Ergodicity of the dynamical systems on 2-adic spheres

(joint talk with V. Anashin and A. Khrennikov)

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Foundations of p-adic theory and more general non-Archimedean dynamical systems were presented in:

K. Mahler, *p*-adic numbers and their functions, Cambridge Univ. Press, 1981

W. H. Schikhof, Ultrametric calculus. An introduction to *p*-adic analysis, Cambridge University Press, Cambridge, 1984

V. Anashin and A. Khrennikov, Applied Algebraic Dynamics, vol. 49, Walter De Gruyter, 2009

P-adic ergodicity was studied by

V. Anashin, Uniformly distributed sequences of *p*-adic integers,II, iscrete Math. Appl., 12(6), 2002, pp. 527–590 and **Matthias Gundlach, Andrei Khrennikov, Karl-Olof Lindahl**, see e.g. Ergodicity on *p*-adic sphere, University of Hamburg Press (2000), pp. 15–21

Applications:

V. Anashin, Pseudorandom number generation by *p*-adic ergodic transformations, http://arxiv.org/abs/cs/0401030v1

A. Khrennikov, Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models, KLuwer, 1997

A. Khrennikov, Information dynamics in cognitive, psychological, social, and anomalous phenomena, Fundamental Theories of Physics, Kluwer, Dordreht, 2004

Overview

p-**adic dynamics** attract remarkable interest to their applications in a number of various domains as:

1. Pure mathematics
2. Physics
3. Biology, Genetics
4. Cognitive science, Neurophysiology
5. Computer science, Cryptology

Non-Archimedean fields

Let K be a field. An **absolute value** on K is a function $|\cdot| : K \to \mathbb{R}$ such that

- $|x| \geq 0$ for all $x \in K$,
- $|x| = 0$ if and only if $x = 0$,
- $|xy| = |x||y|$, for all $x, y \in K$,
- $|x + y| \leq |x| + |y|$, for all $x, y \in K$.

If $|.|$ in addition satisfies the **strong triangle inequality**

$$|x + y| \leq \max(|x|, |y|)$$

for all $x, y \in K$ then we say that $|.|$ is **non-Archimedean**.

p-adic absolute value

Let p be a fixed prime number. Each non-zero integer n can be written uniquely as

$$n = p^{\operatorname{ord}_p n} \hat{n},$$

where $\hat{n}$ is a non-zero integer, $p \nmid \hat{n}$, and $\operatorname{ord}_p n$ is a unique non-negative integer.
The function $\operatorname{ord}_p : \mathbb{Z} \setminus \{0\} \to \mathbb{N}_0$ is called the p-**adic valuation**.
If $a, b \in \mathbb{Z}$ then the p-adic valuation of $x = a/b$ is

$$\operatorname{ord}_p x = \operatorname{ord}_p a - \operatorname{ord}_p b.$$

Definition

The p-**adic absolute value** *of* $x \in \mathbb{Q} \setminus \{0\}$ *is given by*

$$|x|_p = p^{-\operatorname{ord}_p x}$$

and $|0|_p = 0$.

The p-adic absolute value is non-Archimedean. It induces a metrics

$$\rho(x, y) = |x - y|_p.$$

Definitions and notions from p-adic dynamics

Dynamical system theory study trajectories

$$x_0,\ x_1 = f(x_0), \dots, x_{i+1} = f(x_i) = f^{i+1}(x_0), \dots$$

Consider $\langle \mathbb{Z}_p, \mu_p, f \rangle$, where

- $\mathbb{Z}_p$ is a space of p-adic integers,
- **the normalized Haar measure** $\mu_p(B_{p^{-k}}(a)) = p^{-k}$, $B_{p^{-k}}(a) = a + p^k\mathbb{Z}_p$ are elementary measurable subsets (balls) in $\mathbb{Z}_p$ of radius p^{-k} centered at the point $a \in \mathbb{Z}_p$,
- $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is a μ_p-measurable function that is continuous with respect to p-adic metric.

Definitions and notions from p-adic dynamics

Recall that a **sphere** $S_{p^{-k}}(a)$ of radius p^{-k} centered at $a \in \mathbb{Z}_p$ is a disjoint union of $p-1$ balls of radius $\frac{1}{p^{k+1}}$ each:

$$S_{p^{-r}}(a) = \bigcup_{s=1}^{p-1} \left(a + p^k + p^{k+1}\mathbb{Z}_p\right), a \in \mathbb{Z}_p, r = 1/p^k, k = 1, 2, \ldots$$

Definitions and notions from p-adic dynamics

A function $f: \mathbb{Z}_p \to \mathbb{Z}_p$ is called **compatible** iff the congruence $a \equiv b \pmod{p^k}$ implies the congruence $f(a) \equiv f(b) \pmod{p^k}$, for all $a, b \in \mathbb{Z}_p$.
In other words, the function f is compatible if and only if f satisfies **Lipschitz condition** with a constant 1:

$$|f(a) - f(b)|_p \leq |a - b|_p \,,$$

for all $a, b \in \mathbb{Z}_p$.
Here $|\cdot|_p$ stands for p-adic absolute value.

