Introduction to Model Theory

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Outline

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First order languages

A first order language $\mathcal{L}$ is given by

- **constant symbols** $\{c_i\}_{i \in I}$;
- **relation symbols** $\{R_j\}_{j \in J}$ (R_j of some fixed arity n_j);
- **function symbols** $\{f_k\}_{k \in K}$ (f_k of some fixed arity n_k);
- a distinguished binary relation "=" for **equality**;
- an infinite set of **variables** $\{v_i \mid i \in \mathbb{N}\}$ (we also use x, y etc.);
- the **connectives** $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and
- the **quantifiers** $\forall, \exists$.

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First order languages (continued)

$\mathcal{L}$-formulas are built inductively (in the obvious manner).

Let φ be an $\mathcal{L}$-formula.

- A variable x is **free** in φ if it is not bound by a quantifier.
- φ is called a **sentence** if it contains no free variables.
- We write $\varphi = \varphi(x_1, \ldots, x_n)$ to indicate that the free variables of φ are among $\{x_1, \ldots, x_n\}$.

In what follows, we will only consider **countable** languages.

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First order structures

Definition

An $\mathcal{L}$-structure $\mathcal{M}$ is a tuple $\mathcal{M} = (M; c_i^{\mathcal{M}}, R_j^{\mathcal{M}}, f_k^{\mathcal{M}})$, where

- M is a non-empty set, the **domain** of $\mathcal{M}$;
- $c_i^{\mathcal{M}} \in M$, $R_j^{\mathcal{M}} \subseteq M^{n_j}$, and $f_k^{\mathcal{M}} : M^{n_k} \to M$
  are **interpretations** of the symbols in $\mathcal{L}$.

To interpret an $\mathcal{L}$-formula φ in $\mathcal{M}$, note that the quantified variables **run over** M.

Let $\varphi(x_1, \ldots, x_n)$ and $\bar{a} \in M^n$ be given.

We set $\mathcal{M} \models \varphi(\bar{a})$ if and only if φ holds for $\bar{a}$ in $\mathcal{M}$.

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Examples of languages and structures

- $\mathcal{L}_{rings} = \{0, 1, +, -, \cdot\}$ **(language of rings)**.
  Any (unitary) ring is naturally an $\mathcal{L}_{rings}$-structure, e.g. $\mathcal{C} = (\mathbb{C}; 0, 1, +, -, \cdot)$ and $\mathcal{R} = (\mathbb{R}; 0, 1, +, -, \cdot)$.
  $\varphi \equiv \forall x \exists y\, y \cdot y = x$ is an $\mathcal{L}_{rings}$-formula (even a sentence), with $\mathcal{C} \models \varphi$ and $\mathcal{R} \models \neg\varphi$.
- $\mathcal{L}_{\mathrm{oag}} = \{0, +, <\}$ **(language of ordered abelian groups)**
  Let $\mathcal{Z} = (\mathbb{Z}; 0, +, <)$ and $\mathcal{Q} = (\mathbb{Q}; 0, +, <)$.
  Let $\psi(x, y) \equiv \exists z(x < z \wedge z < y)$.
  Then $\mathcal{Q} \models \psi(1, 2)$, $\mathcal{Z} \not\models \psi(1, 2)$ and $\mathcal{Z} \models \psi(0, 2)$.

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First order theories

An $\mathcal{L}$-theory T is a set of $\mathcal{L}$-sentences.

- An $\mathcal{L}$-structure $\mathcal{M}$ is a **model** of T if $\mathcal{M} \models \varphi$ for every $\varphi \in T$. We denote this by $\mathcal{M} \models T$.
- T is called **consistent** if it has a model.

Examples

1. The usual field axioms, in $\mathcal{L}_{rings}$, give rise a theory T_{fields}, with $\mathcal{M} \models T_{fields}$ if and only if $\mathcal{M} = (M; 0, 1, +, -, \cdot)$ is a field.
2. Let $\varphi_n \equiv \forall z_0 \cdots \forall z_{n-1} \exists x\, x^n + z_{n-1}x^{n-1} + \ldots + z_0 = 0$. **ACF**$= T_{fields} \cup \{\varphi_n \mid n \geq 2\}$. (Models are **alg. closed fields**.)
3. There is an $\mathcal{L}_{oag}$-theory **DOAG** whose models are preciseley the **non-trivial divisible ordered abelian groups**.
4. If $\mathcal{M}$ is an $\mathcal{L}$-structure, $\mathrm{Th}(\mathcal{M}) = \{\varphi\ \mathcal{L}\text{-sentence} \mid \mathcal{M} \models \varphi\}$.

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The expressive power of first order logic

Theorem (Compactness Theorem)

Let T be a theory. Suppose that any finite subtheory T_0 of T has a model. Then T has a model.

Corollary

1. *If T has arbitrarily large finite models, it has an infinite model. Thus, there is e.g. no theory whose models are the finite fields.*
2. *If T has an infinite model, it has models of arbitrarily large cardinality. In particular, an infinite $\mathcal{L}$-structure is not determined (up to $\mathcal{L}$-isomorphism) by its theory.*

To prove (1), consider $\psi_n \equiv \exists x_1, \ldots, x_n \bigwedge_{i<j} x_i \neq x_j$, and apply compactness to $T' = T \cup \{\psi_n \mid n \in \mathbb{N}\}$.

