Second International Conference and Workshop on Valuation Theory

Abstracts of lectures and talks

18th-29th July 2011

1 Aim of the conference

The conference is intended to cover some of the recent developments in valuation theory and its applications. Valuation theory developed in the first half of this century as a part of algebraic number theory, the theory of ordered fields and the theory of ordered abelian groups. Then, through the work of Zariski and Abhyankar, it witnessed important applications in algebraic geometry. Later, as the tools of algebraic geometry shifted away from valuation theory, and algebraic number theory did not provide any new valuation theoretical questions, research in valuation theory became less frequent. However, mathematicians from real algebra and from model theory kept an interest in valuation theory, and there was a remarkable, but somewhat clandestine development of the subject. In recent years, valuation theory has found its way back into algebraic geometry and has witnessed many important applications in various branches of mathematics. It is fascinating to see how fundamental principles of valuation theory are discovered to play a role in various topics which seem to be only loosely related.

In particular, valuation theory has been discovered to be extremely useful in the theory of complex dynamical systems, and in the study of non-oscillating trajectories of real analytic vector fields in three dimensions. Analogues of the Riemann-Zariski valuation spaces have been found to be the proper framework for questions of intersection theory in algebraic geometry.

In a different direction, the relation between Berkovich geometry, tropical geometry and valuations spaces, on the one hand, and the geometry of arc spaces and valuation spaces, on the other, have begun to deepen and clarify.

The well established topic of the model theory of valued fields is also being transformed, in particular through the study of valued fields with operators. Simultaneously, the work on the "classical" aspects of the subject has increased and substantial progress has been made toward the solution of the local uniformization problem in arbitrary characteristic, the study of ramification of valuations in arbitrary dimension, in particular the nature and role of the mysterious "defect". All this corresponds to new insights and new approaches to old problems. This is why right now is a perfect time to hold a conference onthe newly growing and flourishing subject of valuation theory.

The organizers of the conference have chosen to privilege the topics listed above among a great diversity of topics where valuation theory plays a role, but some of those other topics will be represented as well, and some contributed lectures will be accepted. The intended audience includes graduate students and researchers in all aspects of this subject. The conference is intended to bring together the experts of several branches of valuation theory and of related topics in order to strengthen the relations between these branches. At the same time it shall introduce non-experts to the principles and results of valuation theory, in particular if they are coming from areas of mathematics in which valuation theory has recently witnessed important contributions (e.g., algebraic geometry, Galois theory, asymptotic analysis, dynamical systems) or in which the role of valuation theory still has to be determined (e.g., C*-algebras). One main goal of our conference is to point out and study the relations between different branches of valuation theory, and their applications to other branches of mathematics. Another main goal is to state and discuss the many important open problems in valuation theory in order to provide an optimal basis for future research conducted by experts and students.

Contents

2 Abstracts of lectures and talks

2.1 Dicritical divisors of pencils

Shreeram S. Abhyankar

The dicritical divisors of a pencil at a simple point of a surface constitute an important tool in affine algebraic geometry, i.e., in the study of polynomial rings. In this talk it is shown how these dicritical divisors may be viewed as certain nodes of the singularity tree of a generic member of the pencil.

2.2 Pencils, dicriticals and curvettes

Enrique Artal

This talk is devoted to the analytic part of a joint work with S.S. Abhyankar and D. Shannon. We use the notion of curvettes in a regular local ring of dimension 2 to get a better understanding of the behaviour of the dicritical divisors of a pencil

2.3 Extremal Valued Fields

Salih Azgin (joint work with F.-V. Kuhlmann and Florian Pop)

Let $f(x_1, ...x_k)$ be a polynomial over $\mathbb{R}[[t]]$. If f has a zero modulo(t^n) for every n then f has an actual zero in $R[[t]]^k$. The notion of extremality may be seen as a generalization of this fact to arbitrary valued fields. A valued field K is extremal if the values of every polynomial $f(x_1, ..., x_k)$ over K reaches a maximum (possibly $\infty = v(0)$) when evaluated at tuples from the valuation ring.

Henselian valued fields were expected to be extremal, when residue characteristic is 0. We present a hands on account of how this expectation fails and provide a surprising classification theorem for extremal valued fields. This is joint work with F.-V. Kuhlmann and F. Pop.

2.4 F_1-geometry

Vladimir Berkovich

I'll talk on work in progress on algebraic and analytic geometry over the field of one element F_1. This work originates in non-Archimedean analytic geometry as a result of a search for appropriate framework for so called skeletons of analytic spaces and formal schemes. I'll explain the notion of a scheme over F_1 and its relation to the notion of a logarithmic scheme.

2.5 Differential calculus with integers

Alexandru Buium

An analogue of differential calculus can be developed [1] in which functions $x(t)$ are replaced by integers α in various number fields and the derivation operator $\delta_t = d/dt$ on functions is replaced by a Fermat quotient operator δ_p with respect to a prime p; on rational integers α the operator δ_p acts as $\delta_p\alpha = \frac{\alpha-\alpha^p}{p}$. One can introduce then arithmetic analogues of various constructions in differential geometry such as: jet spaces, prolongations, Cartan distributions, contact transformations, etc. Cf. [2]. Applications of this theory include diophantine results (on Manin-Mumford, on Heegner points, on modular forms), constructions of quotient spaces that don't exist in usual algebraic geometry (such as modular curves modulo actions of Hecke correspondences), and liftings to characteristic zero that are "impossible" in usual algebraic geometry. The talk is an introduction to this circle of ideas.

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2.6 Valuations, multiindex filtrations and topology
Antonio Campillo

One considers filtrations by mutiindex on germs of analytical varieties which are associated to valuations not neccessary centered at the orignin, as well as filtrations induced on analytical subspaces. We show how a Poincaré series can be associated to those filtrations and who in many cases one explicity can reach topological invariants of the germ from the Poincaré series. The talk include results in cooperation with Félix Delgado, Sabir Gussein Zade, Fernando Hernando and Ann Lemahieu.

2.7 Dicritical divisors (after S. Abhyankar and I. Luengo)
Vincent Cossart (joint work with Mickaël Matusinski)

S. Abhyankar and I. Luengo presented this year in the American Journal of Mathematics a new Theory of dicritical divisors valid in the most general case. We give new proofs and we generalize their results.

We will recall the definitions of dicritical divisors, reprove their existence in the local case, study the *polynomial dicritical divisors* and, at the end, we give a necessary and sufficient condition for a dicritical divisor v associated to an element $z \in$QF(R) such that $zx^a y^b \in R$, (R regular local ring of dimension 2, (x, y) r.s.p.) to be polynomial.

