Second International Conference on Valuation Theory

El Escorial (Spain)

On $\mathbb{R}$−places and related topics

Danielle Gondard (Université Paris 6)

gondard@math.jussieu.fr

24 juillet 2011

1. Background in Real Algebra

2. The space of $\mathbb{R} - places$

3. Valuation fans

4. The real holomorphy ring

5. On connected components of M(K)

6. Towards abstract $\mathbb{R}-places$

7. Open questions

Part I

Background in Real Algebra

Artin-Schreier paper in Crelle Journal (1927) defines real fields and real-closed fields. Is a basis for Real Agebra and Real Algebraic Geometry.

Almost no change after ; see Moderne Algebra by Van der Waerden (1930), Lectures in Abstract Algebra by N. Jacobson (1964), and Algebra by S. Lang (1965). But the notion of cone associated to an order came later : J.P. Serre (1949).

This 1927 paper allowed Artin to solve Hilbert's 17th problem again in Crelle 1927. This result was reproved by S.Lang in 1953 using valuations and real places.

K commutative field s.t. $-1 \notin \sum K^2$

$\Leftrightarrow$ exists total ordering compatible with field structure

From order $\leq_P$ to **positive cone** $P \subset K$:

$a \leq b \Leftrightarrow b - a \in P$; then P satifies

$P + P \subset P$, $P.P \subset P$, $P \cup -P = K$, $-1 \notin P$

Implying : 0, 1 $\in P$, $P \cap -P = \{0\}$, $\sum K^2 \subset P$

Any real field must have $char K = 0$

Examples : $\mathbb{Q}(\sqrt{2})$, $\mathbb{R}((X))$, $\mathbb{R}(X)$...

Nice theorem : $\sum K^2 = \cap P_i$, P_i orderings of K

Krull valuation : $v : K^* \twoheadrightarrow \Gamma$, Γ a totally ordered abelian group (*the value group*) such that

(1) $v(xy) = v(x) + v(y)$ for any x, y in K^*

(2) $v(x+y) \geq \min\{v(x), v(y)\}$, for any x, y in K^*, with $x+y$ in K^*

Valuation ring : $A = \{a \in K \mid a = 0 \text{ or } v(a) \geqslant 0\}$

Maximal ideal : $I = \{a \in K \mid a = 0 \text{ or } v(a) > 0\}$

Residue field : $k = A/I$; *Group of units* : $A^* = A \backslash I$

A valuation v **is real** if and only if the residue field k is formally real ($-1 \notin \sum k^2$). And a field K admits real valuations if and only if it is a formally real field ($-1 \notin \sum K^2$)

Given K a formally real field,

P an ordering of K , v a valuation

v **is compatible with** P

$\Leftrightarrow 1 + I_v \subset P$

$\Leftrightarrow A_v$ is convex with respect to P

$\Leftrightarrow I_v$ is convex with respect to P

$\Leftrightarrow 0 <_P a <_P b \Rightarrow v(a) \geq v(b)$ (in value group Γ).

The trivial valuation is compatible with any P

K a formally real field, P an ordering, and v a valuation compatible with P, then $\overline{P}$, induced by P in the residue field k_v, is an ordering of k_v.

Important example is :

$$A(P) = \{x \in K \mid \exists r \in \mathbb{Q}, \quad -r \leq_P x \leq_P r\}$$

$A(P)$ is a valuation ring, with maximal ideal

$$I(P) = \{a \in K \mid \forall r \in \mathbb{Q}^*, -r \leq_P a \leq_P r\}$$

v valuation associated to $A(P)$ is compatible with P.

$\overline{P}$ induced by P in the residue field $k_v = A(P)/I(P)$, is an **archimedean** ordering of k_v.

The valuation rings compatible with a given P form a chain under inclusion with **smallest element** $A(P)$.

Part II

The space of $\mathbb{R}-$ places

Places on a field K :

$$\varphi : K \longrightarrow F \cup \{\infty\}$$

where F is a field, sum and product are extended to $F \cup \{\infty\}$ by $x + \infty = \infty$ for $x \in F$, and $x\infty = \infty$ for $x \in F^* \cup \{\infty\}$

φ is such that $\varphi(1) = 1$, and satisfies homomorphism rules $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(xy) = \varphi(x)\varphi(y)$ whenever right side terms are defined.

