On Numerical Semigroups and Their Applications to Algebraic Geometry Codes

Maria Bras-Amorós

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A. Acute
B. Symmetric
C. Arf
D. Classification

$\nu$, $\tau$ and Improved Codes
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Further on Semigroups
A. Characterization
B. Counting

# On Numerical Semigroups and Their Applications to Algebraic Geometry Codes

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# Numerical Semigroups

## Definition

*A numerical semigroup is a subset $\Lambda$ of $\mathbb{N}_0$ satisfyig*

- $0 \in \Lambda$
- $\Lambda + \Lambda \subseteq \Lambda$
- $|\mathbb{N}_0 \setminus \Lambda|$ *is finite (genus:=$g$:= $|\mathbb{N}_0 \setminus \Lambda|$)*

# Numerical Semigroups

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- $\Lambda + \Lambda \subseteq \Lambda$
- $|\mathbb{N}_0 \setminus \Lambda|$ *is finite (genus:=g:= $|\mathbb{N}_0 \setminus \Lambda|$)*

The third condition implies that there exist

- Conductor := the unique integer $c$ with $c - 1 \notin \Lambda$, $c + \mathbb{N}_0 \subseteq \Lambda$
- Frobenius number := the largest gap$= c - 1$
- Dominant := the non-gap previous to $c$.

# Numerical Semigroups

### Definition

*A numerical semigroup is a subset $\Lambda$ of $\mathbb{N}_0$ satisfyig*

- $0 \in \Lambda$
- $\Lambda + \Lambda \subseteq \Lambda$
- $|\mathbb{N}_0 \setminus \Lambda|$ *is finite* (*genus*$:=g:=|\mathbb{N}_0 \setminus \Lambda|$)

The third condition implies that there exist

- Conductor $:=$ the unique integer $c$ with $c-1 \notin \Lambda$, $c+\mathbb{N}_0 \subseteq \Lambda$
- Frobenius number $:=$ the largest gap$= c-1$
- Dominant := the non-gap previous to $c$.

The inclusion $\Lambda \subseteq \mathbb{N}_0$ implies that there exists

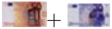
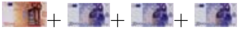
- Enumeration $:=$ the unique bijective increasing map $\lambda : \mathbb{N}_0 \to \Lambda$

# Cash Point

**Example**

The amounts of money one can obtain from a cash point (divided by 10)

# Cash Point

| amount | | amount/10 |
|---|---|---|
| 0 |  | 0 |
| *10* | *impossible!* | |
| 20 |  | 2 |
| *30* | *impossible!* | |
| 40 |  | 4 |
| 50 |  | 5 |
| 60 |  | 6 |
| 70 |  | 7 |
| 80 |  | 8 |
| 90 |  | 9 |
| 100 |  | 10 |
| 110 |  | 11 |
| ⋮ | ⋮ | ⋮ |

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# Cash Point

| amount | | amount/10 |
|---|---|---|
| 0 |  | 0 |
| | | gap |
| 20 |  | 2 |
| | | gap |
| 40 |  +  | 4 |
| 50 |  | 5 |
| 60 |  +  +  | 6 |
| 70 |  +  | 7 |
| 80 |  +  +  +  | 8 |
| 90 |  +  +  | 9 |
| 100 |  +  | 10 |
| 110 |  +  +  +  | 11 |
| $\vdots$ | $\vdots$ | $\vdots$ |

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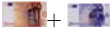
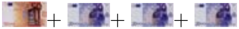
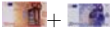
# Cash Point

| amount | | amount/10 |
|---|---|---|
| 0 |  | 0 |
| 20 |  | 2 |
| | | (3) |
| 40 |  +  | 4 |
| 50 |  | 5 |
| 60 |  +  +  | 6 |
| 70 |  +  | 7 |
| 80 |  +  +  +  | 8 |
| 90 |  +  +  | 9 |
| 100 |  +  | 10 |
| 110 |  +  +  +  | 11 |
| ⋮ | ⋮ | ⋮ |

# Cash Point

| amount | | amount/10 |
|---|---|---|
| 0 |  | 0 |
| 20 |  | 2 |
| 40 |  | 4 |
| 50 |  | 5 |
| 60 |  | 6 |
| 70 |  | 7 |
| 80 |  | 8 |
| 90 |  | 9 |
| 100 |  | 10 |
| 110 |  | 11 |
| ⋮ | ⋮ | ⋮ |

# Cash Point

| amount | | amount/10 |
|---|---|---|
| 0 |  | 0 |
| 20 |  | 2 |
| 40 |  | 4 |
| 50 |  | 5 |
| 60 |  | 6 |
| 70 |  | 7 |
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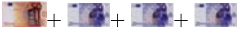
| 110 |  | 11 |
| $\vdots$ | $\vdots$ | $\vdots$ |

