QUICKSTEP ON NORMALIZATION

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Motivation

- $a = 1 + \sqrt[3]{2}$ is integral over $\mathbb{Z}$ since $(a-1)^3 - 2 = 0$.
- $s(x) = \sqrt{1+x}$ is integral over $\mathbb{R}[x]$ since $s^2 - (1+x) = 0$.
- $u(t) = t$ is integral over $\mathbb{C}[t^2, t^3]$ since $u^2 - t^2 = 0$.
- $u(x,y) = \frac{x}{y}$ is integral over $\mathbb{C}[x,y]/\langle x^2 - y^3\rangle$ since $u^2 - y = 0$.

In order to give sense to the algebraic equations satisfied by the integral elements, these have to belong to some chosen ring extension.

Algebraic Side

Let $R \subseteq S$ be a ring extension of commutative rings with $1_R = 1_S$ (often assumed to be Noetherian). An element $a \in S$ is called *integral over* R (or *integrally dependent on* R) if and only if a satisfies an equation

$$a^n + c_{n-1}a^{n-1} + \ldots + c_1 a + c_0 = 0 \qquad (*)$$

with all $c_i \in R$. Equivalent: a is integral over R if and only if $R[a]$ is a finitely generated R-module. The equation $(*)$ is called an *equation of integral dependence* satisfied by a over R.

- The extension $R \subseteq S$ is called *integral* if every element $a \in S$ is integral over R.
- R is called *integrally closed in* S, if ervery element in S which is integral over R already belongs to R.
- R is called *integrally closed* if R is integrally closed in Quot(R), the total quotient ring of R.
- R is called *normal* if R is reduced and integrally closed.

Theorem 1. (Finite extensions) *Let $R \subseteq S$ be a ring extension and let $a_1, \ldots, a_k \in S$ be integral over R. Then $R[a_1, \ldots, a_k]$ is a finitely generated R-module.*

Proof. Induction on k. Zariski-Samuel, Commutative Algebra, vol. I, p. 255.

Corollary. *Let $R \subseteq S$ be a ring extension. Then $\overline{R}^S = \{a \in S; a$ integral over $R\}$ is a subring of S which contains R, and $\overline{R}^S$ is integrally closed in S.*

The ring $\overline{R}^S$ is called the *integral closure* of R in S. In the case that S is the total quotient ring of R we denote the integral closure of R by $\overline{R}$. The most important case is the one in which R is an integral domain, its total quotient ring being then its quotient field.

When dealing with an integrally closed integral domain R (that is, with an integral domain which is integrally closed in its quotient field) we shall refer to R as an *integrally closed domain*. In the case that R is reduced, we call $\overline{R}$ the *normalization* of R. Notice that the normalization of a ring is normal.

Corollary. (Transitivity of integral dependence) *Let $R \subseteq S \subseteq T$ be ring extensions, with $R \subseteq S$ and $S \subseteq T$ integral. Then $R \subseteq T$ is an integral extension.*

Proof. Let a be an element of T, and let

$$a^n + c_{n-1}a^{n-1} + \ldots + c_0 = 0$$

($c_i \in S$) be an equation of integral dependence for a over S. Then the ring $S' := R[c_0, \ldots, c_{n-1}]$ is a finite R-module (Theorem 1). Since a is integral over S', $S'[a]$ is a finite S'-module, and therefore a finite R-module. Therefore a is integral over R.

Example 1. A unique factorization domain R is integrally closed. In fact, let $\frac{x}{y} \in \mathrm{Quot}(R)$ for $x, y \in R$ be integral over R; we may assume that gcd$(x,y) = 1$. Then there exist $c_i \in R$ with

$$\left(\frac{x}{y}\right)^n + c_{n-1}\left(\frac{x}{y}\right)^{n-1} + \ldots + c_1\left(\frac{x}{y}\right) + c_0 = 0.$$

We multiply the equation with y^n and conclude

$$x^n = -y(c_{n-1}x^{n-1} + \ldots + c_0 y^n).$$

Hence y divides x^n. Since gcd$(x, y) = 1$ it follows that y is invertible, and so $\frac{x}{y} \in R$.