Real places : same with F formally real (or real-closed).

$\mathbb{R}-$**places** : same with $F = \mathbb{R}$.

P ordering of K a formally real field, v valuation associated to $A(P) = \{a \in K \mid \exists r \in \mathbb{Q}, -r \leq_P a \leq_P r\}$ and $I(P) = \{a \in K \mid \forall r \in \mathbb{Q}^*, -r \leq_P a \leq_P r\}$ the maximal ideal of $A(P)$.

P induces over the residue field $k_v = A(P)/I(P)$ an archimedean order $\overline{P}$; hence $(k_v,\overline{P})$ can be embedded uniquely in $(\mathbb{R}, \mathbb{R}^2)$

The $\mathbb{R}-place$ **associated to** P is $\lambda_P : K \to \mathbb{R} \cup \{\infty\}$

defined by the following commutative diagram :

$$K \xrightarrow{\lambda_P} \mathbb{R} \cup \{\infty\}$$

$$\pi \searrow \quad \nearrow i$$

$$k_v \cup \{\infty\}$$

Let $M(K)$ denote the **space of** $\mathbb{R}-$**places** of K then $M(K) = \{ \lambda_P \mid P \in \chi_K\}$ because any $\mathbb{R}-place$ can be defined as before using an ordering P of K.

The space $M(K)$ is given the coarsest **topology** such that the evaluation applications defined $\forall a \in K$ below are continuous :

$$e_a : M(K) \longrightarrow \mathbb{R} \cup \{\infty\}$$

$$\lambda_P \mapsto \lambda_P(a)$$

Explicitely : $\lambda_P(a) = \infty$ if $a \notin A(P)$

and if $a \in A(P)$, $\lambda_P(a) = \inf\{r \in \mathbb{Q} \mid a \leq_P r\}$

$$= \sup\{r \in \mathbb{Q} \mid r \leq_P a\}$$

The **space of orderings** of a formally real field K is usually denoted by χ_K.

It is equipped with the Harrison topology generated by the open-closed sets :

$$H(a) = \{P \in \chi_K \mid a \in P\}.$$

χ_K is a compact, Hausdorff and totally disconnected space hence a Boolean space.

Craven has shown that every boolean space is isomorphic to the space of orderings χ_K of some field K.

The following map Λ

$$\Lambda : \chi_K \longrightarrow M(K)$$

$$P \mapsto \lambda_P$$

is continuous, closed and surjective:

$M(K)$ is a compact Hausdorff space.

This topology on the space $M(K)$ of $\mathbb{R} - places$ is also the **quotient topology** inherited from that of χ_K the space of orderings equipped with the usual Harrison topology.

Theorem (Becker) *:*

the following are equivalent :

(1) Λ *is a bijection ;*

(2) $\forall a \in K \quad a^2 \in \sum K^4$ *;*

(3) every real valuation of K *has a 2-divisible value group ;*

(4) K *does not admit any ordering of exact level* 2 *.*

Theorem : The following are equivalent :

(1) $\lambda_P = \lambda_Q$

(2) $P \cap Q$ *is a valuation fan.*

(3) P *and* Q *belong to a 2-primary chain of orderings (2-power levels).*

Definition (Harman) : a 2-primary chain of orderings is $(P_n) = (P_0, P_1, ..., P_n, ...)$, P_0 *being a usual order,* P_n *an ordering of level* 2^{n-1} *such that*

$$P_n \cup - P_n = (P_0 \cap P_{n-1}) \cup -(P_0 \cap P_{n-1})$$

All the P_n induce the same archimedean order $\overline{P_n}$ on the residue field k_v of the valuation v associated to the valuation ring $A(P_0) = \left\{ a \in K \mid \exists r \in \mathbb{Q} \; - r \leqslant_{P_0} a \leqslant_{P_0} r \right\}$