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2.8 On transfer principles for motivic exponential integrals
Raf Cluckers

Transfer principles for motivic integrals allow one, for several kinds of statements, to deduce results over fields of p-adic numbers from results over local fields of positive characteristic, and vice versa. A first such transfer principle was obtained by F. Loeser and the author in [1], and treats equalities between integrals which come from motivic exponential integrals. A second transfer principle is work in progress with J. Gordon and I. Halupczok [2] and treats integrability conditions for integrals which come from motivic exponential integrals. Both transfer principles have applications in the Langlands program. I will present and explain this to a broad audience, including for an integral what it means to come from a motivic exponential integral.

References

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2.9 Semigroups of valuations dominating a Noetherian local ring
Steven Dale Cutkosky

The semigroups which are obtained by a valuation dominating a Noetherian local domains are remarkably rich and diverse. We discuss the classification of valuations dominating Noetherian local domains of small dimension, and give some general necessary conditions, including some bounds on their growth. We also give a number of examples showing the subtly of this problem. Some references on this problem are: [7], [6], [5], [2], [3], [1], [4].

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2.10 Introduction to Berkovich analytic spaces

Antoine Ducros

At the beginning of the 90's, Berkovich has proposed a new approach to p-adic analytic geometry, which is based in an essential way upon the notion of absolute value (rather than abstract Krull valuation). I will explain how the Berkovich space associated to an algebraic variety is defined, and then study carefully a basic (but crucial) example: the Berkovich projective line; after that, I will say some words about higher genus curves, trying to show why and how their homotopy types encodes deep arithmetical properties.

Then I will come to several applications, in various area (spectral theory, dynamics,...), most of which will illustrate the following principle: it often happens that a p-adic avatar of a given complex-analytic geometrical object does actually exist, but only in the Berkovich setting, because naive p-adic analytic spaces don't contain enough points to detect it.

2.11 Artin-Schreier defect extensions and Strong Monomialization

Samar Elhitti (joint work with Dale Cutkosky, Laura Ghezzi and Franz-Viktor Kuhlmann)

Inspired by Kuhlmann's classification of Artin-Schreier defect extensions as dependent or independent [2], and based on our work on Cutkosky and Piltant's example [1] that fails Strong Monomialization, and on Temkins' work [3], we present partial results supporting the belief that the dependent Artin-Schreier defect extensions are the more "harmful" ones, in the sense of failing Cutkosky's Strong Monomialization.

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[3] Temkin, M., *Inseparable local uniformization*, submitted.

2.12 Corner loci of piecewise polynomials, and polynomially weighted tropical varieties

Alexander Esterov

Counting Euler characteristics of the discriminant of the quadratic equation in terms of Newton polytopes in two different ways, G. Gusev found an unexpected relation for mixed volumes of two polytopes and the convex hull of their union.

We give an elementary proof of this equality, deducing it from the following fact, conjectured by A. G. Khovanskii: the mixed volume of a collection of polytopes only depends on the product of their support functions (rather than on the individual support functions). This dependence is essentially a certain specialization of the isomorphism between two well-known combinatorial models for the cohomology of toric varietes. We provide a new description of this isomorphism, which also may be of interest because of a new object and operation (tropical variety with polynomial weights and its corner locus) that appears in our construction. We also discuss its possible applications to the study of intersection theory on tropical varieties.

2.13 Domains having a unique Kronecker function ring

Alice Fabbri

We study the class of integrally closed domains having a unique Kronecker function ring. Such domains have fairly simple sets of valuation overrings and are a generalization of Prüfer domains (i.e. domains having the property that each localization at a nonzero prime ideal yields a valuation domain). We give characterizations by studying Zariski-Riemann spaces of valuation overrings and integral closure of finitely generated ideals. We provide new examples of such domains and show that for several well-known classes of integral domains the property of having a unique Kronecker function ring makes them fall into the class of Prüfer domains.

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2.14 Riemann-Zariski Spaces

Charles Favre

Our aim is use Riemann-Zariski spaces to relate the study of regular local rings (including the definition of Rees' valuations, and Izumi's theorem) to pluripotential analysis on non-archimedean analytic spaces. These lectures are based on my joint work with S. Boucksom and M. Jonsson.

2.15 Hilbertian fields and families of valuations

Arno Fehm (joint work with Elad Paran)

Hilbert's irreducibility theorem for number fields is of central importance in Galois theory and arithmetic geometry and led to the notion of a Hilbertian field, that is, a field that satisfies the consequence of Hilbert's theorem. This talk will discuss conditions on the family of valuations on a field K that imply that K is Hilbertian or non-Hilbertian. Our main result here (in [1]) is a generalization of a theorem of Weissauer that states that the quotient field of a generalized Krull domain of dimension exceeding one is Hilbertian. The key ingredient in our proof is an approximation result for valuations on non-Hilbertian fields by R. Klein from 1982.

As an immediate application of this generalization we deduce Hilbertianity of certain fields of (generalized) power series. For example, if R is a domain contained in a rank-1 valuation ring of its quotient field, then any ring lying between $R[X]$ and $R[[X]]$ has a Hilbertian quotient field. The talk will also discuss for which domains the condition of being contained in a rank-1 valuation ring is satisfied, and what happens when it is not.

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2.16 Nash problem for surfaces

Javier Fernández de Bobadilla

In the late 60s Nash proposed to study a natural correspondence between the irreducible components of the arc space of a singularity and the essential components of a resolution of singularities. In the case of surfaces he predicted that this correspondence would be a bijection.

In a recent joint work M. Pe Pereira and the speaker have settled Nash prediction for surfaces in the affirmative. I will explain the proof in the talk.

2.17 The ultrafilter topology on spaces of valuation domains and applications

Carmelo-Antonio Finocchiaro (joint work with Marco Fontana and K. Alan Loper)

The aim of this talk is to present some results obtained in a joint paper with Marco Fontana and K. Alan Loper.

Let K be a field and A be a subring of K. I will denote by $\mathrm{Zar}(K|A)$ the set of all the valuation domains having K as the quotient field and containing A as a subring. Following Zariski's approach, it is possible to endow $\mathrm{Zar}(K|A)$ with a natural topological structure, by taking as basic open sets the sets of the type $\mathrm{Zar}(K|B)$, where B runs in the family of all the subalgebras of K that are of finite type over A. It is well known that this topology, usually called *the Zariski topology*, makes $\mathrm{Zar}(K|A)$ a compact topological space, but it is almost never Hausdorff. In this talk, I will present a new topology on $\mathrm{Zar}(K|A)$, called *the ultrafilter topology*, that refines the Zariski topology on $\mathrm{Zar}(K|A)$ and makes it a compact Hausdorff space.

I will give applications of this topology to characterize some classes of integrally closed domains. Moreover, after recalling the basic background on semistar operations and Kronecker function rings, I will show that the *completion* (in the sense of Zariski) of an `e.a.b.` semistar operation corresponds to the compactification of a certain subspace of $\mathrm{Zar}(K|A)$. Finally, I will introduce the *dual topology* (in the sense of Hochster) on $\mathrm{Zar}(K|A)$ and I will characterize, in terms of the ultrafilter topology and of the dual topology, when two complete semistar operations are the same.