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# Weierstrass Semigroups

Consider

- $\chi$ a projective curve without singularities
- $P$ a rational point in $\chi$
- $A$ the ring of fuctions on $\chi$ with poles only at $P$
- $\Lambda = \{-v_p(f) : f \in A \setminus \{0\}\}$ the pole orders of $A$ at $P$

# Weierstrass Semigroups

Consider

- $\chi$ a projective curve without singularities
- $P$ a rational point in $\chi$
- $A$ the ring of fuctions on $\chi$ with poles only at $P$
- $\Lambda = \{-v_p(f) : f \in A \setminus \{0\}\}$ the pole orders of $A$ at $P$

It holds:

- $0 = -v_P(1) \in \Lambda$
- $(-v_P(f)) + (-v_P(g)) = (-v_P(fg)) \in \Lambda$ for all $f, g \in A$
- $|\mathbb{N}_0 \setminus \Lambda| =$ genus of $\chi$ (finite)

# Weierstrass Semigroups

**Example**

**Klein quartic**

$\chi : x^3y + y^3 + x = 0$

$P$ : rational point with affine coordinates $x = 0, y = 0$

| $i$ | $\lambda_i$ |
|---|---|
| 0 | 0 |
| 1 | 3 |
| 2 | 5 |
| 3 | 6 |
| 4 | 7 |
| 5 | 8 |
| 6 | 9 |
| 7 | 10 |
| 8 | 11 |
| $\vdots$ | $\vdots$ |

# Weierstrass Semigroups

**Example**

**Hermitian Curve Over $\mathbb{F}_4$**

$\chi : x^5 = y^4 + y$

$P$ : the unique point at infinity

| $i$ | $\lambda_i$ |
|---|---|
| 0 | 0 |
| | |
| 1 | 4 |
| 2 | 5 |
| | |
| 3 | 8 |
| 4 | 9 |
| 5 | 10 |
| | |
| 6 | 12 |
| 7 | 13 |
| $\vdots$ | $\vdots$ |

# Acute Semigroups

**Definition**

*A numerical semigroup is ordinary if it is equal to*

$$\{0\} \cup \{i \in \mathbb{N}_0 : i \geqslant c\},$$

*for some non-negative integer $c$.*

*A numerical semigroup is acute if it is ordinary or if its last interval of gaps is smaller than or equal to the previous one.*

# Acute Semigroups

**Example**

The Weierstrass semigroup at point $P$ of the Klein quartic is acute.

| $i$ | $\lambda_i$ | |
|---|---|---|
| 0 | 0 | |
| | | $\leftarrow$ 2 gaps |
| 1 | 3 | |
| | | $\leftarrow$ 1 gap |
| 2 | 5 | |
| 3 | 6 | |
| 4 | 7 | |
| 5 | 8 | |
| 6 | 9 | |
| 7 | 10 | |
| 8 | 11 | |
| 9 | 12 | |
| $\vdots$ | $\vdots$ | |

# Acute Semigroups

**Example**

The Weierstrass semigroup at point $P$ of the Hermitian curve is acute.

| $i$ | $\lambda_i$ | |
|---|---|---|
| 0 | 0 | |
| | | |
| 1 | 4 | |
| 2 | 5 | |
| | | $\leftarrow$ 2 gaps |
| 3 | 8 | |
| 4 | 9 | |
| 5 | 10 | |
| | | $\leftarrow$ 1 gap |
| 6 | 12 | |
| 7 | 13 | |
| 8 | 14 | |
| $\vdots$ | $\vdots$ | |

# Symmetric Semigroups

**Definition**

*A numerical semigroup with conductor $c$ and genus $g$ is symmetric if $c = 2g$.*

**Example:**

The Weierstrass semigroup at point $P$ of the Hermitian curve is symmetric.

Its conductor is $c = 12$ and its genus is $g = 6$.

| $i$ | $\lambda_i$ | |
|---|---|---|
| 0 | 0 | |
| | | $\longleftarrow$ 3 gaps |
| 1 | 4 | |
| 2 | 5 | |
| | | $\longleftarrow$ 2 gaps |
| 3 | 8 | |
| 4 | 9 | |
| 5 | 10 | |
| | | $\longleftarrow$ 1 gap |
| 6 | 12 | $\longleftarrow c = 12$ |
| 7 | 13 | |
| 8 | 14 | |
| 9 | 15 | |
| 10 | 16 | |
| $\vdots$ | $\vdots$ | |

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# Symmetric Semigroups

## Lemma

*A numerical semigroup $\Lambda$ is symmetric if and only if for any non-negative integer $i$,*

$$i \notin \Lambda \Longleftrightarrow c - 1 - i \in \Lambda.$$

| $i$ | $\lambda_i$ | |
|---|---|---|
| 0 | 0 | |
| | | 11-10 |
| | | 11-9 |
| | | 11-8 |
| 1 | 4 | |
| 2 | 5 | |
| | | 11-5 |
| | | 11-4 |
| 3 | 8 | |
| 4 | 9 | |
| 5 | 10 | |
| | | 11-0 |
| 6 | 12 | |
| $\vdots$ | $\vdots$ | |