Part III

Valuation fans

K a real field, $T \subset K$ is a **preordering** iff

$T + T \subset T$, $T.T \subset T$

$0, 1 \in T$ and $-1 \notin T$

$T^* = T \setminus \{0\}$ is a subgroup of K^*

T is a **quadratic** preordering $\Leftrightarrow K^2 \subset T$

T is a **level** n preordering $\Leftrightarrow K^{2n} \subset T$

T is possibly without level

T is **compatible with a valuation** $v \Leftrightarrow 1 + I_v \subset T$

A preordering T is a **fan**

$\Longleftrightarrow \forall a \notin -T$ holds $T + aT \subset T \cup aT$

These preorderings are well behaved for compatibility : Given v a valuation of a real field K, and T a preordering compatible with v : T is a fan in $K \Leftrightarrow \overline{T}$ is a fan in k_v.

Trivialization theorem : *T a fan in K then there exists a valuation v compatible with T inducing in the residue field k_v a trivial fan $\overline{T}$* (meaning an ordering P, or the intersection of two orderings $P_1 \cap P_2$)

Quadratic case proved by Bröcker

Torsion preordering case proved by Becker

A valuation fan *is a preordering* T *such that there exists a real valuation* v, *compatible with* T, *inducing an ordering on the residue field* k_v.

One can even ask T induces an archimedean ordering using $A(T) = \{x \in K \mid \exists r \in \mathbb{Q} \;\; r \pm x \in T\}$

- usual orderings (level 1) ;

- higher level orderings (level n, Becker 1978) ;

- $T = (A(P)^* \cap P) \cup \{0\}$ (no level).

Original and **alternative definition for valuation fan** (Bill Jacob, 1981) is : T preordering such that

$\forall x \notin \pm T$ holds $1 \pm x \in T$ or $1 \pm x^{-1} \in T$

K a formally real field, $P \subset K$ is an **ordering of exact level** n iff :

$\sum K^{2n} \subset P$, $P + P \subset P$, $P.P \subset P$ (hence P^* is a subgroup of K^*)

and we have $K^*/P^* \simeq \mathbb{Z}/2n\mathbb{Z}$.

Level 1 orderings are the total usual orders.

Theorem (Becker) *:* $\sum K^{2n} = \bigcap_{\text{level of } P \text{ divides } n} P$

Theorem (Becker) : *Let p be a prime,*

$\sum K^2 \neq \sum K^{2p} \Longleftrightarrow K$ *admits orderings of level* p.

Example : $K = \mathbb{R}((X))$

The two usual orders are :

$P_+ = K^2 \cup XK^2$ and $P_- = K^2 \cup -XK^2$

For any prime p there exist two orderings of level p :

$P_{p,+} = K^{2p} \cup X^p K^{2p}$ and $P_{p,-} = K^{2p} \cup -X^p K^{2p}$

A signature of level n is a morphism of abelian groups

$$\sigma : K^* \to \mu_{2n}$$

such that the kernel is additively closed.

Then $P = \ker \sigma \cup \{0\}$ is an ordering of higher level, and its level divides n.

A generalized signature (N. Schwartz 1990) *is a morphism of abelian groups*

$$\sigma : K^* \to G$$

such that the kernel is a valuation fan

Important level 1 valuation fans are the

$$T = \cap P_i$$

where the P_i are obtained from the $\mathbb{R}-place$ λ by

$$\Lambda^{-1}(\lambda) = \{P_i \mid \lambda_{P_i} = \lambda\}$$

$\mathbb{Q}(2^{\frac{1}{2}})$ and $\mathbb{R}((X))$ have isomorphic spaces of orderings, but the first one has two $\mathbb{R} - places$ and no ordering of level 2 and the second one has only one $\mathbb{R} - place$ but has one 2-primary chain of higher level orderings.

The Baer-Krull Theorem :

Let K be a formally real field, v be a real valuation of K, and $\overline{P}$ be an ordering in the residue field k_v.

Denote by $\chi_{v,\overline{P}}$ the set of all orderings P_i of K inducing the given $\overline{P}$ in k_v.

There is a bijection between $\chi_{v,\overline{P}}$ and $Hom(\Gamma_v, \mathbb{Z}/2)$, where Γ_v denotes the value group of v.