2.18 On $\mathbb{R}$-places and related topics

Danielle Gondard-Cozette

The talk will provide some background on real algebra, for which we refer for example to the book [15], and deal with the space of $\mathbb{R}$-places ([1], [6], [8], [12], [14], [16]). After giving the basic properties of the space of $\mathbb{R}$-places, we explore its links with other notions among which the real holomorphy ring ([2], [4], [21]) and valuation fans ([3], [5], [13]).

We shall also show that this space of $\mathbb{R}$-places has interesting applications in real algebraic geometry ([4], [20]) and in abstract spaces of orderings in the sense of Marshall ([7], [10], [18]), and finally mention some open problems ([4], [10], [11]).

Recent results on the topology of spaces of $\mathbb{R}$-places, and on the realizability of topological spaces as spaces of $\mathbb{R}$-places ([9], [17], [19]) will be discussed in K. Osiak's talk.

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2.19 Real valuations on skew polynomial rings

Ángel Granja

Let D be a division ring, T be a variable over D, σ be an endomorphism of D, δ be a σ-derivation on D (i.e. $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in D$) and $R = D[T; \sigma, \delta]$ be the left skew polynomial ring over D such that $Ta = \sigma(a)T + \delta(a)$.

Furthermore, let $Val(R)$ be the set of functions $\mu : R \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ satisfying the standard axioms of valuations, whose restriction to D is no trivial and which are σ-compatible (i.e. $\mu(\sigma(a)) = \mu(a)$ for each $a \in D$). We consider the partial order $\preceq$ on $Val(R)$ given by $\mu \preceq \widetilde{\mu}$ if and only if

$\mu(f) \leq \widetilde{\mu}(f)$ for all $f \in R$ and $\mu, \widetilde{\mu} \in Val(R)$. We first note that $\mu \preceq \widetilde{\mu}$ implies that μ and $\widetilde{\mu}$ have the same restriction to D, thus $Val(R)$ is a disjoint union of the sets $Val_\nu(R)$ such that ν is a real valuation on D.

In this talk, we will show that $(Val_\nu(R), \preceq)$ has a natural structure of parameterized complete non-metric tree. This is known in many commutative situations, see [Ber], [FJ], [Gra], ...

It is worth pointing out that our techniques are similar to MacLane one's in [M], but we first put our attention on the study of the partial order $\preceq$ rater than generalize MacLane's concept of key polynomial to skew polynomial rings, which we shall do at the end of talk.

On the other hand, our results can be used to study rings that are particular cases of (iterated) skew polynomials rings (Weyl algebras, quantum spaces or even some subalgebras of Lie algebras, ..., see [GW]); as well to give irreducibility criterions for skew polynomials in a similar setting out as for polynomial rings over commutative fields equipped with a proper real valuation. (See for example, [GMR] and [HOS].)

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2.20 On equivariant versions of Poincaré series of filtrations and monodromy zeta functions

Sabir M. Gusein-Zade

In a number of papers it was shown that the Poincaré series of some natural filtrations (defined by valuations) on the rings of germs of functions on singularities are related (sometimes coincide) with appropriate monodromy zeta functions. A generalization of these results to cases equivariant with respect to an action of a finite group G could help to understand reasons for these relations. One meets the problem of the lack of appropriate definitions of equivariant versions of Poincaré series and monodromy zeta functions. Recently such version were defined as elements of Grothendieck rings of G-sets with some additional structures. We shall discuss these approaches and some results obtained on this way. The talk is based on joint works with A.Campillo, F.Delgado and W.Ebeling.

2.21 Whitney stratifications in valued fields

Immanuel Halupczok

In real algebraic geometry, Whitney stratifications are a useful tool to understand algebraic or semi-algebraic subsets $X \subset \mathbb{R}^n$ (or other, more general categories of sets); see e.g. [1]. A *Whitney stratification* of X is a partition of X into (semi-algebraic) subsets $S_0, \dots S_{\dim X}$ such that $\dim S_i = i$, each S_i is smooth, and moreover, S_i "behaves well" in the neighbourhood of S_j for each $j < i$. (This "behaving well" are the two so-called *Whitney conditions*.)

I will present a variant of Whitney stratifications for Henselian valued fields K with residue characteristic 0. As in the real setting, different categories of subsets $X \subset K^n$ can be considered, e.g.

subvarieties, or definable sets in the sense of model theory. (Different languages can be used; this also includes sub-analytic sets, for example.) In the talk, I will define and explain this notion of Whitney stratification. The main result is, as in the real setting, that Whitney stratifications exist for any given X in our category.

At first sight, the valued field Whitney stratifications seem rather different from the classical ones. To see the relation, one should work in a suitable real closed extension K of the valued field $\mathbb{R}((t))$, and this field K should be thought of as $\mathbb{R}$ with an additional infinitesimal element t. (For model theorists: K should be an ω_1-saturated elementary extension of $\mathbb{R}$.) The classical Whitney conditions can be reformulated using this infinitesimal element; then, the valued field Whitney conditions turn out to be a strenghening of this reformulation.

A bit surprisingly, valued field Whitney stratifications can also be seen from a completely different point of view. The usual notion of an isometry is that the valuation of differences is preserved. We obtain a strenghening of this by additionally requiring that the leading term is preserved; let me call such maps "risometries" (the r stands for "residue field", in analogy to the relation between the valuation map v and the map rv to the "leading term structure" $K^{\times}/(1+\mathcal{M})$). In other words, risometries are "translations up to smaller terms". Now valued field Whitney stratifications can be seen as a description of sets up to risometry.

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2.22 Patching and local-global principles

David Harbater (joint work with Julia Hartmann and Daniel Krashen)

Local-global principles for global fields have been stated in terms of discrete valuations, e.g. in the contexts of quadratic forms and the Tate-Shafarevich group. In joint work, we study the analogous situation of one-variable function fields over a complete discretely valued field. Here the situation can be viewed as two-dimensional, since a model over the valuation ring is an arithmetic surface. This provides a richer environment, in which local-global principles can be considered either with respect to discrete valuations (corresponding to curves on an arithmetic surface) or in terms of points on the closed fiber of a model. We prove results concerning both types of local-global principles, using the technique of patching, which the speaker had previously used to obtain results in Galois theory and which was extended in work with Julia Hartmann to permit applications to other types of problems. In particular, we obtain such principles for quadratic forms and for torsors, as well as describing the obstructions to such principles in cases where they do not hold.