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**Lemma**

*All symmetric semigroups are acute.*

**Proof**

*Let $\Lambda$ be a non-ordinary symmetric semigroup.*

*Since $1 \notin \Lambda$, by the previous Lemma $c - 2 \in \Lambda$.*

*Thus, the last interval of gaps consists of one gap $(c - 1)$.*

*The semigroup must therefore be acute.* $\square$

# Arf Semigroups

**Definition**

*A numerical semigroup $\Lambda$ is Arf if for any $a, b, c \in \Lambda$ with $a \geqslant b \geqslant c$ we have $a + b - c \in \Lambda$.*

**Example**

The Weierstrass semigroup at point $P$ of the Klein quartic is Arf.

| $i$ | $\lambda_i$ |
|---|---|
| 0 | 0 |
| 1 | 3 |
| 2 | 5 |
| 3 | 6 |
| 4 | 7 |
| 5 | 8 |
| 6 | 9 |
| 7 | 10 |
| $\vdots$ | $\vdots$ |

$7 + 5 - 3 = 9 \in \Lambda$

# Arf Semigroups

## Lemma

*All Arf semigroups are acute.*

## Proof

*Let $\Lambda$ be a non-ordinary Arf semigroup.*

*Consider $c, c', d, d'$ as in the example, where $c', c'+1, \dots, d$ is the last interval of non-gaps before the conductor.*

| $i$ | $\lambda_i$ | |
|---|---|---|
| 0 | 0 | $\longleftarrow d'$ |
| | | $\longleftarrow c'$ |
| 1 | 3 | |
| | | $\longleftarrow d$ |
| 2 | 5 | $\longleftarrow c$ |
| 3 | 6 | |
| 4 | 7 | |
| 5 | 8 | |
| 6 | 9 | |
| 7 | 10 | |
| $\vdots$ | $\vdots$ | |

$$d \geqslant c' > d' \Longrightarrow d + c' - d' \in \Lambda.$$

$$\left.\begin{array}{l} d + c' - d' \in \Lambda \\ d + c' - d' > d \end{array}\right\} \Longrightarrow d + c' - d' \geqslant c \Longrightarrow c - d \leqslant c' - d'. \qquad \square$$

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# Acute Semigroups

## Theorem

- *The set of acute semigroups is a proper subset of the whole set of numerical semigroups.*
- *It properly includes*
  - *Symmetric and pseudo-symmetric semigroups,*
  - *Arf semigroups,*
  - *Semigroups generated by an interval.*

# Near-Acute Semigroups

Let $[a, b] = \{a, a+1, \dots, b-1, b\}$ and let

$$\Lambda = 0 \cup [c_m, d_m] \cup [c_{m-1}, d_{m-1}] \cup \cdots \cup [c_2, d_2] \cup [c_1, d_1] \cup [c, \infty)$$

## Definition

$\Lambda$ is near-acute if $c - d_1 \leqslant d_1 - d_2$ or $2d_1 - c + 1 \notin \Lambda$.

## Theorem

*[Munuera, Torres, 2007]*

- *The set of near-acute semigroups is a proper subset of the whole set of numerical semigroups.*
- *It properly includes the set of acute semigroups*

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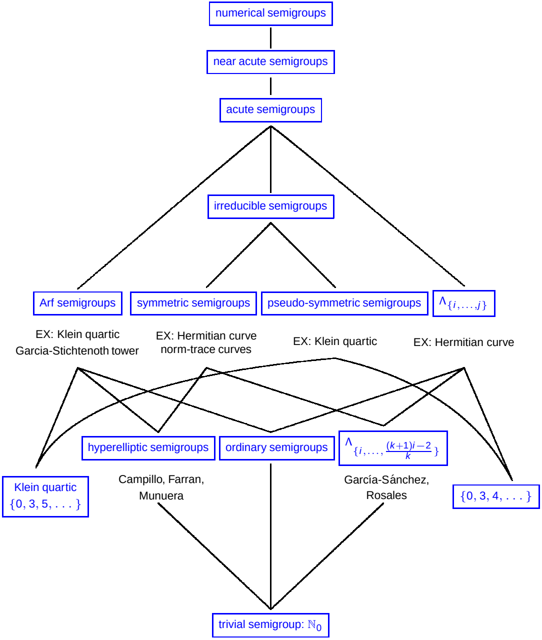

# $\nu$ Sequence

## Definition

*Given a numerical semigroup $\Lambda$ define its $\nu$ sequence as*

$$\nu_i = \#\{j \in \mathbb{N}_0 : \lambda_i - \lambda_j \in \Lambda\}$$

## Example
## Klein quartic

| $i$ | $\lambda_i$ | $\nu_i$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 3 | 2 |
| 2 | 5 | 2 |
| 3 | 6 | 3 |
| 4 | 7 | 2 |
| 5 | 8 | 4 |
| 6 | 9 | 4 |
| 7 | 10 | 5 |
| 8 | 11 | 6 |
| 9 | 12 | 7 |
| 10 | 13 | 8 |
| ⋮ | ⋮ | ⋮ |