As a consequence of the Baer-Krull theorem, if $\Gamma_v/2\Gamma_v$ has, as vector space over $\mathbb{Z}/2$, a basis of n classes, then $\chi_{v,\overline{P}}$ **has 2^n elements** P_i .

Hence the lifting of $\overline{P}$ to K is unique if and only if Γ_v is 2−divisible.

Part IV

The real holomorphy ring

The real holomorphy ring $H(K)$ of a formally real field K is defined as the intersection of the real valuation rings of K.

We have also :

(1) $H(K) = \cap A(P)$ where P ranges over $\chi(K)$ the space of orderings of K and

$A(P) = \{a \in K \mid \exists r \in \mathbb{Q} \ \ -r \leqslant_P a \leqslant_P r\}$ is the smallest valuation ring compatible with P.

(2) $H(K) = \{a \in K \mid \exists n \in \mathbb{N}$ s. t. $n \pm a \in \sum K^2\}$

(3) $H(K) = \mathbb{Q}[\{\frac{1}{1+q} \mid q \in \sum K^2\}]$

The real spectrum of $H(K)$:

$SperH(K) = \{\alpha \mid \alpha$ prime ordering of $H(K)$ $\}$

A prime ordering is $\alpha \subset H(K)$ satisfying : $\alpha + \alpha \subset \alpha$, $\alpha\alpha \subset \alpha$, $H(K)^2 \subset \alpha$, $-1 \notin \alpha$, $\alpha \cup -\alpha = H(K)$, $\alpha \cap -\alpha = \mathfrak{p}$ a prime ideal.

Then α induces on $H(K)/\mathfrak{p}$ a prime ordering $\overline{\alpha}$ such that $\overline{\alpha} \cap -\overline{\alpha} = \{0\}$, therefore $\overline{\alpha}$ extends uniquely to an ordering of qf($H(K)/\mathfrak{p}$). Conversely given a prime ideal $\mathfrak{p}$ and an ordering $\overline{\alpha}$ on qf($H(K)/\mathfrak{p}$), we get by restriction to $H(K)/\mathfrak{p}$ a prime ordering with support $\{0\}$, the preimage in $H(K)$ is a prime ordering with support $\mathfrak{p}$.

Hence we have **another description of** $SperH(K)$:

$\{(\mathfrak{p},\overline{\alpha}) \mid \mathfrak{p} \in specH(K), \overline{\alpha}$ order of $quot(H(K)/\mathfrak{p})\}$

A **basis of open sets** of the topology on $H(K)$ is

$D(a_1, .., a_n) = \{\alpha \in SperH(K) \mid a_i \notin -\alpha \text{ for all } i\}$

The topology is compact (does not mean Hausdorff)

α is **minimal** if there is no β such that $\beta \neq \alpha$ and $\beta \subset \alpha$

β is **maximal** if there is no α such that $\beta \neq \alpha$ and $\alpha \subset \beta$

$MinSperH(K) = \{\alpha \in SperH(K) \mid \alpha \text{ minimal}\}$

$MaxSperH(K) = \{\beta \in SperH(K) \mid \beta \text{ maximal}\}$

β is a **specialization** of α whenever $\alpha \subset \beta$.

The specializations of a prime ordering form a chain under inclusion and a prime ordering is contained in a unique maximal specialization.

Theorem : *the following diagram is commutative*

$$\begin{array}{ccc} \chi(K) & \xrightarrow{speri} & MinSperH(K) \\ \downarrow \Lambda & & \downarrow sp \\ M(K) \overset{res}{\to} Hom(H(K),\mathbb{R}) & \overset{j}{\to} & MaxSperH(K) \end{array}$$

The horizontal mappings are **homeomorphisms**, and the vertical one are **continuous surjections**. The spaces are compact and $M(K)$ and $MaxSperH(K)$ have quotient topologies inherited from Λ and sp.

$speri$ is defined by $P \mapsto P \cap H(K)$,

sp by $\alpha \mapsto \alpha^{\max}$,

res by $\lambda \mapsto \lambda_{|H(K)}$ and j by $\varphi \mapsto \varphi^{-1}(\ \mathbb{R}^2)$.