Definitions and notions from p-adic dynamics

The mapping $f\colon \mathbb{S} \to \mathbb{S}$ of a measure space $\mathbb{S}$ that is endowed with a probability measure μ is said to **preserve the measure** μ iff $\mu(f^{-1}(S)) = \mu(S)$ for every measurable subset $S \subset \mathbb{S}$.
A μ-preserving mapping f is said to be **ergodic** iff $\mu(S) = 1$ or $\mu(S) = 0$ for every measurable $S \subset \mathbb{S}$ such that $f^{-1}(S) = S$.

Definitions and notions from p-adic dynamics

A compatible mapping $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is said to be **bijective** (resp., **transitive**) **modulo** p^k iff the induced mapping $f \bmod p^k : x \mapsto f(x) \bmod p^k$ is a permutation (resp., a permutation with a single cycle) on the ring $\mathbb{Z}/p^k\mathbb{Z}$ of residues modulo p^k.
The following theorem holds:

Theorem (V. Anashin)

A compatible function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is measure-preserving (or, accordingly, ergodic) if and only if it is bijective (accordingly, transitive) modulo p^k for all $k = 1, 2, 3, \ldots$

Practical importance of p-adic dynamics

In 1992 by V.Anashin and co-authors was constructed a computer program that produces random-looking sequence of numbers (pseudorandom generator).

A **pseudorandom number generator** is an algorithm that takes a short random string and stretches it to a much longer string that looks like random.

PRNG as an autonomous dynamical system $\langle Z_p, \mu, f\rangle$:

- f is a **state update**, G is **output function**, x_0 is **initial state** (key);
- sequence of states = orbits:

$$x_0, x_1 = f(x_0), \dots, x_{i+1} = f(x_i) = f^{i+1}(x_0), \dots$$

- output sequence = observables:

$$G(x_0), G(x_1), \dots, G(x_i + 1), \dots$$

Practical importance of p-adic dynamics

A PRNG must meet the following conditions to be considered as good:

- The output sequence must be pseudorandom, i.e. f is transitive modulo p^n
- The output function G must not spoil pseudorandomness, i.e. be bijective modulo p^k

So by theorem (V. Anashin) we should take as G **measure-preserving** function and as f **ergodic** compatible function.
Therefore a problem arises to describe such functions.

Practical importance of p-adic dynamics

Another interesting problem is to study ergodic behavior of p**-adic monomial dynamical systems** that is described by iterations of

$$f(x) = x^n, n \in \mathbb{N}, n \geq 2. \tag{1}$$

In paper of [M. Gundlach, A. Khrennikov, K.-O. Lindahl] was stated the need to study ergodic stability for **pertubations** of monomial systems

$$x \mapsto x^n + q(x),$$

where $q(x)$ is a 'small' polynom, on p-adic spheres $f : S_r(a) \to S_r(a)$, where

$$S_r(a) = \{x \in Q_p : |x - a|_p = r\}, a \in Z_p, r = 1/2^k, k = 1, 2, \ldots$$

van der Put series

Given a continuous function $g\colon \mathbb{Z}_p \to \mathbb{Z}_p$, there exists a unique sequence $B_0, B_1, B_2, \ldots$ of p-adic integers such that

$$g(x) = \sum_{m=0}^{\infty} B_m \chi(m, x)$$

for all $x \in \mathbb{Z}_p$, where

$$\chi(m, x) = \begin{cases} 1, & \text{if } |x - m|_p \leq p^{-n} \\ 0, & \text{otherwise} \end{cases}$$

and $n = 1$ if $m = 0$; n is uniquely defined by the inequality $p^{n-1} \leq m \leq p^n - 1$ otherwise. This series is called the **van der Put series** of the function g.

van der Put series

The number n in the definition of $\chi(m,x)$ is just the number of digits in a base-p expansion of $m \in \mathbb{N}_0$:
Given $m \in \mathbb{N}_0$ denote via $\lfloor \log_p m \rfloor$ the largest rational integer that is either less than, or equal to, $\log_p m$; then

$$\lfloor \log_p m \rfloor = (\text{the number of digits in a base-}p\text{ expansion for } m) - 1;$$

henceforth $n = \lfloor \log_p m \rfloor + 1$ for all $m \in \mathbb{N}_0$ (we put $\lfloor \log_p 0 \rfloor = 0$).