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# $\tau$ Sequence

## Definition

*Given a numerical semigroup $\Lambda$ define its $\tau$ sequence as*

$$\tau_i = \max\{j \in \mathbb{N}_0 : \textit{ exists } k \textit{ with } j \leqslant k \leqslant i \textit{ and } \lambda_j + \lambda_k = \lambda_i\}$$

## Example
**Klein quartic**

| $i$ | $\lambda_i$ | $\tau_i$ | |
|---|---|---|---|
| 0 | 0 | 0 | $0+0=0$ |
| 1 | 3 | 0 | $0+3=3$ |
| 2 | 5 | 0 | $0+5=5$ |
| 3 | 6 | 1 | $3+3=6$ |
| 4 | 7 | 0 | $0+7=7$ |
| 5 | 8 | 1 | $3+5=8$ |
| 6 | 9 | 1 | $3+6=9$ |
| 7 | 10 | 2 | $5+5=10$ |
| 8 | 11 | 2 | $5+6=11$ |
| 9 | 12 | 3 | $6+6=12$ |
| 10 | 13 | 3 | $6+7=13$ |
| $\vdots$ | $\vdots$ | $\vdots$ | |

# One-Point Codes

## Definition

*Consider*

- $\chi$ *a projective curve without singularities over* $\mathbb{F}_q$
- $P$ *a rational point in* $\chi$
- $A$ *the ring of functions on* $\chi$ *with poles only at* $P$
- $\Lambda$ *the Weierstrass semigroup at* $P$
- $\lambda$ *enumeration of* $\Lambda$ ($\Lambda = \{\lambda_0 = 0 < \lambda_1 < \lambda_2 \dots\}$)
- $\{f_i : i \in \mathbb{N}_0, f_i \in A, -v_P(f) = \lambda_i\}$.
- $P_1, P_2, \dots, P_n$ *rational points in* $\chi$ *different from* $P$
- $ev : A \longrightarrow \mathbb{F}_q^n,\ f \mapsto (f(P_1), \dots, f(P_n))$.

*Define*

$$C_i = \{ev(f) : f \in A, -v_P(f) \leqslant \lambda_i\}^\perp \quad \text{for } i \in \mathbb{N}_0 \text{ (standard)}$$
$$C_W = \langle ev(f_i) : i \in W\rangle^\perp \quad \text{for } W \subseteq \mathbb{N}_0$$

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# Generic Errors

**Definition**

*The points $P_{i_1}, \dots, P_{i_t}$ are generically distributed if no element $f \in A$ with $-v_P(f) < \lambda_t$ vanishes in all of them. Generic errors are those errors whose non-zero positions correspond to generically distributed points.*

Generic errors of weight $t$ can be a very large proportion of all possible errors of weight $t$ [Hansen, 2001].

By restricting the errors to be corrected to generic errors the decoding requirements will be weaker and we still will be able to correct almost all errors.

# Correction Capability of Berlekamp-Massey-Sakata algorithm with Majority Voting

**Theorem**

*[Feng, Rao, 1995]*

*All error vectors of weight $t$ can be corrected by $C_W$ if $W$ contains all $i$ with $\nu_i < 2t + 1$.*

**Theorem**

*[Bras-Amorós, O'Sullivan, 2006]*

*All generic error vectors of weight $t$ can be corrected by $C_W$ if $W$ contains all $i$ with $\lambda_i \notin \{\lambda_j + \lambda_k : j, k \geqslant t\}$.*

**Remark**

*$\lambda_i \notin \{\lambda_j + \lambda_k : j, k \geqslant t\}$ is equivalent to $\tau_i < t$.*

# Correction Capability Optimized Codes

**Codes for correcting $t$ errors of any kind**

Take $W$ equal to

$\widetilde{R}(t) = \{i \in \mathbb{N}_0 : \lfloor \frac{\nu_i - 1}{2} \rfloor < t\}$ (Feng–Rao improved codes)

$R(t) = \{i \in \mathbb{N}_0 : i \leqslant i(t)\}$, where $i(t) = \max \widetilde{R}(t)$ (standard)

**Codes for correcting $t$ generic errors**

Take $W$ equal to

$\widetilde{R}^*(t) = \{i \in \mathbb{N}_0 : \tau_i < t\}$

$R^*(t) = \{i \in \mathbb{N}_0 : i \leqslant i^*(t)\}$, where $i^*(t) = \max \widetilde{R}^*(t)$ (standard)

Increasingness of $\nu$ $\longleftrightarrow$ Compare $\widetilde{R}(t)$ and $R(t)$

Increasingness of $\tau$ $\longleftrightarrow$ Compare $\widetilde{R}^*(t)$ and $R^*(t)$