Part V

On connected components of $M(K)$

The space of $\mathbb{R}-places\ M(K)$ of a formally real field K can be **totally disconnected** (e.g. totally archimedean fields).

$M(K)$ can be **connected** like in the case of $K = \mathbb{R}(X)$ or $K = \mathbb{R}((X))$.

Theorem (Harman) : *$M(K)$ is connected $\Leftrightarrow$ $M(K(X))$ connected $\Leftrightarrow$ $M(K((X))$ connected.*

Theorem (Schulting) : *$M(K)$ and $M(K(X))$ have the same number of connected components.*

Theorem (Becker-G.) : *when the number of connected components of the space $M(K)$ is finite then it is*

$$1 + \log_2[K^{*2} \cap \sum K^4 : (\sum K^{*2})^2].$$

Denote by $\pi_0(M(K))$ **the set of connected components of the space of real places**, using the units of the real holomorphy ring we have :

$$| \pi_0(M(K)) | = \log_2[E : E^+]$$

where E are the units of the real holomorphy ring $H(K)$ and where $E^+ = E \cap \sum K^2$.

For any formally real field K we can even get :

$$| \pi_0(M(K)) | = 1 + \log_2[K^{*2} \cap \sum K^4 : (\sum K^{*2})^2]$$

by proving that $K^{*2} \cap \sum K^4 / (\sum K^{*2})^2$ is isomorphic to $E/(E^+ \cup -E^+)$.

Behind this is the separation of connected components of $M(K)$ by elements $\beta \notin \sum K^2$ such that $\beta^2 \in \sum K^4$.

Theorem (Becker-G.) *: λ_P and λ_Q are in two distinct connected components of $M(K)$ if and only if*

$$\exists \beta \in K^* \; (\beta \in P \cap -Q \; et \; \beta^2 \in \sum K^4)$$

Lemma : *if exists b, s. t. $b \notin \sum K^2$ and $b^2 \in \sum K^4$, then does not exist $P \in H(b)$ and $Q \in H(-b)$ such that $\lambda_P = \lambda_Q$.*

Otherwise $b \notin (P \cap Q) \cup -(P \cap Q)$ and $\lambda_P = \lambda_Q$ imply that there exists an ordering of level 2, P_2, such that $P_2 \cup -P_2 = (P \cap Q) \cup -(P \cap Q)$ with $b \notin P_2$ hence $b \notin \sum K^4 = \cap P_{2,i}$.

$\Leftarrow$ Suppose that λ_P and λ_Q are in the same connected component C of $M(K)$ ($P \neq Q$), and that there exists $b \in P \cap -Q$ with $b^2 \in \sum K^4$, Λ being closed $C \cap \Lambda(H(b))$ and $C \cap \Lambda(H(-b))$ are a partition of C into two non empty closed sets, impossible.

$\Rightarrow$ If λ_P and λ_Q are in C and C' two distinct connected components, $M(K)$ being compact and Hausdorff, there exists an open-closed set $U \supset C$ and $U^c = M(K) \backslash U \supset C'$.

Let $X = \Lambda^{-1}(U)$ and $Y = \Lambda^{-1}(U^c)$, X and Y form a partition of $\chi(K)$; Λ being surjective

$$\Lambda^{-1}(\Lambda(\Lambda^{-1}(U))) = \Lambda^{-1}(U)$$

hence $\Lambda^{-1}(\Lambda(X)) = X$ and also $\Lambda^{-1}(\Lambda(Y)) = Y$.

By Harman's following lemma "*If* $\chi(K) = \chi_1 \cup \chi_2$ *where* χ_1 *and* χ_2 *are open-closed disjoint sets such that* $\Lambda^{-1}(\Lambda(\chi_1)) = \chi_1$ *and* $\Lambda^{-1}(\Lambda(\chi_2)) = \chi_2$ *then there exists* a *such that* $\chi_1 = H(a)$ *and* $\chi_2 = H(-a)$" then there exits b such that $X = H(b)$ and $Y = H(-b)$ with $b^2 \in \sum K^4$, then we have $b \in P \cap -Q$ and $b^2 \in \sum K^4$.