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1. *If T has arbitrarily large finite models, it has an infinite model. Thus, there is e.g. no theory whose models are the finite fields.*
2. *If T has an infinite model, it has models of arbitrarily large cardinality. In particular, an infinite $\mathcal{L}$-structure is not determined (up to $\mathcal{L}$-isomorphism) by its theory.*

To prove (1), consider $\psi_n \equiv \exists x_1, \ldots, x_n \bigwedge_{i<j} x_i \neq x_j$, and apply compactness to $T' = T \cup \{\psi_n \mid n \in \mathbb{N}\}$.

Complete theories

Let T be a theory. A sentence ψ is a **consequence** of T, denoted $T \models \psi$, if every model of T is also a model of ψ.

$\mathcal{M}$ and $\mathcal{N}$ are called **elementarily equivalent** if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. We write $\mathcal{M} \equiv \mathcal{N}$.

A consistent theory T is complete if all its models are elementarily equivalent. Alternatively, for every φ, either $T \models \varphi$ or $T \models \neg\varphi$.

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Definable sets

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A set $D \subseteq M^n$ is said to be **definable** if there is a formula $\varphi(\overline{x}, \overline{y})$ and parameters $\overline{b}$ from M such that

$$D = \varphi(\mathcal{M}, \overline{b}) := \left\{ \overline{a} \in M^n \mid \mathcal{M} \models \varphi(\overline{a}, \overline{b}) \right\}.$$

If $\overline{b}$ may be taken from $B \subseteq M$, we say D is B-definable.

Convenient to add parameters, passing to $\mathcal{L}_B = \mathcal{L} \cup \{c_b \mid b \in B\}$. Then $\mathcal{M}$ expands naturally to an $\mathcal{L}_B$-structure $\mathcal{M}_B$.

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   Let $K \models ACF$, and let $V = V(K) \subseteq K^n$ be an affine variety. Then V is definable in $\mathcal{L}_{rings}$ by a quantifier-free formula. More generally, this is the case for every constructible subset of K^n.

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Elementary substructures

- $\mathcal{M} \subseteq \mathcal{N}$ is a **substructure** if

$$c^{\mathcal{M}} = c^{\mathcal{N}}, f^{\mathcal{N}} \restriction_{M^n} = f^{\mathcal{M}} \text{ and } R^{\mathcal{N}} \cap M^n = R^{\mathcal{M}}.$$

- We say $\mathcal{M}$ is an **elementary** substructure of $\mathcal{N}$, $\mathcal{M} \preccurlyeq \mathcal{N}$ if for every $\mathcal{L}$-formula $\varphi(\overline{x})$ and every tuple $\overline{a} \in M^n$ one has

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In other words, the embedding respects all definable sets.

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Quantifier elimination

Definition

A theory T has quantifier elimination (QE) if for every formula $\varphi(\overline{x})$ there is a quantifier free (q.f.) formula $\psi(\overline{x})$ such that

$$T \models \forall \overline{x}\, (\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))\,.$$

Proposition

Let T be a (consistent) theory with QE.

- *In $\mathcal{M} \models T$, every definable set is q.f. definable. Equivalently, projections of q.f. definable sets are q.f. definable.*
- *Let $\mathcal{M}$ and $\mathcal{N}$ be models of T. Then $\mathcal{M} \subseteq \mathcal{N} \Rightarrow \mathcal{M} \preccurlyeq \mathcal{N}$. ($T$ is* **model complete***).*
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Examples of theories with QE

Theorem (Chevalley-Tarski Theorem)

ACF has quantifier elimination.

Corollary

In algebraically closed fields, a set is definable iff it is constructible.

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ACF_p *is complete and* **strongly minimal**: *in every model* $\mathcal{M} \models \mathrm{ACF}_p$, *every definable subset of* M *is finite or cofinite.*

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Model-completeness of ACF

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The theory of the real field $\mathcal{R} = (\mathbb{R}, 0, 1, +, -, \cdot)$ does not have QE. (The set of squares is not q.f.-definable.)

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Let $\mathcal{L}_{o.rings} = \mathcal{L}_{rings} \cup \{<\}$, and let **RCF** (the **theory of real closed fields**) be the $\mathcal{L}_{o.rings}$-theory whose models are

- **ordered fields** F such that
- every positive element in F is a square in F and
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Theorem (Tarski 1951)

RCF is complete (so equal to $Th(\mathbb{R})$) and has QE.

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The definable sets in RCF are precisely the **semi-algebraic sets** *(sets defined by boolean combinations of polynomial inequalities).*

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0-minimal theories

Definition

Let $\mathcal{L} = \{<, \ldots\}$. An $\mathcal{L}$-theory T is *o*-**minimal** if in any $M \models T$, any definable subset of M is a finite union of intervals and points.

Corollary

RCF is an o-minimal theory.

Proof.

Clearly, $p(X) \geq 0$ defines a set of the right form, for p a polynomial. We are done by Tarski's QE result. □

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The notion of a complete type

Definition

Let $\mathcal{M}$ be a structure and $B \subseteq M$. A set $p(\overline{x})$ of $\mathcal{L}_B$-formulas $\varphi(x_1, \ldots, x_n)$ is a (complete) ***n*-type over B** if

- $p(\overline{x})$ is finitely satisfiable, i.e. for any $\varphi_1, \ldots, \varphi_k \in p$ there is $\overline{a} \in M^n$ such that $\mathcal{M} \models \varphi_i(\overline{a})$ for all i;
- $p(\overline{x})$ is maximal with this property.

Example

Let $\mathcal{N} \succcurlyeq \mathcal{M}$. For $\overline{a} \in N^n$, $\mathrm{tp}(\overline{a}/B) := \{\varphi(\overline{x}) \in \mathcal{L}_B \mid \mathcal{N} \models \varphi(\overline{a})\}$ is a complete n-type over B, the **type of $\overline{a}$ over** B.