Compare $\nu$ and $\tau$ $\longleftrightarrow$ Compare $\widetilde{R}^*(t)$ and $\widetilde{R}(t)$

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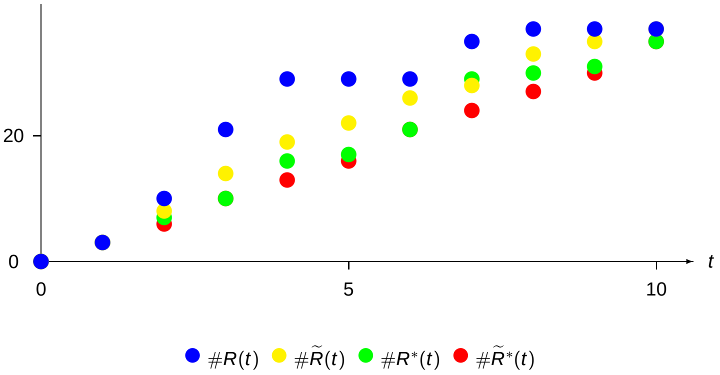
# Hermitian Codes Redundancy ($\mathbb{F}_{7^2}$)



# Increasingness of $\nu$

The $\nu$ sequence of the ordinary semigroup $\{0\} \cup [c, \infty)$ is

$$1, 2, \stackrel{(c)}{\ldots}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$$

**Proposition**

*If $\nu$ is non-decreasing then $\Lambda$ is Arf.*

**Proposition**

*Let $\Lambda$ be the non-ordinary near-acute semigroup*

$$0 \cup [c_m, d_m] \cup \cdots \cup [c_2, d_2] \cup [c_1, d_1] \cup [c, \infty)$$

*and let $m = \min\{\lambda^{-1}(c + c_1 - 2), \lambda^{-1}(2d)\}$. Then*

- $\nu_m > \nu_{m+1}$
- $\nu_i \leqslant \nu_{i+1}$ *for all* $i > m$.

# Increasingness of $\nu$

## Corollary

- *The unique semigroup for which $\nu$ is strictly increasing is $\mathbb{N}_0$.*
- *The only numerical semigroups for which $\nu$ is non-decreasing are ordinary-semigroups.*

## Corollary

- $\widetilde{R}(t) = R(t)$ *for all* $t \in \mathbb{N}_0$ *if and only if the associated numerical semigroup is ordinary.*

# Increasingness of $\nu$

**Lemma**

*If $i \geqslant 2c - g - 1$ then $\nu_i = i - g + 1$.*

**Corollary**

*For any numerical semigroup,*

- $\widetilde{R}(t) = R(t)$ *for all* $t \geqslant c - g$.
- $\#\widetilde{R}(t) = \#R(t) = \lambda_t + t$ *for all* $t \geqslant c - g$.

# Increasingness of $\tau$

The $\tau$ sequence of $\mathbb{N}_0$ is

$$0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \dots$$

The $\tau$ sequence of the semigroup $\{0\} \cup [c, \infty)$ with $c > 0$ is

$$0, \overset{(c+1)}{\dots}, 0, 1, 1, 2, 2, 3, 3, 4, 4, \dots$$

## Proposition

*For a non-ordinary semigroup with conductor $c$, genus $g$ and dominant $d$ (non-gap previous to $c$) let $m = \lambda^{-1}(2d)$. Then*

- $\tau_m = c - g - 1 > \tau_{m+1}$
- $\tau_i \leqslant \tau_{i+1}$ *for all* $i > m$.

## Corollary

1. *The unique numerical semigroups with non-decreasing $\tau$ sequence are ordinary semigroups.*

# Increasingness of $\tau$

The $\tau$ sequence of $\mathbb{N}_0$ is

$$0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots$$

The $\tau$ sequence of the semigroup $\{0\} \cup [c, \infty)$ with $c > 0$ is

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2. $\widetilde{R}^*(t) = R^*(t)$ *for all* $t \in \mathbb{N}_0$ *if and only if the associated numerical semigroup is ordinary.*

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- $\tau_m = c - g - 1 > \tau_{m+1}$
- $\tau_i \leqslant \tau_{i+1}$ *for all* $i > m$.

## Corollary

3. *For any numerical semigroup,* $\widetilde{R}^*(t) = R^*(t)$ *for all* $t \geqslant c - g$.

# More on $\tau$

## Lemma

$\#\{i \in \mathbb{N}_0 : \tau_i < t\} = \lambda_t + t - \#\{h \in \mathbb{N}_0 \setminus \Lambda : h = \lambda_i + \lambda_j - \lambda_t, i, j \geqslant t\}$

## Corollary

1. $\#\widetilde{R}^*(t) \leqslant \lambda_t + t$ *for all* $t \in \mathbb{N}_0$.
2. $\#\widetilde{R}^*(t) = \#R^*(t) = \lambda_t + t$ *for all* $t \geqslant c - g$.