Theorem (Becker-G.) : *Let Y be a smooth projective irreducible variety over $\mathbb{R}$ with function field $K = \mathbb{R}(Y)$, and denote $|\pi_0(Y(\mathbb{R}))|$ the number of connected components of $Y(\mathbb{R}) \neq \emptyset$ then :*

$$| \pi_0(Y(\mathbb{R})) | = 1 + \log_2[K^{*2} \cap \sum K^4 : (\sum K^{*2})^2]$$

The original proof uses the connected components of the space of $\mathbb{R}-places$ $M(K)$ since we can prove that:

$$| \pi_0(Y(\mathbb{R})) | = | \pi_0(M(\mathbb{R}(Y))) |$$

and that for the real field $K = \mathbb{R}(Y)$:

$$| \pi_0(M(K)) | = 1 + \log_2[K^{*2} \cap \sum K^4 : (\sum K^{*2})^2]$$

Recall **Harnack's inequality** for algebraic curves :

$| \pi_0(Y(\mathbb{R})) | \leq g + 1$ with $g = (n-1)(n-2)/2$ for an algebraic smooth curve of degree n.

Sketch of proof of

$$| \pi_0(Y(\mathbb{R}))| = | \pi_0(M(\mathbb{R}(Y)))|$$

We use the center map $c : M(K) \to Y(\mathbb{R})$, defined by $c(\lambda) = c(V_\lambda)$ the unique point x (Y projective) whose local ring $\mathfrak{o}_x$ is dominated by V_λ the valuation ring associated to the $\mathbb{R}-$place. In that case c is surjective, the central points being the closure of the regular points. And one can prove that c is continuous.

Bröcker proved that the fiber of a central point has a finite number of connected components, and if x is regular then the fiber is connected.

Lemma: *if an application, from a compact space X to another compact space Y, is surjective and continuous, and if each fiber is connected, then it induces a bijection beetween $\pi_0(X)$ and $\pi_0(Y)$.*

Part VI

Towards abstract $\mathbb{R}$−places

The space of orderings of a field, studied in relation with quadratic forms and real valuations, have been the origin of the theory of abstract spaces of orderings (1979-80) and of **Marshall's problem** :

"is every abstract space of orderings the space of orders of some field ?"

We have shown that -under certain conditions- one can associate to an abstract space of orderings a "$P-structure$" (partition of the space of orderings into subspaces which are fans and such that any fan intersects only one or two classes) corresponding to the space of $\mathbb{R}-places$ in the field case.

An abstract space of orderings is (X, G) where G is a group of exponent 2 (hence abelian), -1 a distinguished element of G, X a subset of $Hom(G, \{1, -1\})$ such that :

(1) X is a closed subset of $Hom(G, \{1, -1\})$

(2) $\forall \sigma \in X \quad \sigma(-1) = -1$

(3) $\bigcap_{\sigma \in X} \ker \sigma = 1 \quad (\ker \sigma = \{a \in G \mid \sigma(a) = 1\})$

(4) f, g quadratic forms over G

$$\begin{aligned} D_X(f \oplus g) &= \\ \cup\{D_X \langle x, y \rangle &\mid \quad x \in D_X(f),\ y \in D_X(g)\} \end{aligned}$$

where $D_X(f) = \{a \in G$ represented by $f\}$, i.e. there exits g such that $f \equiv_X \langle a \rangle \oplus g \quad (f \equiv_X h$ iff f and h have same dimension, and $\forall \sigma \in X$ same signature)

Fans can be seen as sets of signatures, then a level 1 fan of four elements is characterized by :

$$\sigma_0\sigma_1\sigma_2\sigma_3 = 1$$

and it corresponds to the fan $T = \bigcap_{i=1}^{4} \ker \sigma_i \cup \{0\}$.