2.23 Introduction to model theory

Martin Hils

In the first lecture, we will introduce the basic concepts of model theory. In particular, we will discuss *first order structures, (complete) theories, definable sets, quantifier elimination*, as well as *type spaces, saturation* etc. These notions will be illustrated using algebraic examples (algebraically closed fields, real closed fields, and valued fields). We will also give a quick overview of the model theory of valued fields, with particular emphasis on the *Ax-Kochen-Ershov principle.* It states that, in equicharacteristic 0, the first order theory of a henselian valued field is determined by the theory of the value group together with the theory of the residue field.

In the second lecture, more advanced topics will be treated. First, we will talk about *imaginary elements*, i.e. finite tuples modulo a definable equivalence relation. These serve as *codes* for definable sets. In some theories, one may dispense with imaginaries (e.g. in algebraically closed or real closed fields). In the valued context, the situation is more complicated. We will mention the classification of imaginaries in algebraically closed valued fields by Haskell-Hrushovski-Macpherson [1], a result which initiated geometric model theory in valued fields (cf. [2]).

We will then discuss the notion of a *definable type*, a key device from geometric model theory, and its relationship to *stability*. (Stable theories are precisely those theories in which every type is definable.) Finally, we will show how one may treat the space of definable types as a prodefinable set.

The tutorial is designed to provide some model-theoretic background for the courses given by Tom Scanlon and Ehud Hrushovski.

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2.24 Constructible sets over valued fields
Ehud Hrushovski

2.25 A model theoretic view of non-archimedean topology
Ehud Hrushovski

2.26 Some generalizations of Eisenstein-Schönemann Irreducibility Criterion
Sudesh K. Khanduja

One of the results generalizing Eisenstein Irreducibility Criterion states that if $\phi(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is a polynomial with coefficients from the ring of integers such that a_s is not divisible by a prime p for some $s \leqslant n$, each a_i is divisible by p for $0 \leqslant i \leqslant s-1$ and a_0 is not divisible by p^2, then $\phi(x)$ has an irreducible factor of degree at least s over the field of rational numbers. We have observed that if $\phi(x)$ is as above, then it has an irreducible factor $g(x)$ of degree s over the ring of p-adic integers such that $g(x)$ is an Eisenstein polynomial with respect to p. In this lecture, we discuss an analogue of the above result for a wider class of polynomials which will extend the classical Schönemann Irreducibility Criterion as well as Generalized Schönemann Irreducibility Criterion and yields irreducibility criteria by Akira, Panaitopol and Stefănescu (cf. J. Number Theory, 25 (1987) 107-111).

2.27 Spaces of valuations
Manfred Knebusch

A valuation $v : R \to \Gamma \cup \{\infty\}$ on a commutative ring R (in the sense of Bourbaki, Alg. comm. VI) can also be viewed as a pair $(\mathfrak{p}; \hat{v})$ consisting of a prime ideal $\mathfrak{p}$, namely the support $v^{-1}(\infty)$ of v, and a Krull valuation $\hat{v}$ on qf$(R/\mathfrak{p})$. The first part of the lecture is devoted to various possibilities to turn the set Spev(R) of (equivalence classes of) valuations on R into a spectral space such that the support Spev$(R) \to$ Spec(R), $v \to \mathfrak{p}$ is a spectral map. At least three of these spectral topoogies have proved to be useful in applications. They all are coarsenings of the same constructible topology on Spev(R), which is dictated by model theory. They are determined by their specializations (giving the closures of one-point sets). There are two basic types of specializations, the primary and the secondary specializations. All three spectral topologies on Spev(R) have the same primary specializations but different secondary specializations.

Such a valuation spectrum Spev R is a huge space in general. In the second part of the lecture we focus on two sorts of subspaces of Spev R. One of them consists only of special valuations, i.e., valuations which do not admit a proper primary specialization. They come up by fixing a Prüfer subring A of R (cf. by book with Digen Zhang, Springer LNM 1791).

2.28 Ramification

Franz-Viktor Kuhlmann

I will give a quick introduction to the ramification theory of valued fields, covering decomposition, inertia and ramification fields and their properties. Building on the classical theory, the following notions will be introduced and discussed: tame extensions and tame fields, purely wild extensions, structure of minimal purely wild extensions, field complements of the ramification field, immediate algebraic extensions, the defect, algebraically maximal and defectless fields, elimination of ramification, the generalized stability theorem, henselian rationality over tame and over perfect fields, Artin-Schreier defect extensions and their classification.

2.29 Ultrametrics and Implicit Function Theorems

Franz-Viktor Kuhlmann

The usual Implicit Function Theorem over a valued field is closely related to the multi-dimensional Hensel's Lemma. Both hold if and only if the field is henselian, that is, admits a unique extension of its valuation to its algebraic closure. Hensel's Lemma was originally proved by Hensel for the fields of p-adic numbers. The common proof used here is the valuation theoretical version of the Newton algorithm. But if one attempts to generalize the proof to other valued fields, for instance, to any maximal field, one may be forced to use transfinite induction, and the proof becomes much less elegant and effective. This fact, as well as the desire to formulate a Hensel's Lemma for other valued structures than valued fields, triggered the search for a more universal underlying principle. It turned out that this principle does not need addition or multiplication, it can already be formulated on the level of ultrametric spaces. (Every valuation induces an ultrametric.) This paves the way to proving Hensel's Lemmas not only in the case of less algebraic structure, but also for valued fields with more structure, such as derivations or automorphisms.

The principle I formulated in [1] is a "Main Theorem" which gives a criterion for the surjectivity of so-called *immediate mappings* on ultrametric spaces. The concept of immediate mappings is a natural generalization of the notion of immediate extensions of valued fields. The Main Theorem provides a uniform tool to prove all sorts of Hensel's Lemmas and generalized Hensel's Lemmas, including an easy proof of the multi-dimensional Hensel's Lemma on maximal fields, from where it can be pulled down to any henselian field.

But the Main Theorem also allows us to push the limits further: it can be used to prove an infinite-dimensional Implicit Function Theorem over suitable valued rings. The need for such infinite-dimensional versions has come up in the work of Bernard Teissier on local uniformization.

I will give a short introduction to ultrametric spaces, immediate mappings and the Main Theorem. Then I will describe its applications to the proofs of the multi-dimensional Hensel's Lemma and of an infinite-dimensional Implicit Function Theorem.

My work has been inspired by work of Paulo Ribenboim and Sibylla Priess-Crampe, and the Main Theorem has close connections with their Fixed Point and Attractor Theorems.

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2.30 Spaces of $\mathbb{R}$-places

Katarzyna Kuhlmann

n $\mathbb{R}$-place is a place of some field with residue field inside the field $\mathbb{R}$ of real numbers. In our talk, we will give a survey on some recent developments in the theory of $\mathbb{R}$-places.