# Comparing $\nu$ and $\tau$

## Proposition

- $\tau_i \geqslant \lfloor \frac{\nu_i - 1}{2} \rfloor$ *for all* $i \in \mathbb{N}_0$
- $\tau_i = \lfloor \frac{\nu_i - 1}{2} \rfloor$ *for all* $i \geqslant 2c - g - 1$
- $\tau_i = \lfloor \frac{\nu_i - 1}{2} \rfloor$ *for all* $i \in \mathbb{N}_0$ *if and only if* $\Lambda$ *is Arf.*

## Proof of 1.

Let

$$\begin{aligned} N_i &= \{j \in \mathbb{N}_0 : \lambda_i - \lambda_j \in \Lambda\} \\ &= \{N_{i,0} < N_{i,1} < N_{i,2} < \cdots < N_{i,\nu_i - 1}\}. \end{aligned}$$

Then

- $N_{i,j} \geqslant j$
- $\tau_i = N_{i, \lfloor \frac{\nu_i - 1}{2} \rfloor}$

$\square$

# Comparing $\nu$ and $\tau$

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## Corollary

1. $\widetilde{R}^*(t) \subseteq \widetilde{R}(t)$ *for all* $t \in \mathbb{N}_0$.
2. $\widetilde{R}^*(t) = \widetilde{R}(t)$ *for all* $t \in \mathbb{N}_0$ *if and only if the associated numerical semigroup is Arf.*

# Comparing $\nu$ and $\tau$

## Lemma

*For a numerical semigroup with conductor $c > 2$,*

- $\tau_{(2c-g-2)+2i} = \tau_{(2c-g-2)+2i+1} = c - g - 1 + i$ *for all* $i \geqslant 0$
- *At least one of the following statements holds*
  - $\tau_{(2c-g-2)-1} = c - g - 1$
  - $\tau_{(2c-g-2)-2} = c - g - 1$

## Corollary

3. $\widetilde{R}^*(t) = \widetilde{R}(t)$ *for all* $t \geqslant c - g$.

# Comparing $\nu$ and $\tau$

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## Corollary

3. $\widetilde{R}^*(t) = \widetilde{R}(t)$ *for all* $t \geqslant c - g$.

## Corollary

*The genus and the conductor are determined by the $\tau$ sequence.*

# Semigroup Characterization by $\tau$

## Theorem

*A numerical semigroup is completely determined by its $\tau$ sequence.*

## Proof

*We can construct a numerical semigroup $\Lambda$ from its $\tau$ sequence as follows:*

- *Let $k$ be the minimum integer such that for all $i \in \mathbb{N}_0$,*
  - $\tau_{k+2i} = \tau_{k+2i+1}$
  - $\tau_{k+2i+2} = \tau_{k+2i+1} + 1$
- *Set*
  - $c = k - \tau_k + 1$
  - $g = k - 2\tau_k$

  *This determines $\lambda_i$ for all $i \geqslant c - g$*
- *For $i = c - g - 1$ to $1$, $\lambda_i = \frac{1}{2}\min\{\lambda_j : \tau_j = i\}$*

$\square$

On Numerical Semigroups and Their Applications to Algebraic Geometry Codes

Maria Bras-Amorós

Semigroup Families
0. Preliminaries
A. Acute
B. Symmetric
C. Arf
D. Classification

$\nu$, $\tau$ and Improved Codes
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Further on Semigroups
A. Characterization
B. Counting

# Semigroup Characterization by $\nu$

## Theorem

*A numerical semigroup is completely determined by its $\nu$ sequence.*

## Proof

*We can construct a numerical semigroup $\Lambda$ from its $\nu$ sequence as follows:*

- *If $\nu_i = i + 1$ for all $i \in \mathbb{N}_0$ then $\Lambda = \mathbb{N}_0$*
- *Otherwise let $k = \max\{j : \nu_j = \nu_{j+1}\}$ (it exists and it is unique)*
- *Let $g = k + 2 - \nu_k$ and $c = \frac{k+g+2}{2}$*
  - $0 \in \Lambda$, $1, c - 1 \notin \Lambda$
  - *For all $i \geqslant c$, $i \in \Lambda$*
- *For $i = c - 2$ to $i = 2$,*
  - *Define $\tilde{D}(i) = \{l \in \Lambda^c : c - 1 + i - l \in \Lambda^c, i < l < c - 1\}$*
  - *$i \in \Lambda$ if and only if $\nu_{c-1+i-g} = c + i - 2g + \#\tilde{D}(i)$*

$\square$

# Semigroup Characterization by $\oplus$

## Definition

*Given a numerical semigroup $\Lambda$ with enumeration $\lambda$ define the binary operation*

$$i \oplus j = \lambda^{-1}(\lambda_i + \lambda_j).$$

*Equivalently,*

$$\lambda_{i \oplus j} = \lambda_i + \lambda_j.$$

## Theorem

*A numerical semigroup is completely determined by the $\oplus$ operation.*

# Semigroup Characterization

**Theorem**

*No numerical semigroup can be determined by a finite subset of*

- $\nu$ *values*
- $\tau$ *values*
- $\oplus$ *values.*

# Counting

Let $n_g$ denote the number of numerical semigroups of genus $g$.

