Inseparable local uniformization

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Outline

General paradigm of resolution of singularities

- The main aim of resolution of singularities is to approximate a singular scheme X with a regular scheme Y by finding a cover map $f : Y \to X$. It is called a *(local, altered, etc.) desingularization*, depending on the type of the covering.
- Grothendieck proved that quasi-excellent (or qe) schemes form the widest class where one can hope to have a consistent desingularization theory. Conversely, it is widely hoped/believed that qe schemes do possess desingularization of a very strong form.

Desingularization

Definition

A *(weak) desingularization* of an integral scheme X is a proper birational morphism $f : Y \to X$ with a regular Y.

Known cases:

- Characteristic zero is the ideal situation: any qe $X/\mathbf{Q}$ possesses a desingularization by blowing up regular centers over X_{sing}. This can be done functorially in all regular morphisms and extends to qe formal schemes and analytic spaces.
- Varieties over k of characteristic $p > 0$: only for $\dim(X) \leq 3$. Perfect k and $p > 3!$ is due to Abhyankar (1966); $[k : k^p] < \infty$ is due to Cossart-Piltant (2008).
- General qe schemes (e.g. over $\mathbf{Z}$): only for surfaces.

Altered desingularization

A very successful way to weaken the classical notion of desingularization was found by de Jong in 1996. It is easy enough to be proved by current methods and yet it covers most of cohomological applications (mainly with divisible coefficients).

Theorem (de Jong)

Any integral scheme X of finite type over a base qe surface S admits an altered desingularization: there exists a regular integral Y with an alteration $f : Y \to X$ (i.e. f is proper and generically finite).

Altered local desingularization

There are weaker but useful versions of desingularization.

Definition

An *altered local desingularization* of an integral scheme X is a morphism $f : Y = \sqcup_{i=1}^{n} Y_i \to X$ such that $Y_1, \dots, Y_n$ are integral and regular, each $L_i = k(Y_i)$ is finite over $K = k(X)$ and any valuation of K with center on X lifts to some Y_i.

For cohomological applications with non-divisible coefficients one wants to control $[L_i : K]$.

- If $L_i = K$ then f is a *local desingularization*.
- f is an *inseparable local desingularization* if L_i/K are purely inseparable.
- f is an *l'-altered local desingularization* if $([L_i : K], l) = 1$.

From local to global

An additional motivation for studying weaker desingularizations is that sometimes they serve as an input for a patching/descent that produces a usual desingularization. Here are two examples.

- For threefolds, local desingularization implies global (Zariski, 1940).
- In characteristic zero, altered desingularization implies desingularization via toroidal quotients (Abramovich-de Jong, 1997).

Known cases

Altered local desingularization is known in the following cases.

- l'-altered local desingularization for any prime l invertible on X: for **all** qe schemes X (Gabber, around 2007).
- Inseparable local desingularization: for all varieties (Temkin, 2008, the main topic of this talk).
- Local desingularization: for varieties X of dimension at most 3 and with $[k(X):k(X)^p] < \infty$ (Cossart-Piltant, 2008).

Inseparable local uniformization

Zariski observed in 1930ies that the Riemann-Zariski space of all valuations on X is quasi-compact, and hence (altered) local desingularization reduces to (altered) desingularization of X along a single valuation. The latter is called *(altered) local uniformization*. Here is the local formulation of our main result.

Theorem (Inseparable local uniformization)

For any valuation ν on an integral variety X there exists a regular integral variety Y with a dominant morphism $Y \to X$ such that ν lifts to Y and $L = k(Y)$ is finite and purely inseparable over $K = k(X)$.

An application to local uniformization

Let p be the exponential characteristic (i.e. $p \in \{1, 2, 3, 5, \dots\}$). Choose n so that $L^{p^n} \subset K$, then K is obtained from L^{p^n} by successive adjoining p-th roots, and the integral closure of $\mathcal{O}_Y^{p^n}$ in K gives rise to a model of ν. It easily follows that

Corollary

To prove local uniformization of ν it suffices to resolve along ν hypersurface singularities of the form $x_{n+1}^p = f(x_1, \dots, x_n)$.

Informally, this type of singularities was always recognized as the important test case, where all "bad things" can happen, but here we do prove a rigorous statement.

Work in progress

The same method can be used to establish the following generalization. We write "conjecture" since most of the proof (a couple of articles) was not written down yet.

Conjecture

For any valuation ν of residue characteristic p on an integral qe scheme X there exists a regular integral scheme Y with a dominant morphism $Y \to X$ such that ν lifts to Y and $L = k(Y)$ is of degree p^n over $K = k(X)$. In equal characteristic one can take L/K either abelian or purely inseparable. In mixed characteristic one can take $L \subset K(x_1^{1/p^n}, \ldots, x_m^{1/p^n})$.

Conventiones

Let us fix some notation.