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Every complete type p is of the form $p(\overline{x}) = \mathrm{tp}(\overline{a}/B)$. Such a tuple $\overline{a}$ is called a **realisation of** p.

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Every complete type p is of the form $p(\overline{x}) = \mathrm{tp}(\overline{a}/B)$.
Such a tuple $\overline{a}$ is called a **realisation of** *p.*

The notion of a complete type

Definition

Let $\mathcal{M}$ be a structure and $B \subseteq M$. A set $p(\overline{x})$ of $\mathcal{L}_B$-formulas $\varphi(x_1, \ldots, x_n)$ is a (complete) ***n*-type over B** if

- $p(\overline{x})$ is finitely satisfiable, i.e. for any $\varphi_1, \ldots, \varphi_k \in p$ there is $\overline{a} \in M^n$ such that $\mathcal{M} \models \varphi_i(\overline{a})$ for all i;
- $p(\overline{x})$ is maximal with this property.

Example

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- For $B \subseteq M$, let $S_n^{\mathcal{M}}(B)$ be the set of complete n-types over B.
- $\mathcal{M} \preccurlyeq \mathcal{N} \Rightarrow S_n^{\mathcal{M}}(B) = S_n^{\mathcal{N}}(B)$ canonically, so we write $S_n(B)$.
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  The sets U_φ form a **basis of clopen sets** for a topology on $S_n(B)$, the space of complete n-types over B, a profinite space.

Example (Type spaces in ACF)

Let $K \models \mathrm{ACF}$ and let $K_0 \subseteq K$ be a subfield. Then, by QE,

$$S_n(K_0) \cong \mathrm{Spec}(K_0[x_1, \ldots, x_n]), \text{ via}$$

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Let T be *o*-minimal (e.g. $T = \mathrm{DOAG}$ or RCF) and $\mathcal{D} \models T$.

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Definition

Let κ be an infinite cardinal. An $\mathcal{L}$-structure $\mathcal{M}$ is κ-**saturated** if for every $B \subseteq M$ with $|B| < \kappa$, every $p \in S_n(B)$ is realised in $\mathcal{M}$.

Remark

It is enough to check the condition for $n = 1$.

Examples

1. $K \models \mathrm{ACF}$ is κ-saturated if and only if $\mathrm{tr.deg}(K) \geq \kappa$.
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Let $K \models \mathrm{ACF}$. Then K is $|K|$-homogeneous.

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Let κ and $\mathcal{M}$ be given. There exists an elementary extension $\mathcal{N} \succcurlyeq \mathcal{M}$ which is κ-saturated and κ-homogeneous.

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The Universe

Let T be complete and κ a very big cardinal.

A universe $\mathcal{U}$ for T is a κ-saturated and κ-homogeneous model.

When working with a universe $\mathcal{U}$,

- "**small**" means "of cardinality $< \kappa$";
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Definable and algebraic closure I

Definition

Let $B \subseteq \mathcal{U}$ be a set of parameters and $a \in \mathcal{U}$.

- a is **definable over** B if $\{a\}$ is a B-definable set;
- a is **algebraic over** B if there is a finite B-definable set containing a.
- The **definable closure of** B is given by

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$$\mathrm{dcl}(B) = \{a \in \mathcal{U} \mid a \text{ definable over } B\}.$$

- Similarly define $\mathrm{acl}(B)$, the **algebraic closure of** B.

Definable and algebraic closure II

Examples

- In **ACF**, if K denotes the field generated by B, then $\mathrm{dcl}(B) = K^{1/p^\infty}$ and $\mathrm{acl}(B) = K^{alg}$.
- In **DOAG**, $\mathrm{dcl}(B) = \mathrm{acl}(B)$ is the divisible hull of $\langle B \rangle$.
- In **RCF**, $\mathrm{dcl}(B) = \mathrm{acl}(B)$ equals the real closure of the field generated by B.

Fact

1. $a \in \mathrm{dcl}(B)$ *if and only if* $\sigma(a) = a$ *for all* $\sigma \in \mathrm{Aut}_B(\mathcal{U})$
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A criterion for QE

The following criterion is often useful in practice.

We will use it in the context of valued fields.

Theorem

Let T be a theory and κ an infinite cardinal. TFAE:

1. *T has QE.*
2. *Let $\mathcal{A} \subseteq \mathcal{M}, \mathcal{N} \models T$. Assume*
   - *$|M| < \kappa$ and*
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   Then $\mathcal{M}$ may be embedded into $\mathcal{N}$ over $\mathcal{A}$.

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Outline

Valued fields: notations and choice of a language

Let K be a valued field. We use standard notation:

- $\mathrm{val} : K^\times \to \Gamma$ (the **valuation map**)
- $\Gamma = \Gamma_K$ is an ordered abelian group (written additively), plus a distinguised element ∞ ($+$ and $<$ are extended as usual);
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- For $a \in K$ and $\gamma \in \Gamma$ denote $B_{\geq\gamma}(a)$ (resp. $B_{>\gamma}(a)$) the **closed** (resp. **open**) **ball** of radius γ around a.
- K gives rise to an $\mathcal{L}_{\mathrm{div}} = \mathcal{L}_{rings} \cup \{\,\mathrm{div}\,\}$-structure, via
$$x \,\mathrm{div}\, y :\Leftrightarrow \mathrm{val}(x) \leq \mathrm{val}(y).$$
- $\mathcal{O}_K = \{x \in K \,:\, x \,\mathrm{div}\, 1\}$, so $\mathcal{O}_K$ is $\mathcal{L}_{\mathrm{div}}$-definable
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QE in algebraically closed valued fields

ACVF: $\mathcal{L}_{\mathrm{div}}$-theory of **alg. closed non-trivially valued fields**

Theorem (Robinson)

The theory ACVF has QE. Its completions are given by $\mathrm{ACVF}_{p,q}$*, for* $(p, q) = (char(K), char(k))$.