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# Counting

Let $n_g$ denote the number of numerical semigroups of genus $g$.

- $n_0 = 1$, since the unique numerical semigroup of genus 0 is $\mathbb{N}_0$
- $n_1 = 1$, since the unique numerical semigroup of genus 1 is $\mathbb{N}_0 \setminus \{1\}$
- $n_2 = 2$. Indeed the unique numerical semigroups of genus 2 are

$$\{0, 3, 4, 5, \dots\},$$

$$\{0, 2, 4, 5, \dots\}.$$

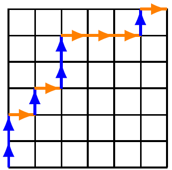
# Dyck Paths

**Definition**

*A Dyck path of order $n$ is a staircase walk from $(0,0)$ to $(n,n)$ that lies over the diagonal $x = y$*

To each Dyck path it corresponds a unique tree.

**Example**



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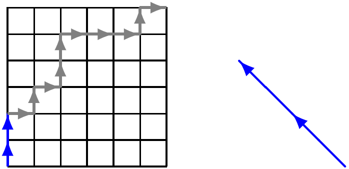

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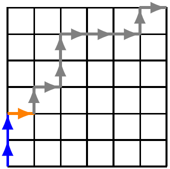

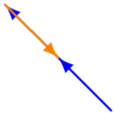

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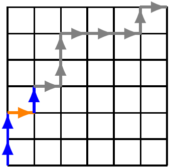

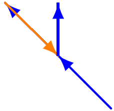

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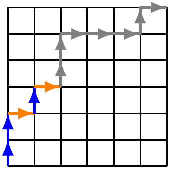

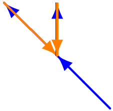

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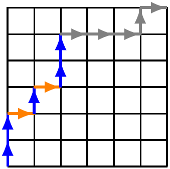

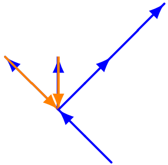

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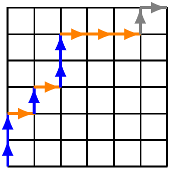

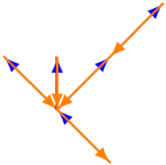

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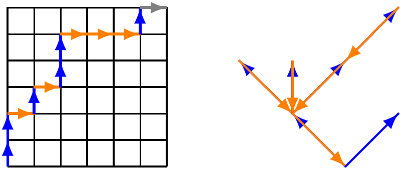

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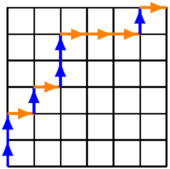

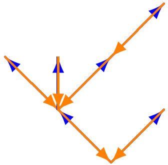

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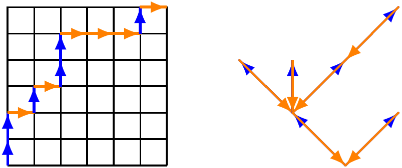


The number of Dyck paths of order $n$ is given by the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

# Dyck Paths

## Definition

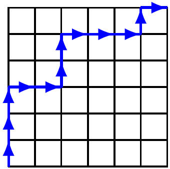
*The square diagram of a numerical semigroup is the path*

$$e(i) = \begin{cases} \rightarrow & \text{if } i \in \Lambda, \\ \uparrow & \text{if } i \not\in \Lambda, \end{cases} \qquad \text{for } 1 \leqslant i \leqslant 2g.$$

It always goes from $(0,0)$ to $(g,g)$.

## Example

The square diagram of the numerical semigroup $\{0,4,5,8,9,10,12,\dots\}$ is



# Dyck Paths

## Proposition

*The square diagram of a numerical semigroup is a Dyck path.*

## Corollary

*Each numerical semigroup can be represented by a different tree.*

## Corollary

*The number of numerical semigroups of genus $g$ is bounded by the Catalan number $C_g = \frac{1}{g+1}\binom{2g}{g}$.*

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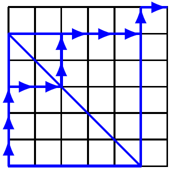
# Dyck Paths

## Proposition

*A numerical semigroup is symmetric if and only if its square diagram is symmetric with respect to the counterdiagonal of the subsquare* $[0, g-1]^2$.



## Proposition

*The weight of a numerical semigroup* $\left(\sum_{l_i : i\text{th gap}} (l_i - i)\right)$ *is the area over the path of the numerical semigroup in the square* $[0, g]^2$.