T. Craven showed in 1975 that every boolean space can be realized as the space of orderings of some field. In contrast to this, it is an open problem whether every compact Hausdorff space can be realized as the space of $\mathbb{R}$-places of some field. Here, the topology on the space of $\mathbb{R}$-places is the one induced by the Harrison topology of the space of orderings of that field, via the map that associates to every ordering its canonical $\mathbb{R}$-place.

The talk will concentrate on two problems: metrizability and realizibility of spaces of $\mathbb{R}$-places. We will show that the class of compact Hausdorff spaces which can be realized as spaces of $\mathbb{R}$-places is closed under finite disjoint unions, closed subsets, and direct products with Boolean spaces (joint work with Ido Efrat, [1]). It is obvious that the space of $\mathbb{R}$-places of any countable field if metrizable. The same is true for any function field of transcendence degree 1 over a totally Archimedean field. We will show that for a function field F over a real closed field R of higher transcendence degree than 1 the space $M(F)$ is metrizable iff R is countable (joint work with Murray Marshall and Michal Machura, [2]). For transcendence degree 1 the situation is more complicated. The necessary condition for metrizability of the space $M(F)$ is that R contains a countable dense subfield. It is also a sufficient condition in the case of F being a rational function field over R (joint work with F.-V. Kuhlmann and M. Machura, [3]). We will end our talk by a description of the structure of the space $M(R(X))$ over any non-Archimedean real closed field R and show that its topological dimension is 1 (joint work with F.-V. Kuhlmann, [4]). Finally, we will mention some open problems.

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2.31 Hardy type derivations on generalised series fields

Mickaël Matusinski (joint work with Salma Kuhlmann)

We consider the valued field $\mathbb{R}((\Gamma))$ of generalised series (with real coefficients and monomials in a totally ordered multiplicative group Γ). We investigate how to endow $\mathbb{R}((\Gamma))$ with a series derivation, that is a derivation that satisfies some natural properties such as commuting with infinite sums (strong linearity) and (an infinite version of) Leibniz rule. We characterize when such a derivation is of Hardy type, that is, when it behaves like differentiation of germs of real valued functions in a Hardy field. We provide a necessary and sufficient condition for a series derivation of Hardy type to be surjective.

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2.32 An application of generating sequences of valuations on comparison of divisorial valuations

Mohammad Moghaddam

We recall the notion of generating sequences of valuations and the theory of MacLane key polynomials. We show that this theory can be used to give a proof of the Izumi theorem on comparison of divisorial valuations.

2.33 Algebraic foliations and dicritical valuations

Francisco Monserrat (joint work with C. Galindo)

A classical problem, proposed by H. Poincaré, consists of determining whether a differential equation with polynomial coeficients, of first order and degree 1, is algebraically integrable or, equivalently, whether an algebraic foliation of the projective plane has a rational first integral. We shall show that, assuming certain conditions related with the dicritical divisors that appear in the resolution of singularities, there exists an algorithm that povides a solution to this problem whose input is the resolution of the (dicritical) singularities of the foliation. We shall see also that these assumed conditions are not necessary if there is only one dicritical divisor. In addition, we shall show that, when the number of dicritical divisors is two, it is possible to determine (without the additional conditions) whether a foliation has a rational first integral of fixed genus (different from 1).

2.34 Arc spaces and valuations

Hussein Mourtada

We will describe relations between divisorial valuations centered on a smooth algebraic variety X defined over $\mathbb{C}$ and some constructible subsets in the space of arcs X_∞ of X. If time permits, we will talk about extensions of these relations to singular varieties.

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2.35 Extending a valuation centered in a local domain to the formal completion

M. A. Olalla Acosta (joint work with F. J. Herrera-Govantes, M. Spivakovsky and B. Teissier)

In this talk I want present our paper *Extending a valuation centered in a local domain to the formal completion* (arXiv: 1007.4658v1 [math.AG])

Let (R, m, k) be a local noetherian domain with field of fractions K and R_ν a valuation ring, dominating R (not necessarily birationally). Let $\nu|_K : K^* \to \Gamma$ be the restriction of ν to K; by definition, $\nu|_K$ is centered at R. Let $\hat{R}$ denote the m-adic completion of R. In the applications of valuation theory to commutative algebra and the study of singularities, one is often induced to replace R by its m-adic completion $\hat{R}$ and ν by a suitable extension $\hat{\nu}_-$ to $\frac{\hat{R}}{P}$ for a suitably chosen prime ideal P, such that $P \cap R = (0)$. The first reason is that the ring $\hat{R}$ is not in general an integral domain, so that we can only hope to extend ν to a *pseudo-valuation* on $\hat{R}$, which means precisely a valuation $\hat{\nu}_-$ on a quotient $\frac{\hat{R}}{P}$ as above. The prime ideal P is called the *support* of the pseudo-valuation. It is well known and not hard to prove that such extensions $\hat{\nu}_-$ exist for some minimal prime ideals P of $\hat{R}$. Although, as we shall see, the datum of a valuation ν determines a unique minimal prime of $\hat{R}$ when R is excellent, in general there are many possible primes P as above and for a fixed P many possible extensions $\hat{\nu}_-$. This is the second reason to study extensions $\hat{\nu}_-$.

In the case of valuations of rank one, we can define the ideal

$$H := \bigcap_{\beta \in \Phi} (\mathcal{P}_\beta \hat{R}),$$

where $\mathcal{P}_\beta$ denote the ν-ideal of R of value β.

And then we can prove that H is a prime ideal of $\hat{R}$ and ν extends uniquely to a valuation $\hat{\nu}_-$, centered at $\frac{\hat{R}}{H}$.

When studying extensions of ν to the completion of R, one is led to the study of its extensions to the henselization $\tilde{R}$ of R as a natural first step. This, in turn, leads to the study of extensions of ν to finitely generated local strictly étale extensions R^e of R. We therefore start out by letting $\sigma : R \to R^\dagger$ denote one of the three operations of completion, (strict) henselization, or a finitely generated local strictly étale extension:

$$\begin{aligned} R^\dagger &= \hat{R} \quad \text{or} && (1)\\ R^\dagger &= \tilde{R} \quad \text{or} && (2)\\ R^\dagger &= R^e. && (3)\end{aligned}$$

The ring $R^\dagger$ is local; let $m^\dagger$ denote its maximal ideal. The homomorphisms

$$R \to \tilde{R} \quad \text{and} \quad R \to R^e$$

are regular for any ring R; by definition, if R is an excellent ring then the completion homomorphism is regular.

In the general case of valuations of rank r, we define a chain of $2r+1$ *prime* ideals, that we call implicit prime ideals,

$$H_0 \subset H_1 \subset \cdots \subset H_{2r} = H_{2r+1} = mR^\dagger,$$

satisfying $H_{2\ell} \cap R = H_{2\ell+1} \cap R = P_\ell$ and such that $H_{2\ell}$ is a minimal prime of $P_\ell R^\dagger$ for $0 \le \ell \le r$. Moreover, if $R^\dagger = \tilde{R}$ or $R^\dagger = R^e$, then $H_{2\ell} = H_{2\ell+1}$. We call H_i the **i-th implicit prime ideal** of $R^\dagger$, associated to R and ν.