Corollary

1. *In ACVF, a set is definable iff it is* **semi-algebraic***, i.e. a boolean combination of sets given by polynomial equations and valuation inequalities.*
2. *In particular, definable sets in 1 variable are (finite) boolean combinations of singletons and balls.*
3. *If* $K_0 \subseteq K \models \mathrm{ACVF}$ *is a subfield, then* $\mathrm{acl}(K_0) = K_0^{alg}$ *and* $\mathrm{dcl}(K_0) = \left(K_0^{1/p^\infty}\right)^h$.

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Classification of purely transcendental extensions

For $i = 1, 2$, let $L_i = K(t_i)$ be valued fields, with $t_i \notin K = K^{alg}$.

- (**residual** case) If $\mathrm{val}(t_i) = 0$ and $\mathrm{res}(t_i) \notin k_K$ for $i = 1, 2$, then $t_1 \mapsto t_2$ induces an isomorphism $L_1 \cong_K L_2$.
- (**ramified** case) If $\gamma_i = \mathrm{val}(t_i) \notin \Gamma_K$ for $i = 1, 2$, and γ_1 and γ_2 define the same cut in Γ_K, then $L_1 \cong_K L_2$ via $t_1 \mapsto t_2$.
- (**immediate** case) If there is a pseudo-Cauchy sequence (a_ρ) in K without pseudo-limit in K such that $a_\rho \Rightarrow t_i$ for $i = 1, 2$, then $L_1 \cong_K L_2$ via $t_1 \mapsto t_2$.

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The proof of QE in ACVF

We use the criterion.

Let $L, L^* \models \mathrm{ACVF}$, and $A \subseteq L, L^*$ a common $\mathcal{L}_{\mathrm{div}}$-substructre. Assume L is countable and L^* is $\aleph_1$-saturated. We have to show that L embeds into L^* over A.

- WMA $A = K$ is a field. (Easy)
- WMA $K = K^{alg}$. (Extensions of $\mathcal{O}_K$ to K^{alg} are $\mathrm{Gal}(K)$-conj.)

  $\Rightarrow$ Enough to K-embed $K(t)$ into L^*, for $t \notin K = K^{alg}$:
- $K(t)/K$ is either residual, or ramified, or immediate.
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- The other cases are treated similarly. $\square$

The proof of QE in ACVF

We use the criterion.

Let $L, L^* \models \mathrm{ACVF}$, and $A \subseteq L, L^*$ a common $\mathcal{L}_{\mathrm{div}}$-substructre. Assume L is countable and L^* is $\aleph_1$-saturated. We have to show that L embeds into L^* over A.

- WMA $A = K$ is a field. (Easy)
- WMA $K = K^{alg}$. (Extensions of $\mathcal{O}_K$ to K^{alg} are Gal(K)-conj.)

  $\Rightarrow$ Enough to K-embed $K(t)$ into L^*, for $t \notin K = K^{alg}$:
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Multi-sorted languages and structures

A multi-sorted language $\mathcal{L}$ is given by

- a non-empty family of **sorts** $\{S_i \mid i \in I\}$;
- **constants** c, where c specifies the sort $S_{i(c)}$ it belongs to;
- **relation symbols** $R \subseteq S_{i_1} \times \cdots \times S_{i_n}$, for $i_1, \ldots, i_n \in I$;
- **function symbols** $f : S_{i_1} \times \cdots \times S_{i_n} \to S_{i_0}$;
- **variables** $(v_j^i)_{j \in \mathbb{N}}$ running over the sort S_i (for every i).

$\mathcal{L}$-formulas are built in the obvious way.

An **$\mathcal{L}$-structure** $\mathcal{M}$ is given by

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A variant: valued fields in a three-sorted language

Let $\mathcal{L}_{k,\Gamma}$ be the following 3-sorted language, with sorts K, Γ and k:

- Put $\mathcal{L}_{rings}$ on K, $\{0, +, <, \infty\}$ on Γ and $\mathcal{L}_{rings}$ on k;
- $\mathrm{val} : K \to \Gamma$, and
- $\mathrm{RES} : K^2 \to k$ as additional function symbols.

A valued field K is naturally an $\mathcal{L}_{k,\Gamma}$-structure, via

$$\mathrm{RES}(x, y) := \begin{cases} \mathrm{res}(xy^{-1}), & \text{if } \mathrm{val}(x) \geq \mathrm{val}(y) \neq \infty; \\ 0 \in k, & \text{else.} \end{cases}$$

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ACVF in the three-sorted language

Theorem

ACVF eliminates quantifiers in $\mathcal{L}_{k,\Gamma}$.

Remark

The proof is similar to the one in the one-sorted context (in $\mathcal{L}_{\mathrm{div}}$).

Corollary

In ACVF, the following holds:

1. Γ *is a* **pure divisible ordered abelian group**: *any definable subset of Γ^n is $\{0,+,<\}$-definable (with parameters from Γ).*
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Lemma

The class of henselian valued fields is axiomatisable in $\mathcal{L}_{k,\Gamma}$.