- k is a trivially valued ground field, $p = \mathrm{exp.char.(k)} \in \{1, 2, 3, 5, \dots\}$.
- K/k is a finitely generated extension of valued fields
- ν is the valuation on K (or K_ν) and h_ν is its height (rank).
- X is a *model* of K_ν (or ν), i.e. X is integral and separated k-variety, $k(X) = K$ and ν is centered on X.
- K_ν° is the ring of integers, $|K_\nu^\times|$ is the group of values and $\widetilde{K}_\nu$ is the residue field.
- $D_\nu = D_{K/k} = \dim(X) - \mathrm{tr.deg.}_k(\widetilde{K}_\nu) - \mathrm{rk}_{\mathbf{Q}}(|K_\nu^\times| \otimes \mathbf{Q})$ is the *transcendental defect* of ν; it is non-negative by Abhyankar's inequality and ν is called *Abhyankar* if $D_\nu = 0$.

Main steps

Proof of the inseparable local uniformization theorem runs in three steps; each step serves as the induction base for the next one:

- Step 1: the case of Abhyankar ν (only the height one case will be used in Step 2).
- Step 2: the case of $h_\nu = 1$ is done by induction on D_ν.
- Step 3: the general case is done by induction on h_ν.

Step 2 is the crucial one, and it is the only step that prevents us from proving the local uniformization. We will briefly discuss each step and then concentrate on Step 2.

Relation to toroidal geometry

- Abhyankar valuations are much easier to work with because the valued field (K, ν) is *defectless*, i.e. any finite extension L/K satisfies $\sum e_i f_i = [L : K]$.
- Local desingularization is known for Abhyankar ν over perfect k (Kuhlmann-Knaf, 2005), but we have to prove slightly more (see the next slide), so we have to provide a new proof.
- The key technique is log-geometry (or toroidal geometry). One easily finds K_0 such that $[K : K_0] < \infty$ and $\nu_0 = \nu|_{K_0}$ is toric, and the main idea is to show that sufficiently fine toric model of ν_0 induces a toroidal model of ν. One essentially uses that K/K_0 is defectless.

Descent and simultaneous local uniformization

Theorem (Simultaneous local uniformization of Abhyankar valuations)

If k is perfect, $K = K_\nu$ is Abhyankar, X is a model of K_ν and $K_1, \dots, K_n$ are finite valued extensions of K, then there exists a finer model $Y \to X$ of K_ν such that the lift of ν to each K_i is centered at a toroidal point $y_i \in Y_i = \mathrm{Nor}_{K_i}(Y)$, and y_1 is even regular.

- One cannot have all y_i's regular (Abhyankar's example).
- We will only need *descent* version in which $n = 1$, but the rest comes for free.
- Equivariant local uniformization is obtained when $n = 1$ and L_1/K is Galois.

Induction on the transcendental defect

- Direct induction on dimension: fix curve fibrations $X = X_d \to \cdots \to X_1 \to X_0$ where $d = D_\nu$ and $\nu|_{k(X_0)}$ is Abhyankar. The defect rank of $\nu_i = \nu|_{k(X_i)}$ is i.
- To uniformize ν_{i+1} apply the induction assumption on ν_i and a (sort of) relative uniformization of ν_i over $S_i = \mathrm{Spec}(k(X_i)^\circ_{\nu_i})$. The latter is possible only after a purely inseparable (or other) alteration of S_i (or X_i), so we have to replace X_i's with their inseparable alterations.
- A similar induction scheme was also used by Kuhlmann and (recently and in other context) by Kedlaya.
- de Jong's and Gabber's works use full fibration by curves (to dimension 0), and D_ν plays no role there.

Induction on height

- ν is composed from ν_1 of height one and a valuation on $\widetilde{K}_{\nu_1}$. Find a fibration $X \to B$ such that ν_1 is centered over the generic point $\eta \in B$ and $\widetilde{\nu} = \nu|_{k(B)}$ is of height $h_\nu - 1$.
- Combine inseparable local uniformizations of ν_1 on the generic fiber X_η and of $\widetilde{\nu}$ on B into an inseparable local uniformization of ν. This involves a lot of approximation technique from [EGA IV, §8] (see next slide).
- A similar argument reduces local uniformization to the height one case.
- For technical reasons (e.g. log-geometry is not developed yet for Berkovich spaces), one loses stronger properties (e.g. descent strengthening) when running Step 3 in the current paper.

Approximation of schemes

- Roughly speaking, [EGA IV, §8] (which I call approximation theory) states that $S = \mathrm{proj}\lim X_i$ exists whenever the transition morphisms are affine, and then f.p. (finitely presented) geometry over S is the direct limit of f.p. geometries over X_i.
- Often, this is the only way to generalize known results to non-noetherian schemes, e.g. $S = \mathrm{Spec}(K_\nu^\circ)$.
- One makes an essential use of a simple but very important observation that $S \widetilde{\to} \mathrm{proj}\lim X_i$ where X_i are all (affine) models of ν. In particular, one can use approximation to pass from varieties to f.p. S-schemes and vice versa.