The ideals H_i behave well under local blowing ups along ν (that is, birational local homomorphisms $R \to R'$ such that ν is centered in R'). This means that given any local blowing up along ν or ν-extension $R \to R'$, the i-th implicit prime ideal H'_i of $R'^\dagger$ has the property that $H'_i \cap R^\dagger = H_i$.

Let $(0) \subsetneqq \mathbf{m}_1 \subsetneqq \cdots \subsetneqq \mathbf{m}_{r-1} \subsetneqq \mathbf{m}_r = \mathbf{m}_\nu$ be the prime ideals of the valuation ring R_ν. By definitions, our valuation ν is a composition of r rank one valuations $\nu = \nu_1 \circ \nu_2 \cdots \circ \nu_r$, where ν_ℓ is a valuation of the field $\kappa(\mathbf{m}_{\ell-1})$, centered at $\frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}}$.

If $R^\dagger = \tilde{R}$, we will prove that there is a unique extension $\tilde{\nu}_-$ of ν to $\frac{\tilde{R}}{H_0}$. If $R^\dagger = \hat{R}$, the situation is more complicated.

We describe the set of extensions $\nu^\dagger_-$ of ν to $\lim\limits_{\overrightarrow{R'}} \frac{R'^\dagger}{P' R'^\dagger}$, where P' is a tree (the notion of *tree* will be defined during the talk) of prime ideals of $R'^\dagger$ such that $P' \cap R' = (0)$. We show that specifying such a valuation $\nu^\dagger_-$ is equivalent to specifying the following data:

(1) a chain of trees of prime ideals $\tilde{H}'_i$ of $R'^\dagger$ (where $\tilde{H}'_0 = P'$), such that $H'_i \subset \tilde{H}'_i$ for each i and each $R' \in \mathcal{T}$, satisfying one additional condition

(2) a valuation $\nu^\dagger_i$ of the residue field $k_{\nu^\dagger_{i-1}}$ of $\nu^\dagger_{i-1}$, whose restriction to the field $\lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_{i-1})$ is centered at the local ring $\lim\limits_{\overrightarrow{R'}} \frac{R'^\dagger_{\tilde{H}'_i}}{\tilde{H}'_{i-1} R'^\dagger_{\tilde{H}'_i}}$.

If $i = 2\ell$ is even, the valuation $\nu^\dagger_i$ must be of rank 1 and its restriction to $\kappa(\mathbf{m}_{\ell-1})$ must coincide with ν_ℓ.

2.36 Multiplicative Valued Difference Fields

Koushik Pal

A *valued difference field* (K, v, Γ, σ) is a valued field (K, v, Γ) with an automorphism $\sigma : K \to K$ of the field K, that in addition preserves the ring of integers $\mathfrak{O}_K$, i.e., $\sigma(\mathfrak{O}_K) = \mathfrak{O}_K$. The theory of such a structure depends on how the automorphism σ interacts with the valuation function v. If (K, v, Γ, σ) satisfies $v(\sigma(x)) = v(x)$ for all $x \in K$, the valued difference field is called *isometric*. The model theory of isometric valued difference fields have been extensively studied by Thomas Scanlon [1], and Luc Bélair, Angus Macintyre and Thomas Scanlon [2]. If (K, v, Γ, σ) satisfies $v(\sigma(x)) > nv(x)$ for all

$x \in K^{\times}$ with $v(x) > 0$ and for all $n \in \mathbb{N}$, then the valued difference field is called *contractive*. The model theory of such structures has been studied by Salih Azgin [3]. I will talk about a more general case, which incorporates the above two cases, and which we call *multiplicative*. A multiplicative valued difference field satisfies $v(\sigma(x)) = q \cdot v(x)$, where q (> 0) is interpreted as an element of a real-closed field. For example, q could be 2, i.e., $v(\sigma(x)) = 2v(x)$; or q could be an infinite or infinitesimal element as well. I will give axiomatization for such theory, prove an Ax-Kochen-Ershov kind of result and thus prove relative completeness of the theory. I will also show that by expanding the language to include a cross-section, the theory admits quantifier elimination relative to the value group and the residue field.

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2.37 Nonarchimedean geometry, tropicalization, and metrics on curves
Sam Payne

I will discuss the relationship between the nonarchimedean analytification of an algebraic variety and the tropicalizations of its various embeddings in toric varieties, with attention to the metrics on both sides in the special case of curves. This is joint work with Matt Baker and Joe Rabinoff.

2.38 Good reductions for endomorphisms of the projective line
Giulio Peruginelli (joint work with Jung-Kyu Canci and Dajano Tossici)

The notion of Good Reduction for an endomorphism φ of the projective line at a finite place v of a number field K was introduced by Morton and Silverman (see [2]). This notion behaves well with respect to composition of maps and so it has nice applications in the studies of arithmetic of dynamical systems.

Recently Szpiro and Tucker (see [4]) introduced a new notion of Good Reduction, called Critically Good Reduction, which concerns the non-collision modulo v of the ramification points of φ and the non collision modulo v of their images. With this notion they were able to prove a finiteness result for equivalence classes of endomorphisms of the projective line having Critically Good Reduction outside a prescribed finite set of finite places of K. Their result generalizes Shafarevich's result on the finitess of isomorphism classes of elliptic curves having good reduction outside a prescribed finite set of finite places of K (see [3]).

In general this new notion of good reduction behaves really bad under composition. Here we show that under the assumption of separability of the reduction map φ_v, the Critically Good Reduction at v implies the Good Reduction at v.

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2.39 Relative Local Uniformization in Function Fields

Olivier Piltant

While Local Uniformization aims at approximating a valuation ring V of a function field K/k by local rings at regular points of models of K/k, the purpose of Relative Local Uniformization is to approximate an extension W/V of such valuation rings by "good pairs" $R \subset S$ of local rings at points of models.

Let L/K be a finite and separable extension of function fields over k, V be a valuation ring, $k \subset V$, $K = QF(V)$ and W be an extension of V to L. By a (relative) model of W/V, it is meant an inclusion $R \subset S$, where R and S are normal local rings, essentially of finite type over k, with $L = QF(S)$, $K = QF(R)$.

If char$k = 0$, the pair $R \subset S$ can be chosen to be a monomial mapping of regular local rings. Assume for simplicity that W has trivial residue field $LW = k$. Coordinates on R and S are related by monomial equations

$$u_i = \gamma_i v_1^{m_{i1}} \cdots v_n^{m_{in}},$$

with $\gamma_i \in S$ a unit, $1 \leq i \leq n$. The matrix $M = (m_{ij})$ is non singular and $\mathbf{Z}^n/M.\mathbf{Z}^n$ identifies with the inertia group of W/V. These results are refinements of Cutkosky's Monomialization Theorem [1].