Theorem (Ax-Kochen, Eršov)

Let K and K' be henselian valued fields of equicharacteristic 0. Then, the following holds:

1. $K \equiv K'$ *iff* $k \equiv k'$ *and* $\Gamma \equiv \Gamma'$;
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A general transfer principle

Corollary

For any $\mathcal{L}_{k,\Gamma}$-sentence φ there is $N \in \mathbb{N}$ s.t. for any $p > N$,

$$\mathbb{Q}_p \models \varphi \quad \textit{iff} \quad \mathbb{F}_p((t)) \models \varphi.$$

Idea of the proof.

Else, applying compactness, one may find henselian valued fields K, K' of equicharacteristic 0 with $\Gamma \cong \Gamma' \equiv \mathbb{Z}$ and $k \cong k'$ such that $K \models \varphi$ and $K' \models \neg\varphi$, contradicting the AKE principle. □

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Ever since the **approximate solution** *to* **Artin's Conjecture**, *this kind of transfer principle has shown to be extremely powerful.*

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QE in p-adic fields

Let $\mathcal{L}_{\mathrm{Mac}} = \mathcal{L}_{rings} \cup \{P_n \mid n \geq 1\}$, with P_n a new unary predicate.

Any field K gets an $\mathcal{L}_{\mathrm{Mac}}$-structure, letting $P_n(x) \leftrightarrow \exists y\, y^n = x$.

If $K = \mathbb{Q}_p$, then $\mathbb{Z}_p$ is $\mathcal{L}_{\mathrm{Mac}}$-definable in a quantifier-free way:

$$x \in \mathbb{Z}_p \iff \mathbb{Q}_p \models P_2(1 + px^2) \quad (\text{assume } p \neq 2)$$

Theorem (Macintyre)

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Angular component maps

A map $\mathrm{ac} : K \to k$ is an angular component if

- $\mathrm{ac}(0) = 0$;
- $\mathrm{ac} \restriction_{K^\times} : K^\times \to k^\times$ is a group homomorphism;
- $\mathrm{val}(x) = 0 \Rightarrow \mathrm{ac}(x) = \mathrm{res}(x)$.

Example

In $K = k((\Gamma))$, mapping an element to its **leading coefficient** defines an angular component map. (This also works in $\mathbb{Q}_p$.)

Fact

1. *Let $s : \Gamma \to K^\times$ be a cross-section (homomorphic section of val). Then $\mathrm{ac}(a) := \mathrm{res}\left(s(a)^{-1}a\right)$ is an angular component.*
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Relative QE in Pas' language

Let $\mathcal{L}_{\mathrm{Pas}} = \mathcal{L}_{k,\Gamma} \cup \{\mathrm{ac}\}$, where $\mathrm{ac} : K \to k$.

Let T_{Pas} be the $\mathcal{L}_{\mathrm{Pas}}$-theory of **henselian** valued fields of **equicharacteristic 0** with an angular component map.

Theorem (Pas)

T_{PAS} *admits* **elimination of field quantifiers**:

If $\varphi(\overline{x}_K, \overline{x}_\Gamma, \overline{x}_k)$ is an $\mathcal{L}_{\mathrm{Pas}}$-formula, with variables $\overline{x}_K$, $\overline{x}_\Gamma$ and $\overline{x}_k$ running over the sorts K, Γ and k, respectively, there is an $\mathcal{L}_{\mathrm{Pas}}$-formula $\psi(\overline{x}_K, \overline{x}_\Gamma, \overline{x}_k)$ without field quantifiers such that φ and ψ are equivalent modulo T_{Pas}.

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The map ac is not definable in $\mathcal{L}_{k,\Gamma}$. Thus, passing from $\mathcal{L}_{k,\Gamma}$ to $\mathcal{L}_{\mathrm{Pas}}$ leads to more definable sets.

Extensions to valued difference fields

A **valued difference field** is a valued field K together with a distinguished automorphism $\sigma \in \mathrm{Aut}(K)$.

$\Rightarrow$ get induced automorphisms σ_Γ on Γ and σ_{res} on k.

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Outline

Context

- $\mathcal{L}$ is some countable language (possibly many-sorted);
- T is a **complete** $\mathcal{L}$-theory;
- $\mathcal{U} \models T$ is a fixed **universe** (i.e. very saturated and homogeneous);
- all models $\mathcal{M}$ we consider (and all parameter sets A) are **small**, with $\mathcal{M} \preccurlyeq \mathcal{U}$;
- there is a **dominating sort** S_{dom}: for every sort S from $\mathcal{L}$ there is $n \in \mathbb{N}$ and an n-ary function π_S in $\mathcal{L}$,

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such that $\pi_S^{\mathcal{U}}$ is surjective.
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Imaginary Sorts and Elements

Definition

An imaginary element in $\mathcal{U}$ is an equivalence class d/E, where E is a definable equivalence relation on some $D \subseteq_{def} U^n$ and $d \in D(\mathcal{U})$.

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Unordered Tuples

- In any theory, the formula

$$(x = x' \wedge y = y') \vee (x = y' \wedge y = x')$$

defines an equiv. relation $(x, y)E_2(x', y')$ on pairs, with

$$(a, b)E_2(a', b') \Leftrightarrow \{a, b\} = \{a', b'\}.$$

Thus, $\{a, b\}$ may be thought of as an imaginary element.

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Example (Cosets)

Let $(G, \cdot)$ be definable group in $\mathcal{U}$, and let $H \leq G$ a definable subgroup of G. Then any coset $g \cdot H$ is an imaginary.