# Conjecture $n_g/n_{g-1} \to \phi$

## Conjecture

1. $n_g \geqslant n_{g-1} + n_{g-2}$
2. $lim_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1$
3. $lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi$

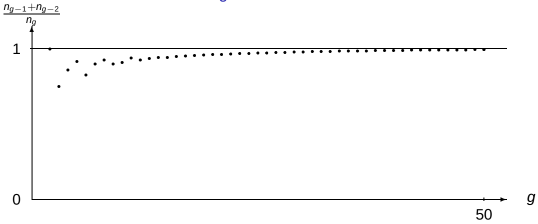
At the moment it has not even been proved that $n_g$ is increasing.

# Conjecture $n_g/n_{g-1} \longrightarrow \phi$

| $g$ | $n_g$ | $n_{g-1}+n_{g-2}$ | $\frac{n_{g-1}+n_{g-2}}{n_g}$ | $\frac{n_g}{n_{g-1}}$ |
|---|---|---|---|---|
| 0 | 1 | | | |
| 1 | 1 | | | 1 |
| 2 | 2 | 2 | 1 | 2 |
| 3 | 4 | 3 | 0.75 | 2 |
| 4 | 7 | 6 | 0.857143 | 1.75 |
| 5 | 12 | 11 | 0.916667 | 1.71429 |
| 6 | 23 | 19 | 0.826087 | 1.91667 |
| 7 | 39 | 35 | 0.897436 | 1.69565 |
| 8 | 67 | 62 | 0.925373 | 1.71795 |
| 9 | 118 | 106 | 0.898305 | 1.76119 |
| 10 | 204 | 185 | 0.906863 | 1.72881 |
| 11 | 343 | 322 | 0.938776 | 1.68137 |
| 12 | 592 | 547 | 0.923986 | 1.72595 |
| 13 | 1001 | 935 | 0.934066 | 1.69088 |
| 14 | 1693 | 1593 | 0.940933 | 1.69131 |
| 15 | 2857 | 2694 | 0.942947 | 1.68754 |
| 16 | 4806 | 4550 | 0.946733 | 1.68218 |
| 17 | 8045 | 7663 | 0.952517 | 1.67395 |
| 18 | 13467 | 12851 | 0.954259 | 1.67396 |
| 19 | 22464 | 21512 | 0.957621 | 1.66808 |
| 20 | 37396 | 35931 | 0.960825 | 1.66471 |
| 21 | 62194 | 59860 | 0.962472 | 1.66312 |
| 22 | 103246 | 99590 | 0.964589 | 1.66006 |
| 23 | 170963 | 165440 | 0.967695 | 1.65588 |
| 24 | 282828 | 274209 | 0.969526 | 1.65432 |
| 25 | 467224 | 453791 | 0.971249 | 1.65197 |
| 26 | 770832 | 750052 | 0.973042 | 1.64981 |
| 27 | 1270267 | 1238056 | 0.974642 | 1.64792 |
| 28 | 2091030 | 2041099 | 0.976121 | 1.64613 |
| 29 | 3437839 | 3361297 | 0.977735 | 1.64409 |
| 30 | 5646773 | 5528869 | 0.97912 | 1.64254 |
| 31 | 9266788 | 9084612 | 0.980341 | 1.64108 |
| 32 | 15195070 | 14913561 | 0.981474 | 1.63973 |
| 33 | 24896206 | 24461858 | 0.982554 | 1.63844 |
| 34 | 40761087 | 40091276 | 0.983567 | 1.63724 |
| 35 | 66687201 | 65657293 | 0.984556 | 1.63605 |
| 36 | 109032500 | 107448288 | 0.98547 | 1.63498 |
| 37 | 178158289 | 175719701 | 0.986312 | 1.63399 |
| 38 | 290939807 | 287190789 | 0.987114 | 1.63304 |
| 39 | 474851445 | 469098096 | 0.987884 | 1.63213 |
| 40 | 774614284 | 765791252 | 0.98861 | 1.63128 |
| 41 | 1262992840 | 1249465729 | 0.98929 | 1.63048 |
| 42 | 2058356522 | 2037607124 | 0.989919 | 1.62975 |
| 43 | 3353191846 | 3321349362 | 0.990504 | 1.62906 |
| 44 | 5460401576 | 5411548368 | 0.991053 | 1.62842 |
| 45 | 8888486816 | 8813593422 | 0.991574 | 1.62781 |
| 46 | 14463633648 | 14348888392 | 0.992067 | 1.62723 |
| 47 | 23527845502 | 23352120464 | 0.992531 | 1.62669 |
| 48 | 38260496374 | 37991479150 | 0.992969 | 1.62618 |
| 49 | 62200036752 | 61788341876 | 0.993381 | 1.6257 |
| 50 | 101090300128 | 100460533126 | 0.99377 | 1.62525 |

# Conjecture $n_g/n_{g-1} \longrightarrow \phi$

## Behavior of $\frac{n_{g-1}+n_{g-2}}{n_g}$




## Behavior of $\frac{n_g}{n_{g-1}}$