Coefficients B_m

Let $m = m_0 + \ldots + m_{n-2}p^{n-2} + m_{n-1}p^{n-1}$ be a base-p expansion for m, i.e., $m_j \in \{0, \ldots, p-1\}$, $j = 0, 1, \ldots, n-1$ and $m_{n-1} \neq 0$, then

$$B_m = \begin{cases} g(m) - g(m - m_{n-1}p^{n-1}), & \text{if } m \geq p; \\ g(m), & \text{otherwise.} \end{cases}$$

It worth notice also that $\chi(m, x)$ is merely a characteristic function of the ball of radius $p^{-\lfloor \log_p m \rfloor - 1}$ centered at $m \in \mathbb{N}_0$.

Simple example of the van der Put series

Canonical expansion	m	$g(m)$	$g(m-q(m))$	B_m
	0			0
	1			1
$01 = 0 \cdot 2^0 + 1 \cdot 2^1$	2	4	0	4
$11 = 1 \cdot 2^0 + 1 \cdot 2^1$	3	9	1	8
$001 = 0 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2$	4	16	0	16
$101 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2$	5	25	1	24
$011 = 0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2$	6	36	4	32
$111 = 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2$	7	49	9	40
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table: For $p = 2$ and $g(x) = x^2$.

Simple example of the van der Put series

The van der Put series:

$$\begin{aligned} g(x) = {} & 0 \cdot \chi(x,0) + 1 \cdot \chi(x,1) + 4 \cdot \chi(x,2) + \\ & + 8 \cdot \chi(x,3) + 16 \cdot \chi(x,4) + +24 \cdot \chi(x,5) + \\ & + 32 \cdot \chi(x,6) + 40 \cdot \chi(x,7) + \dots \end{aligned}$$

Let $x = 5 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2$. The characteristic function $\chi(x, m)$ is equal 1 for $m = 1$ and $m = 5$.
Therefore,

$$g(5) = 1 \cdot \chi(5,1) + 24 \cdot \chi(5,5) = 25.$$

Comment about different basises

To calculate value of the function we should know:

- ∞ coefficients for the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$
- m coefficients for the Mahler series $f(x) = \sum_{i=0}^{\infty} a_i \binom{x}{i}$, where $a_i \in \mathbb{Z}_p$ (respectively, $a_i \in \mathbb{Z}$), $i = 0, 1, 2, \ldots$, and

  $$\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$$

  for $i = 1, 2, \ldots$;

  $$\binom{x}{0} = 1,$$

- $\log_p m$ coefficients for the van der Put series $g(x) = \sum_{m=0}^{\infty} B_m \chi(m, x)$

Results, compatibility

Given a function $f : \mathbb{Z}_p \to \mathbb{Z}_p$, represent f via van der Put series:

$$f(x) = \sum_{m=0}^{\infty} B_m \chi(m, x).$$

Theorem

The function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is **compatible** *if and only if it can be represented as*

$$f(x) = \sum_{m=0}^{\infty} p^{\lfloor \log_p m \rfloor} b_m \chi(m, x),$$

where $b_m \in \mathbb{Z}_p$ for $m = 0, 1, 2, \ldots$.

Note: Schikhof obtained theorem, which describe compatible functions via the van der Put series in [Ultrametric calculus, Sch, 1984]. But we get such theorem represented in more suitable for further results form.

Results, measure-preservation

Let now $p = 2$; then the following criteria are true:

Theorem

The function $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ is **compatible and preserves the measure** *μ_p if and only if it can be represented as*

$$f(x) = b_0\chi(0,x) + b_1\chi(1,x) + \sum_{m=2}^{\infty} 2^{\lfloor \log_2 m \rfloor} b_m \chi(m,x),$$

where $b_m \in \mathbb{Z}_2$ for $m = 0, 1, 2, \ldots$, and

1. $b_0 + b_1 \equiv 1 \pmod 2$,
2. $|b_m|_2 = 1$, *if* $m \geq 2$.

Results, ergodicity

Theorem

The function $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ *is* **compatible and ergodic** (w.r.t. the measure μ_p) *if and only if it can be represented as*

$$f(x) = b_0\chi(0,x) + b_1\chi(1,x) + \sum_{m=2}^{\infty} 2^{\lfloor \log_2 m \rfloor} b_m \chi(m,x)$$

where $b_m \in \mathbb{Z}_2$ *for* $m = 0, 1, 2, \ldots$, *and the following conditions hold simultaneously:*

1. $b_0 \equiv 1 \pmod 2$, $b_0 + b_1 \equiv 3 \pmod 4$, $b_2 + b_3 \equiv 2 \pmod 4$;
2. $|b_m|_2 = 1$ *for* $m \geq 2$;
3. $\sum_{m=2^{n-1}}^{2^n-1} b_m \equiv 0 \pmod 4$ *for* $n \geq 3$.