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There is a canonical way, due to S. Shelah, of expanding

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For any $\emptyset$-definable equivalence relation E on S_{dom}^n we add

- a new **imaginary sort** S_E (S_{dom} is called the **real sort**), a new function symbol $\pi_E : S_{dom}^n \to S_E$
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Existence of codes for definable sets in $\mathcal{U}^{eq}$

Fact

For any definable $D \subseteq \mathcal{U}^n$ there exists $c \in \mathcal{U}^{eq}$ such that $\sigma \in \mathrm{Aut}(\mathcal{U})$ fixes D setwise iff it fixes c.

Proof.
Suppose D is defined by $\varphi(\overline{x}, \overline{d})$. Define an equivalence relation

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Then $c := \overline{d}/E$ serves as a code for D. $\square$

We sometimes write $\ulcorner D \urcorner = \ulcorner \varphi(\overline{x}, \overline{b}) \urcorner$ for this code (it is unique up to interdefinability).

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Galois Correspondence in T^{eq}

The definitions of definable / algebraic closure make sense in $\mathcal{U}^{eq}$.
We write dcl^{eq} or acl^{eq} to stress that we work in $\mathcal{U}^{eq}$.

- For $B \subseteq \mathcal{U}^{eq}$, any $\sigma \in \mathrm{Aut}_B(\mathcal{U})$ fixes $\mathrm{acl}^{eq}(B)$ setwise.
- $\mathrm{Gal}(B) := \{\sigma \restriction_{\mathrm{acl}^{eq}(B)} \mid \sigma \in \mathrm{Aut}_B(\mathcal{U})\}$ is called the absolute Galois group of B.

Theorem (Poizat)

The map

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Elimination of Imaginaries

Definition (Poizat)

The theory T eliminates imaginaries if every imaginary element $a \in \mathcal{U}^{eq}$ is interdefinable with a real tuple $\overline{b} \in \mathcal{U}^n$.

Fact

- *Suppose that for every $\emptyset$-definable equivalence relation E on $\mathcal{U}^n$ there is an $\emptyset$-definable function*

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such that $E(\overline{a}, \overline{a}')$ if and only if $f(\overline{a}) = f(\overline{a}')$.

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Examples of theories which eliminate imaginaries

1. T^{eq} (for an arbitrary theory T)
2. ACF (Poizat)

   This follows from

   - the existence of a **smallest field of definition** of a variety, and
   - the fact that **finite sets** can be coded using **symmetric functions**, e.g. $\{a, b\}$ is coded by $(a + b, ab)$.
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3. RCF (see the following slides)

Theorem (Definable choice in RCF)

Let $R \models \mathrm{RCF}$ and let $(D_a)_{a \in R^k}$ be a definable family of non-empty subsets of R^n. Then there is a definable function $f : R^k \to R^n$ s.t. $f(a) \in D_a \; \forall \, a \in R^k$. Furthermore, if $D_a = D_b$, then $f(a) = f(b)$.

Proof.

Projecting and using induction, it suffices to treat the case $n = 1$.

D_a is a finite union of intervals. Let I be the leftmost interval.

- If I is reduced to a point, we let $f(a)$ be this point;
- if $I = R$, let $f(a) = 0$;
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Elimination of imaginaries in RCF and in DOAG

Corollary

The theory RCF eliminates imaginaries.

In proving definable choice, we only used that the theory is an *o*-minimal expansion of DOAG (with some non-zero element named). From this, one may easily infer the following.

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DOAG eliminates imaginaries. More generally, any o-minimal expansion of DOAG eliminates imaginaries.

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Utility of Elimination of Imaginaries

T has EI $\Rightarrow$ many constructions may be done already in T:

- quotient objects are present in $\mathcal{U}$
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  $\Rightarrow$ easier to classify e.g. interpretable groups and fields in $\mathcal{U}$;
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In search for imaginaries in ACVF

Consider $K \models \mathrm{ACVF}$ (in $\mathcal{L}_{\mathrm{div}}$).

- Clearly, k and Γ are imaginary sorts, i.e. $k, \Gamma \subseteq K^{eq}$.
- More generally, $\mathcal{B}^{o}$ and $\mathcal{B}^{cl}$ (the set of open / closed balls) are imaginary sorts.

Fact

There is no definable bijection between k and a subset of K^n, similarly for Γ instead of k.

Proof idea.

- By QE, any infinite def. subset of K contains an open ball.
- Thus, every infinite definable subset of K^n admits definable maps with infinite image to k as well as to Γ.
- But, using QE in $\mathcal{L}_{k,\Gamma}$, it is easy to see that every definable subset of $k \times \Gamma$ is a finite union of rectangles $D \times E$. □

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In search for imaginaries in ACVF (continued)

Question

Does (K, k, Γ) eliminate imaginaries (in $\mathcal{L}_{k,\Gamma}$)?

- The answer is NO (Holly).
- The answer is NO even if in addition $\mathcal{B}^o$ and $\mathcal{B}^{cl}$ are added. (Haskell-Hrushovski-Macpherson)

Sketch: Let $\gamma > 0$ and let b_1, b_2 be generic elements of $\mathcal{O}$.

Let A_i be the set of open balls of radius γ inside $B_{\geq\gamma}(b_i)$. Then A_i is a definable affine space over k.

It can be shown that a generic affine morphism between A_1 and A_2 cannot be coded in $K \cup \mathcal{B}^o \cup \mathcal{B}^{cl}$.

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- $s \subseteq K^n$ is a lattice if it is a free $\mathcal{O}$-submodule of rank n;
- for $s \subseteq K^n$ a lattice, $s/\mathfrak{m}s \cong_k k^n$.

