gröbner basis. This suggests to speed up the program by avoiding the test, rephrasing of course the problem slightly.

The search can be enhanced to investigate similar questions for order domains of higher transcendence degree over a finite field. Here, we do not have the Hasse-Weil bound for comparison, but can instead employ the footprint bound from Gröbner basis theory.

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