Results, ergodicity

It worth notice that the core of the proof of previous theorem is the following lemma:

Lemma

Let $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ be a function represented by van der Put series $f(x) = \sum_{m=0}^{\infty} B_m \chi(m, x)$. The function f is compatible and ergodic (w.r.t. the measure μ_p) *if and only if the following conditions hold simultaneously:*

1. $B_0 \equiv 1 \pmod 2, B_0 + B_1 \equiv 3 \pmod 4$,
2. $|B_m|_2 = 2^{-\lfloor \log_2 m \rfloor}$, *if* $m \geq 2$;
3. $\left| \sum_{m=2^{n-1}}^{2^n - 1} (B_m - 2^{n-1}) \right|_2 \leq 2^{-(n+1)}$, *if* $n \geq 2$.

Comment. Criteria for measure-preserving and ergodic functions, using Mahler's expansion of the function, was obtained by V. Anashin. However use of the van der Put basis allow us to check properties above faster and easier.

Example of the ergodic function represented via the van der Put series

Given a 2-adic integer $a \in \mathbb{Z}_2$, consider its 2-adic canonical representation $a = \sum_{i=0}^{\infty} \alpha_i 2^i$; that is, $\alpha_i \in \{0, 1\}$ for all $i = 0, 1, 2, \ldots$. Put $\delta_i(a) = \alpha_i$; so $\delta_i \colon \mathbb{Z}_2 \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$.

Example

The following function is ergodic on $\mathbb{Z}_2$:

$$f(x) = 1 + \delta_0(x) + 6\delta_1(x) + \sum_{k=2}^{\infty} 2^k (1 + 2(x \bmod 2^k))\delta_k(x).$$

Here $x \bmod 2^k$ is the least non-negative residue modulo 2^k of the 2-adic integer x.

Ergodicity on 2-adic spheres

New technique to use the van der Put coefficients to study dynamical systems on the 2-adic spheres, where $f: S_{p^{-r}}(a) \to S_{p^{-r}}(a)$, $S_{p^{-r}}(a) = \bigcup_{s=1}^{p-1} \left(a + p^k + p^{k+1}\mathbb{Z}_p\right), a \in \mathbb{Z}_p, r = 1/p^k, k = 1, 2, \ldots$

- Let $p = 2, f: \mathbb{Z}_2 \to \mathbb{Z}_2$ is a compatible function and $S_{2^{-r}}(a)$ is the sphere of radius 2^{-r} with a center at the point $a \in \{0, \ldots, 2^r - 1\}$.
- The function f is invariant on the sphere $S_{2^{-r}}(a)$, i.e. $f: S_{2^{-r}}(a) \to S_{2^{-r}}(a)$.
- For $p = 2$, the sphere $S_{2^{-r}}(a)$ coincide with a ball $U_{2^{-r}}(a + 2^r)$ with center at the point $a + 2^r$, i.e.

$$S_{2^{-r}}(a) = \left\{a + 2^r + 2^{r+1}x | x \in \mathbb{Z}_2\right\} = U_{2^{-r}}(a + 2^r).$$

- Then $f(a + 2^r + 2^{r+1}x) = f(a + 2^r) \bmod 2^{r+1} + 2^{r+1}g(x)$, where the function $g: \mathbb{Z}_2 \to \mathbb{Z}_2$ is compatible because $f(x)$ is compatible by the initial condition.

Ergodicity on 2-adic spheres

Note that $f(a + 2^r + 2^{r+1}x) = f(a + 2^r) \bmod 2^{r+1} + 2^{r+1}g(x)$, where the function $g \colon \mathbb{Z}_2 \to \mathbb{Z}_2$ is compatible because $f(x)$ is compatible by the initial condition.

Theorem

Let $f \colon \mathbb{Z}_2 \to \mathbb{Z}_2$ and

$$f(a + 2^r + 2^{r+1}x) = f(a + 2^r) \bmod 2^{r+1} + 2^{r+1}g(x),$$

where $g \colon \mathbb{Z}_2 \to \mathbb{Z}_2$ is compatible function. The function f is ergodic on the sphere $S_{2^{-r}}(a)$ iff $g(x)$ is ergodic function.