The main purpose of this talk will be to discuss the two-dimensional situation when char$k = p > 0$ [2]. When the value group WL of W is not p-divisible, relative models $R \subset S$ can be chosen with R and S regular; coordinates (u_1, u_2) and (v_1, v_2) are connected by a relation

$$u_1 = f_1(v_1, v_2),\ u_2 = f_2(v_1, v_2)$$

reflecting the ramification of W/V in a way similar to characteristic zero, even when W/V is a defect extension. When WL is p-divisible, the situation is much more complicated and we will sum up the main results and open problems. Joint work with S.D. Cutkosky.

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2.40 Large Fields: Old and New

Florian Pop

The aim of the talk is to review the theory of *large fields,* which were introduced by the author in [9]. The large fields generalize the PAC fields, the henselian (and/or complete fields), the fields which satisfy a universal local-global principle for smooth points on irreducible varieties, etc. It turns out that the class of large fields is the "right" class of fields over which one can do a lot of interesting mathematics, like (inverse) Galois theory [9], [2], [5], [6], [10], etc., arithmetical algebraic geometry [3], [7], [4], etc., model theory of function/valued fields [1], [8], etc. (This is maybe the reason why they acquired several other names...) I plan to present a few new results too, as well as a short list of open problems.

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2.41 Arc spaces and resolution

Ana Reguera

In the midsixties, J. Nash ([Na]) initiated the study of the space of arcs X_∞ of a singular variety X to understand what the various resolutions of singularities of X have in common. His work was developed in the context of the proof of resolution of singularities in characteristic zero by H. Hironaka. From the existence of a resolution of singularities, Nash deduces that the space of arcs X_∞^{Sing} on X centered in the singular locus Sing X of X, has a finite number of irreducible components. More precisely, he defines an injective map $\mathcal{N}_X$, now called the *Nash map*, from the set of *arc families*, i.e. irreducible components of X_∞^{Sing} which are not contained in (Sing $X)_\infty$, to the set of *essential divisors over* X, i.e. exceptional irreducible divisors which appear up to birational equivalence on every resolution of singularities of X. He asks whether this map is surjective, or more generally, how complete is the description of the essential divisors by the image of the Nash map.

In 1980, M. lejeune-Jalabert ([Le]) launched a new idea in relation with the problem: to use arcs in the space of arcs, or equivalently, wedges. However, the "curve selection lemma" does not hold in X_∞ because it is not a Noetherian space. This obstacle was shortcut in 2006 ([Re1]), by introducing the class of *generically stable irreducible subsets of* X_∞, and proving a curve selection lemma for the corresponding *stable points of* X_∞. The stable points are very far from being closed points: the residue field of such points is a transcendental extension of k of infinite transcendence degree. The points P_α of X_∞ corresponding to arc families are stable. In [Re1] the image of the Nash map is characterized in terms of a property of lifting wedges centered at the points P_α to some resolution of singularities of X. Moreover, in a further development, in [Re2], a property which is stronger in principle than the previous one was considered: a resolution of singularities Y of X satisfies the *property of lifting wedges centered at* P_α if every wedge on X whose special arc is P_α lifts to Y. This property holds if and only if P_α is a point of codimension one in X_∞.

The property of lifting wedges centered at any of the points P_α to a desingularization has been proved in all the cases where the Nash map has been proved to be bijective, and fails in the well-known 4-dimensional example given by S. Ishii and J. Kollar ([IK]) in 2003, for which the Nash map is not surjective. The fact that uniruled does not imply birationally ruled for projective varieties of dimension ≥ 3, is crucial in the construction of the example. A deeper study of this idea in [LR] allowed us to reduce the the property of lifting wedges centered at P_α to those essential divisors ν_α

which are uniruled. In particular, this property of lifting wedges for surface singularities over $\mathbb{C}$ is reduced to the case of quasirational surface singularities, i.e. those for which the exceptional curves of a desingularization are rational curves.

Moreover, we have proved recently ([Re3]) that the minimal desingularization Y of a rational surface singularity (S, P_0) over an algebraically closed field of characteristic zero satisfies the property of lifting wedges centered at any of the points P_α defined by the arc families, and in particular, the Nash map for S is surjective. This proves the Nash problem for rational surface singularities and, more generally, reduces the Nash problem for surfaces to quasirational normal singularities which are not rational. In positive characteristic, there are counterexamples to the k-wedge lifting problem for the surface singularity $x^3 + y^5 + z^2 = 0$.

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2.42 Quasi-Valuations Extending a Valuation

Shai Sarussi

Suppose F is a field with valuation v and valuation ring O_v, E is a finite field extension and w is a quasi-valuation on E extending v. We study quasi-valuations on E that extend v; in particular, their corresponding rings and their prime spectrums. We prove that these ring extensions satisfy INC (incomparability), LO (lying over), and GD (going down) over O_v; in particular, they have the same Krull Dimension. We also prove that every such quasi-valuation is dominated by some valuation extending v.

Under the assumption that the value monoid of the quasi-valuation is a group we prove that these ring extensions satisfy GU (going up) over O_v, and a bound on the size of the prime spectrum is given. In addition, a 1:1 correspondence is obtained between exponential quasi-valuations and integrally closed quasi-valuation rings.

Given R, an algebra over O_v, we construct a quasi-valuation on R; we also construct a quasi-valuation on $R \otimes_{O_v} F$ which helps us prove our main Theorem. The main Theorem states that if $R \subseteq E$ satisfies $R \cap F = O_v$ and E is the field of fractions of R, then R and v induce a quasi-valuation w on E such that $R = O_w$ and w extends v; thus R satisfies the properties of a quasi-valuation ring.

To clarify things I write here the definition of a quasi-valuation: a *quasi-valuation* on a ring R is a function $w : R \to M \cup \{\infty\}$, where M is a totally ordered abelian monoid, to which we adjoin an element ∞ greater than all elements of M, and w satisfies the following properties:

(B1) $w(0) = \infty$;

(B2) $w(xy) \geq w(x) + w(y)$ for all $x, y \in R$;

(B3) $w(x + y) \geq \min\{w(x), w(y)\}$ for all $x, y \in R$.