For $n \geq 1$, let

$$S_n := \{\text{lattices in } K^n\},$$

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Fact

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Classification of Imaginaries in ACVF

$\mathcal{G} = \{K\} \cup \{S_n,\ n \geq 1\} \cup \{T_n,\ n \geq 1\}$ are the geometric sorts.
Let $\mathcal{L}_{\mathcal{G}}$ be the (natural) language of valued fields in $\mathcal{G}$.

Theorem (Haskell-Hrushovski-Macpherson 2006)

ACVF eliminates imaginaries down to **geometric sorts**, *i.e. the theory ACVF considered in $\mathcal{L}_{\mathcal{G}}$ has EI.*

Using this result, Hrushovski and Martin were able to classify the imaginaries in the p-adics:

Theorem (Hrushovski-Martin 2006)

$\mathbb{Q}_p$ eliminates imaginaries down to $\{K\} \cup \{S_n,\ n \geq 1\}$.

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$\mathcal{G} = \{K\} \cup \{S_n,\ n \geq 1\} \cup \{T_n,\ n \geq 1\}$ are the geometric sorts.
Let $\mathcal{L}_\mathcal{G}$ be the (natural) language of valued fields in $\mathcal{G}$.

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Classification of Imaginaries in ACVF (cont'd)

Some consequences of the classification of imaginaries in ACVF:

1. May do **Geometric Model Theory** in valued fields.
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   $\Rightarrow$ apply methods from stability outside the stable context.
3. There are striking applications outside model theory:
   - in **representation theory** (Hrushovski-Martin);
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Outline

The notion of a definable type

- As before, T is a **complete** $\mathcal{L}$-theory;
- $\mathcal{U} \models T$ is very saturated and homogeneous.

Definition

Let $\mathcal{M} \models T$ and $A \subseteq M$. A type $p(\overline{x}) \in S_n(M)$ p is *A-definable* if for every $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$ there is an $\mathcal{L}_A$-formula $d_p\varphi(\overline{y})$ s.t.

$$\varphi(\overline{x}, \overline{b}) \in p \;\Leftrightarrow\; \mathcal{M} \models d_p\varphi(\overline{b}) \quad (\text{for every } \overline{b} \in M)$$

We say p is *definable* if it is definable over some $A \subseteq M$.

The collection $(d_p\varphi)_\varphi$ is called a *defining scheme* for p.

Remark

If $p \in S_n(M)$ is definable via $(d_p\varphi)_\varphi$, then the same scheme gives rise to a (unique) type over any $N \succ M$, denoted by $p|N$.

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Definable types: first properties

- **(Realised types are definable)**
  Let $\bar{a} \in M^n$. Then $\mathrm{tp}(\bar{a}/M)$ is definable.
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- **(Preservation under definable functions)**
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Definable 1-types in *o*-minimal theories

Let T be *o*-minimal (e.g. $T = \mathrm{DOAG}$) and $\mathcal{D} \models T$.

- Let $p(x) \in S_1(D)$ be a non-realised type.
- Recall that p is determined by the cut $C_p := \{d \in D \mid d < x \in p\}$.
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Definable 1-types in *o*-minimal theories (cont'd)

Corollary

Let $\mathcal{D} \models \mathrm{DOAG}$ The following are equivalent:

1. $\mathcal{D} \cong (\mathbb{R}, +, <)$*;*
2. *Any $p \in S_1(D)$ is definable;*
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Proof.

1. $\Rightarrow$ 2. Clearly, every cut in $\mathbb{R}$ is rational.

2. $\Rightarrow$ 3. If $p = \mathrm{tp}(a_1, \ldots, a_n/D)$, by QE, p is determined by the 1-types $\mathrm{tp}(a'/D)$, where $a' = \sum_{i=1}^{n} z_i a_i$ for some $z_i \in \mathbb{Z}$.

2. $\Rightarrow$ 1. If $\mathcal{D}$ is non-archimedean, choose $0 < e << d$. Then $\{d \in D \mid d < ne \text{ for some } n \in \mathbb{N}\}$ is an irrational cut. So $\mathcal{D}$ has to be archimedean, and of course equal to its completion. □

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Definable 1-types in ACVF

Let $K \models \mathrm{ACVF}$, $K \preccurlyeq L$, $t \in L \setminus K$, and put $p := \mathrm{tp}(t/K)$.

- If $K(t)/K$ is a residual extension, then p is definable.

Proof.
Replacing t by $at + b$, WMA $\mathrm{val}(t) = 0$ and $\mathrm{res}(t) \notin k_K$.
$\Rightarrow$ Enough to guarantee definably that
$\mathrm{val}(X^n + a_{n-1}X^{n-1} + \ldots + a_0) = 0$ is in p for all $a_i \in \mathcal{O}_K$. $\square$

- If $K(t)/K$ is a ramified extension, up to a translation WMA $\gamma = \mathrm{val}(t) \notin \Gamma(K)$.

  p is definable $\Leftrightarrow$ the cut def. by $\mathrm{val}(t)$ in $\Gamma(K)$ is rational.

  (Indeed, p is determined by $p_\Gamma := \mathrm{tp}_{DOAG}(\gamma/\Gamma(K))$, so p is definable $\Leftrightarrow$ p_Γ is definable.)

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Let $K \models \mathrm{ACVF}$, $K \preccurlyeq L$, $t \in L \setminus K$, and put $p := \mathrm{tp}(t/K)$.