Theorem

Let a compatible function $f \colon \mathbb{Z}_2 \to \mathbb{Z}_2$, represented via the van der Put series

$$f(x) = \sum_{m=0}^{\infty} B_f(m)\chi(m,x) = \sum_{m=0}^{\infty} 2^{\lfloor log_2 m \rfloor} b_m \chi(m,x),$$

Ergodicity on 2-adic spheres

Theorem

Then $f(x)$ is **ergodic on the sphere** $S_{2^{-r}}(a)$ *if and only if the following conditions holds simultaneously:*

1. $$f(a + 2^r + 2^{r+1}x) = a + 2^r \bmod 2^{r+1};$$
2. $$\left|b_f(a + 2^r + m \cdot 2^{r+1})\right|_2 = 2^{-r-1}, m \geq 0;$$
3. $$\frac{b_f(a + 2^r)}{2^{r+1}} + \frac{b_f(a + 2^r + 2^{r+1})}{2^{r+1}} \equiv 3 \bmod 4;$$
4. $$\frac{b_f(a + 2^r + 2^{r+2})}{2^{r+1}} + \frac{b_f(a + 2^r + 3 \cdot 2^{r+1})}{2^{r+1}} \equiv 2 \bmod 4;$$
5. $$\sum_{m=2^{n-1}}^{2^n-1} \frac{b_f(a + 2^r + m \cdot 2^{r+1})}{2^{r+1}} \equiv 0 \bmod 4, n \geq 3.$$

Ergodicity on 2-adic spheres

Example

Let $a \in \{0, 1, \ldots, 2^r - 1\}$. *The function*

$$f(a + 2^r + 2^{r+1}x) = a + 2^r + 2^{r+1}(1 + \delta_0(x) + 6\delta_1(x) + \\ + \sum_{k=2}^{\infty} 2^k (1 + 2(x \bmod 2^k))\delta_k(x)),$$

where $\delta_k(x)$ *is the value of* k*-th binary digit of the number* x, *is ergodic on the sphere* $S_{2^{-r}}(a)$.

Ergodicity on 2-adic spheres

Now we state results for the functions, which are ergodic on the sphere with pertubations of the monomial system.

Theorem

The function $f(x) = x^s + 2^{r+1}u(x)$, *where* $u(x)$ *is compatible function on* $\mathbb{Z}_2$, *is* **ergodic on the sphere** $S_{2^{-r}}(1) = \left\{1 + 2^r + 2^{r+1}x | x \in \mathbb{Z}_2\right\}$ *of sufficiently small radius* ($r \geq 3$) *if and only if* $s \equiv 1 \bmod 4$ *and* $u(1) \equiv 1 \bmod 2$.

Theorem

The function $f(x) = x^s + 2^{r+1}u(x)$, *where* $u(1 + 2^r + 2^{r+1}x) = c + 4d(x)$, *is* **ergodic on the sphere** $S_{2^{-r}}(1) = \left\{1 + 2^r + 2^{r+1}x | x \in \mathbb{Z}_2\right\}, r \geq 3$ *if and only if* $s \equiv 1 \bmod 4$ *and* $c \equiv 1 \bmod 2$ *for any arbitrary function* $d(x)$ *on* $\mathbb{Z}_2$.

Ergodicity on 2-adic spheres

Example

1. *The function* $f(x) = (1+2^3+2^4x)^5 + 2^4$ *is ergodic on the sphere* $S_3(1)$.
2. *The function*

$$f(1+2^r+2^{r+1}x) = (1+2^r+2^{r+1}x)^9 +$$

$$+2^{r+1}\left(1+\sum_{k=2}^{\infty} 2^k \left(1+2(x \bmod 2^k)\right)\delta_k(x)\right)$$

is ergodic on the sphere $S_r(1)$. *It is easy to see that*

$$9 \equiv 1 \bmod 4$$

$$u(1) \equiv 1 \bmod 2 \quad 1+\sum_{k=2}^{\infty} 2^k \left(1+2(x \bmod 2^k)\right)\delta_k(x) \equiv 1 \bmod 2.$$

Ergodicity on 2-adic spheres

Note that

- Here the function $u(x)$ could be any compatible function, i.e. not only differentiable.
- To compare, for the case $p \neq 2$ the ergodicity of the function does not depend on polynom $u(x)$.
- Meanwhile for the case $p = 2$ we have restrictions on the function $u(x)$, namely $u(1) \equiv 1 \bmod 2$.
- And in the case of the monomial function without perturbations, i.e. $u(x) \equiv 0 \bmod 2$, the function $f : x \to x^n$ is not ergodic.
- The transition to the case $p \neq 2$ is still open problem.

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