2.43 Introduction to Motivic Integration

Thomas Scanlon

In a lecture in 1995 Kontsevich introduced motivic integration to give a native geometric account of theorems previously proven via p-adic integration. Denef and Loeser shortly thereafter interpreted and computed these integrals through the model theory of valued fields. In the following years, the theory has developed largely through a finer analysis of definability in valued fields. The most striking recent developments, due independently and with somewhat different formalisms to Cluckers and Loeser and then also Hrushovski and Kazhdan, extend the Ax-Kochen-Ershov principle to integration theories involving exponentials. With these lectures I will discuss the general motivic integration project and expose in greater detail the new transfer theorems. (Please note that while only a few names have been explicitly mentioned in this abstract, many other researchers have proven important theorems in this subject.)

2.44 Key polynomials, generalized Puiseux series and applications to local uniformization in arbitrary characteristic

Mark Spivakovsky

The purpose of this talk is to explain the relationship between generalized power series, associated to a valuation centered in a local noetherian ring (as in Kaplansky's theorem), and the emerging theory of key polynomials for simple extensions of valued fields. The main principle I would like to emphasize is that key polynomials induce algebraic relations on truncations of generalized Puiseux series. These ideas lead to a much more explicit understanding of Kaplansky's theorem on the existence of generalized Puiseux series for valued fields. Finally, I will explain the relevance of these concepts to the problem of local uniformization in arbitrary characteristic.

2.45 Inseparable local uniformization

Michael Temkin

The main result that I will discuss in this lecture is the inseparable local uniformization theorem from [1] stating that local uniformization is possible after a purely inseparable extension of the field of rational functions. More concretely, for any integral variety X over a field k and a valuation ν of $K = k(X)$ with center on X there exist finite purely inseparable extensions k'/k and $K'/k'K$ and an alteration $X' \to X$ with $k(X') = K'$ such that the only lift of ν to $k(X')$ is centered on a k'-smooth point.

I will outline the main strategy of proving this theorem and describe some its generalizations that can be proved using a similar technique but are not yet written down.

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2.46 Extension of a valuation

Michel Vaquié

This talk is an overview of the problem of extending a valuation μ defined on a field K to a cyclic extension L de K, wich is defined by $L = K[X]/(P)$. The first result are obtained by Saunders MacLane in the case where the valuation μ is discrete of rank one ([2] and [3]) . He introduced the notions of *key-polynomial* and of *augmented valuations*. We have extended his results in the case of any valuation, and we have defined *admissible families of valuations*, ([4], [5], [6]). This notion may also be consiered as a generalisation of the *pseudo-Cauchy sequences* introduced by Kaplansky ([1]).

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2.47 Transserial Hardy fields

Joris van der Hoeven

Both Hardy fields and fields of transseries provide interesting models of valued fields with an additional ordered differential structure. In our talk we show that the field of differentially algebraic transseries over the real numbers can be embedded in a Hardy field. The proof relies on the concept of a "transserial Hardy field", which both carries the structure of a Hardy field and of a differential subfield of the field of transseries. We will associate analytic meanings to transseries using a technique of iterated integrals.

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2.48 Valuations with complex values

Oleg Viro

Hyperfields (fields with multivalued addition) provide a wide generailization for the notion of valuation: homomorphisms to hyperfields. Recently I found a degeneration of the field of complex numbers, which is a hyperfield, that is a field with multivalued addition. It allows to define complex tropical varieties some of which can be obtained as degenerations of complex varieties and whose amoebas are tropical varieties. Homomorphisms to the new hyperfield can be used to upgrade Berkovitch constructions.

2.49 Integration in certain o-minimal fields

Yimu Yin

I will first briefly explain Hrushovski-Kazhdan style motivic integration and then outline how to construct such an integration in certain o-minimal fields, namely those fields whose theories are T-convex. This is an ongoing project and not all the details have been thoroughly checked.

2.50 Ergodicity of the dynamical systems on 2-adic spheres

Ekaterina Yurova

Theory of the p-adic dynamical systems are intensively developed in [1], [8], [9], [12]. There are a lot of applications of this theory. For example theoretical p-adic physics [2], psychology [3] and cryptography [9], [11], [12], [13]. Another application of dynamical systems is analysis of the genetic code [4], [5]. One of the applied problems was to study ergodicity of the dynamical systems on the p-adic spheres [6], [7], [10]. The simple dynamical systems were considered first, i.e., iterations of the monomial mappings $x \to x^m, m = 2, 3, ...$ for the invariant spheres. It was shown that this dynamics is already complicated: the natural number $m > 1$ and the prime number $p > 1$ should have nontrivial connection to support a condition of ergodicity.

The past years, there was a big interest to study the p-adic dynamical systems $x \to f(x)$ with non-polynomial functions. Such non-smooth mappings are particularly useful in the construction of the pseudorandom number generators. The author with two collaborators V. Anashin and A. Khrennikov developed a new approach to study ergodicity of the p-adic dynamical systems. It is based on usage of the van der Put basis in the space of continuous functions $f : Z_p \to Z_p$, where Z_p is a ring of p-adic integers. The elements of the van der Put basis are constructed with characteristic functions of the p-adic balls, which are locally constant functions. Such basis enable us to study ergodicity of the p-adic dynamical systems with continuous functions, i.e. not only traditional classes of polynomial, analytic and smooth functions.

In a paper [14] it was obtained a criteria of ergodicity for $f : Z_2 \to Z_2$ in terms of the van der Put basis. This case ($p = 2$) plays an important role in cryptography.

Here we apply the technique based on the van der Put coefficients to study dynamical systems on the 2-adic spheres, $f : S_r(a) \to S_r(a)$, where $S_r(a) = \{x \in Q_p : |x - a|_p = r\}, a \in Z_p, r = 1/2^k, k = 1, 2, ...$ In particular, it solved the problem of stability for the ergodicity for non-smooth perturbations of monomial dynamical systems, i.e. the mappings $x \to x^s + v(x)$, where v is "small perturbations." Thus the problem to get criterion of ergodicity for the small perturbations of monomial dynamical systems on the spheres, which was stated in [6], is completely done for the 2-adic case. The transition to the case $p > 2$ is non-trivial and still open.

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2.51 Approaches to ramification theory in the imperfect residue field case

Igor Zhukov

The talk includes discussion of approaches to ramification theory for extensions of complete discrete valuation fields with imperfect residue fields. Such extensions occur naturally in the study of finite morphisms of schemes over a field of prime characteristic in dimension > 1.

We shall give a brief survey of classical ramification theory (perfect residue field case): lower and upper numbering of ramification groups, conductors, relation to class field theory, local-global formulas (Riemann-Hurwitz, Grothendieck-Ogg-Shafarevich).

In the general case some of the corresponding notions and results are still missing; very simple examples showing the nature of difficulties can be presented. We review several approaches made in attempt to overcome these difficulties:

- Theory of elimination of wild ramification;
- Perfection of residue fields;
- Application of higher class field theory, cohomological duality and rigid geometry;
- Semi-global models and arc spaces.

We shall discuss semi-continuity of ramification invariants on arc spaces and describe a relation of them with invariants of curve singularities.

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