In the abstract situation **an abstract fan** is an abstract space of orderings (X, G) such that

$$X = \{\sigma \in Hom(G, \{1, -1\}) \mid \sigma(-1) = -1\}\,.$$

It is characterized by :

$$\forall \sigma_0, \sigma_1, \sigma_2 \in X \text{ the product } \sigma_0\sigma_1\sigma_2 \in X$$

Definition : *a* $P - structure$ *is an equivalence relation on a space of signatures* (X, G) *such that the canonical mapping* $\Lambda : X \to M$ *(*M *is the set of equivalence classes) satisfies :*

(1) each fiber is a fan ;

(2) if $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$ *then* $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ *has a non empty intersection with at most two fibers*

Marshall has shown that every abstract space of orderings has a $P - structure$.

But unlikely the case of the set of $\mathbb{R}-places$ in a field, this $P - structure$, M, equipped with the quotient topology is not always Hausdorff.

Theorem (Marshall-G) :

Let (X, G) *be a subspace of a space of signatures* (X', G'), *with 2-power exponent. For* $\sigma_0, \sigma_1 \in X$, *define* $\sigma_0 \sim \sigma_1$ *if* $\sigma_0\sigma_1 = \tau^2 \in X'^2$.

Then the following are equivalent :

(1) $\sim$ *defines a* $P - structure$ *on* X.

(2) If $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$, *then :*

σ_0 *is in relation by* $\sim$ *with exactly one of the* $\sigma_1, \sigma_2, \sigma_3$

or σ_0 *is in relation by* $\sim$ *with everyone of the* $\sigma_1, \sigma_2, \sigma_3$.

Moreover in that case the induced $P - structure$ defined on X by $\sim$ has a Hausdorff topology.

In fields, the space of $\mathbb{R}-places$ is determined as soon as one knows the usual orders and the orderings of level 2.

$\mathbb{Q}(2^{\frac{1}{2}})$ and $\mathbb{R}((X))$ have isomorphic spaces of orderings, but the first one has two $\mathbb{R}-places$ and no ordering of level 2 and the second one has only one $\mathbb{R}-place$ but has a 2-primary chain of higher level orderings.

2-primary chain of higher level orderings start with a pair of usual orders P_0, P_1, and corresponding 2-level ordering P_2 satisfies

$$P_2 \cup -P_2 = (P_0 \cap P_1) \cup (-(P_0 \cap P_1))$$

For a 2-level ordering $a^2 \in P_2 \iff a \in P_2 \cup -P_2$; on the side of signatures this gives (τ associated to P_2)

$$\tau(a^2) = \tau^2(a) = \sigma_0(a)\sigma_1(a)$$

Definition : *a space of signatures of level* 2^n *is* (X, G), *where* G *is an abelian group of exponent* 2^n, *and* X *contained in* $Hom(G, \mu_{2^n}) = \chi(G)$ *is such that :*

(0) $\forall \sigma \in X, \forall k \in N$ *with* k *odd,* $\sigma^k \in X$

(1) X *is a closed subset of* $\chi(G)$

(2) $\forall \sigma \in X \;\; \sigma(-1) = -1$ *(*-1 *distinguished element)*

(3) $\bigcap_{\sigma \in X} \ker \sigma = 1$ *(où* $\ker \sigma = \{a \in G \mid \sigma(a) = 1\}$*)*

(4) f, g *forms over* G

$$\begin{aligned} D_X(f \oplus g) &= \\ \cup\{D_X \langle x, y\rangle &\mid x \in D_X(f), y \in D_X(g)\} \end{aligned}$$

Part VII

Open questions

- Does $M(K((X)))$ have the same number of connected components as $M(K)$? (Conjecture : yes)

- In which cases are the connected components of $M(K)$ homeomorphic ? (conjecture : geometric case at least). Study the space $\pi_0(M(K))$ of the connected components of $M(K)$.

- Study the space of valuation fans, and its relation with $SperH(K)$; recall χ_K consists of valuation fans P_i, and to a $\mathbb{R}-place$ λ can be associated a valuation fan $\cap P_i$ where $P_i \in \Lambda^{-1}(\lambda)$. Or same question dealing with signatures.

- Try to define a notion of abstract space of valuation fans and write a theory of abstract $\mathbb{R}-places$. Then use abstract $\mathbb{R}-places$ to solve Marshall's problem of realizability of abstract spaces of orderings.

- Characterize the topological spaces which are realizable as spaces of $\mathbb{R}-places$ (see Osiak's talk for some results)