The induction step in Step 2: an outline

- X is a model of K_ν, $X \to Y$ is a curve fibration, $L = k(Y)$, $\mu = \nu|_L$, $D_{K/L} = 1$.
- The valued field L_μ is inseparably uniformizable by induction. We want to uniformize K_ν using this and relative uniformization of K_ν on the S-curve $X_S = X \times_Y S$ for $S = \mathrm{Spec}(L^\circ_\nu)$. In addition to (standard) approximation arguments this requires two steps.
- Step (a): uniformize the completion $\widehat{K}_\nu$ on the formal $\widehat{S}$-curve $\widehat{X}_S$ (using Berkovich geometry).
- Step (b): use decompletion to uniformize K_ν on X_S and deduce the result for X (the main trouble is non-henselianity of K_ν).

Inseparable uniformization of valued fields

Theorem (Inseparable very local uniformization)

There exist purely inseparable extensions l/k and L/lK and a transcendence basis $x_1, \dots, x_n \in L$ such that L is unramified over $l(x_1, \dots, x_n)$.

- This follows from inseparable local uniformization of ν and can be viewed as its "very local" version.
- Many other local uniformization results/conjectures imply analogous very local results. Converse implication is not automatic, but I expect that the main difficulty is hidden already in the very local versions.
- To simplify exposition, we will use very local formulations to illustrate Steps (a) and (b).

Main analytic ingredient

Theorem (Inseparable uniformization of analytic one-dimensional extensions)

For any one-dimensional extension K/k of equicharacteristic analytic fields with $D_{K/k} = 1$ (e.g. $\widehat{K}_\nu/\widehat{L}_\mu$) there exists a finite purely inseparable l/k and finite separable m/l such that lK is unramified over a subfield of the form $\widehat{m(x)}$ for $x \in lK$.

- A non-trivial extension l/k must be present. So, having purely inseparable (or, as an alternative, abelian of degree p^n) extension l/k is the best we can hope for.
- The proof is a hard computation with analytic fields.
- The same proof shows that in all characteristics the same is true for some $l \subseteq k'$, where k'/k is a fixed extension with deeply ramified k'.

Deeply ramified fields

- An analytic field l of residue characteristic p is *deeply ramified* if (i) l is not discrete, and (ii) $(l^\circ)^p + pl^\circ = l^\circ$ (i.e. frobenius is surjective on $l^\circ/(p)$). (If $\mathrm{char}(l) = p$ then (ii) means that l is perfect.)
- If $\mathrm{char}(k) = p$ then k has both abelian and purely inseparable algebraic extensions l with deeply ramified $\widehat{l}$; if $\mathrm{char}(k) = 0$ then one can take $l = k(x_1^{1/p^\infty}, x_2^{1/p^\infty}, \dots)$.
- We essentially use the following (characterizing) property of a deeply ramified l: if m/l is analytic and $\inf|m-\alpha| < \inf|l-\alpha|$ for $\alpha \in l^a$ then $l(\alpha)$ embeds into m.
- Other such properties are: (i) any algebraic extension of l has zero different (i.e. is almost étale in the sense of Faltings), (ii) $\Omega^1_{m^\circ/l^\circ} = 0$ for finite separable m/l.

Decompletion

Very local analytic uniformization easily implies the following valued analog.

Theorem

If K/k is a one-dimensional extension of valued fields with $D_{K/k} = 1$ then there exists a finite purely inseparable extension l/k, a finite extension m/l, and an unramified extension L/lK, such that L is unramified over a subfield of the form $m(x)$.

- The extension m/l is usually ramified, and we cannot split off m as a subfield of lK because K_ν does not have to be henselian anymore.
- Thus, we only manage to uniformize ν étale-locally, and some descent technique is still required.

The descent trick

- The descent is enabled by the fact that we prove (by induction) the descent version of inseparable local uniformization. In our case, $k = k(Y)$ and we apply the induction assumption to Y and the extension m/k.
- This trick seems to be of global geometric nature. It seems that there is no meaningful formulation of descent very local uniformization.
- Oversimplifying, we find a smooth morphism $T \to Z$ between models of $L = m(x)$ and m. If a finer model $Y' \to Y$ is such that $Z' = \mathrm{Nor}_m(Y')$ refines Z and m is centered on its regular point, then Y' also induces uniformization of L, and hence (by étaleness of $L^\circ/(lK)^\circ$) also of lK.

Conjecture: the strongest form

Conjecture

Assume that X is an integral qe scheme, ν is a valuation of $K = k(X)$ centered on X, $K_1, \dots, K_n$ are finite valued extensions of K_ν, and $\overline{L}/K$ is an algebraic deeply ramified extension. Then:

(i) There exist a finer model $X' \to X$ of K_ν, a finite subextension $L \subseteq \overline{L}$ and extensions of the valuations to $L_i = LK_i$ which are centered on toroidal points $y_i \in \mathrm{Nor}_{L_i}(X')$. Moreover, y_1 can be chosen regular.

(ii) If ν is Abhyankar then can take $L = K$ in (i), i.e. simultaneous local uniformization holds for such ν.