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Replacing t by $at+b$, WMA $\mathrm{val}(t) = 0$ and $\mathrm{res}(t) \notin k_K$.
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Definable 1-types in ACVF (cont'd)

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  (There is no smallest K-definable ball containing t. If p were definable, the intersection of all (closed or open) K-definable balls containing t would be definable.)

Corollary

Let $K \models \mathrm{ACVF}$. The following are equivalent:

1. *K is maximally valued and $\Gamma(K) \cong (\mathbb{R}, +, <)$;*
2. *Any $p \in S_1(K)$ is definable;*
3. *For every $n \geq 1$, any $p \in S_n(K)$ is definable.*

Proof.
1. $\Leftrightarrow$ 2. follows from the above. 1. $\Rightarrow$ 3. follows from the detailed analysis of types in ACVF by Haskell-Hrushovski-Macpherson.

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Definability of types in ACF

Proposition

In ACF, all types over all models are definable.

Proof.

Let $K \models ACF$ and $p \in S_n(K)$.

Let $I(p) := \{f(\overline{x}) \in K[\overline{x}] \mid f(\overline{x}) = 0 \in p\} = (f_1, \ldots, f_r)$.

By QE, every formula is equivant to a boolean combination of polynomial equations. Thus, it is enough to show:

For any d the set of (coefficients of) polynomials $g(\overline{x}) \in K[\overline{x}]$ of degree $\leq d$ such that $g \in I_p$ is definable. This is classical. ∎

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Equivalent definitions of stability

Definition

A theory T is called stable if there is no formula $\varphi(\overline{x}, \overline{y})$ and tuples $(\overline{a}_i, \overline{b}_i)_{i \in \mathbb{N}}$ (in $\mathcal{U}$) such that $\mathcal{U} \models \varphi(\overline{a}_i, \overline{b}_j) \iff i \leq j$.

Theorem (Shelah)

The following are equivalent:

1. *T is stable.*
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3. $\Rightarrow$ 2. There are $\leq |A^{\mathbb{N}}|$ many A-def. types, so $\kappa = 2^{\aleph_0}$ works.
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- ACF, more generally every strongly minimal theory;
- any theory of abelian groups.

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- Every *o*-minimal theory (e.g. DOAG, RCF);
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Uniform definability of types in stable theories

Theorem

Let T be stable and $\varphi(\overline{x}, \overline{y})$ a formula. Then there is a formula $\chi(\overline{y}, \overline{z})$ such that for every type $p(\overline{x})$ (over a model) there is $\overline{b}$ such that $d_p\varphi(\overline{y}) = \chi(\overline{y}, \overline{b})$.

Problem

Is $D_{\varphi,\chi} = \{\overline{b} \in U \mid \chi(\overline{y}, \overline{b})$ is the φ-definition of some type$\}$ always a definable set?

Fact

For T stable, all $D_{\varphi,\chi}$ are definable iff for every formula $\psi(x, \overline{y})$ (in T^{eq}), there is $N_\psi \in \mathbb{N}$ such that whenever $\psi(\mathcal{U}, \overline{b})$ is finite, one has $|\psi(\mathcal{U}, \overline{b})| \leq N_\psi$.

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In ACF, the sets $D_{\varphi,\chi}$ are definable.

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Prodefinable sets

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A prodefinable set is a projective limit $D = \varprojlim_{i \in I} D_i$ of definable sets D_i, with def. transition functions $\pi_{i,j} : D_i \to D_j$ and I some small index set. (Identify $D(\mathcal{U})$ with a subset of $\prod D_i(\mathcal{U})$.)

We are only interested in **countable** index sets $\Rightarrow$ WMA $I = \mathbb{N}$.

Example

1. **(Type-definable sets)** If $D_i \subseteq U^n$ are definable sets, $\bigcap_{i \in \mathbb{N}} D_i$ may be seen as a prodefinable set: WMA $D_{i+1} \subseteq D_i$, so the transition maps are given by inclusion.

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Some notions in the prodefinable setting

Let $D = \varprojlim_{i \in I} D_i$ and $E = \varprojlim_{j \in J} E_j$ be prodefinable.

- There is a natural notion of a prodefinable map $f : D \longrightarrow E$.
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D is strict pro-definable iff $\pi_i(X) \subseteq D_i$ is definable for every relatively definable X and any i.

The set of definable types as a prodefinable set

Assume:

- T has **EI** and
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For any $\varphi(\overline{x}, \overline{y})$ fix $\chi_\varphi(\overline{y}, \overline{z})$ such that for any definable type $p(\overline{x})$ we may take $d_p\varphi(\overline{y}) = \chi_\varphi(\overline{y}, \overline{b})$ for some $\overline{b} = \ulcorner d_p\varphi \urcorner$.

$\Rightarrow$ may identify p (more exactly $p \mid U$) with the tuple $(\ulcorner d_p\varphi \urcorner)_\varphi$.

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1. *With these identifications, the set of definable n-types $S_{def,n}$ is naturally a prodefinable set. Moreover, if $X \subseteq U^n$ is definable, denoting $S_{def,X}(A)$ the set of A-definable types on X, $S_{def,X}$ is a relatively definable subset of $S_{def,n}$.*

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The space of types in ACF as a prodefinable set

Corollary

Let V be an algebraic variety. There is a strict prodefinable set D (in ACF) such that for any field K, $S_V(K) \cong D(K)$ naturally.

Proposition

1. *If V is a curve, then S_V is iso-definable.*
2. *If $\dim(V) \geq 2$, then S_V is not iso-definable.*

Proof sketch.

1. is clear, since S_V is the set of realised types (which is always iso-definable) plus a finite number of generic types.

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