Example 2. Let R be an integrally closed domain and let A be a multiplicatively closed set of non-zero divisors of R. Then the quotient ring R_A is integrally closed.

In fact, let x be an element of the common quotient field of R and R_A which is integral over R_A. Since any finite number of elements of R_A has a common denominator $d \in A$ there exist $c_i \in R$ with

$$x^n + \tfrac{c_{n-1}}{d}x^{n-1} + \ldots + \tfrac{c_1}{d}x + \tfrac{c_0}{d} = 0.$$

We multiply the equation with d^n and conclude that dx is integral over R. Therefore $dx \in R$, since R is integrally closed. Now we set $z := dx \in R$ and get $x = \frac{z}{d} \in R_A$.

Example 3. Let R be an integrally closed domain and I a prime ideal in R. Then the residue class ring R/I is in general *not* integrally closed. In fact, any finite integral domain $K[a_1, \ldots, a_n]$ over a field K is of the form R/I, where R is the polynomial ring $K[x_1, \ldots, x_n]$, but finite integral domains are not in general integrally closed. For this, consider the principal ideal $I = \langle x^2 - y^3 \rangle$. In this case, $\frac{x}{y}$ does not belong to the ring $K[x, y]$, but $\frac{x}{y}$ is integral over that ring since $(\frac{x}{y})^2 = y$.

Theorem 2. (Structure of integrally closed rings) *Let R be a Noetherian integral domain. Then R is integrally closed if and only if $R = \cap_p R_p$, where p runs over all minimal primes of R and the rings R_p are principal valuation rings.*

Proof. Zariski-Samuel I, chap. V, sec. 6; Mumford, The Red Book of Varieties and Schemes, chap. III, sec. 8, p. 272; Bourbaki, Commutative Algebra, Chap. 7.

Theorem 3. (Normalization Lemma) *Let $R = K[x_1, \ldots, x_n]$ be a finite integral domain over a field K, and let d be the transcendence degree of $K(x_1, \ldots, x_n)$ over K. Then there exist d linear combinations $y_1, \ldots, y_d$ of the x_i with coefficients in K, such that R is integral over $K[y_1, \ldots, y_d]$.*

Proof. Zariski-Samuel I, chap. V, sec. 4, p. 266, for infinite fields. The proof for finite fields is due to Nagata, cf. Mumford.

The next theorem is the basis for the existence of the normalization of algebraic varietes.

Theorem 4. (Kronecker-Noether) *Let $R = K[x_1, \ldots, x_n]$ be a finite integral domain over a field k, and let F be a finite algebraic extension of the quotient field $K(x_1, \ldots, x_n)$ of R. Then the integral closure $\overline{R}^F$ of R in F is a finite integral domain over K, and is a finite R-module.*

Proof. Uses the Normalization Lemma. Zariski-Samuel I., chap. V, sec. 4, pp. 264-267; de Jong-Pfister, Local Analytic Geometry, chap. 1.5, p. 41; Lang, Introduction to Algebraic Geometry, p. 120.

Theorem 5. (Zariski's Main Theorem) *Let R be a local (Noetherian) ring, $\hat{R}$ the completion of R and $\overline{R}$ the integral closure of R. Then the integral closure of $\hat{R}$ is the completion of $\overline{R}$, that is*

$$\overline{\hat{R}} = \hat{\overline{R}}.$$

Proof. Zariski-Samuel II, chap. VIII, 13, pp. 313-320. See Thm. 12 below for equivalent formulations.

Notice that the normalization of a local ring is not necessarily a local but only a semi-local ring. Zariski-Samuel II, chap. VIII, sec. 13, Lemma 3, p. 317; de Jong-Pfister, chap. 1.5, Remarks 4.4.4, p. 162.

Theorem 6. (Criterion for Normality, Grauert-Remmert) *Let R be a Noetherian reduced ring. Let $I = \sqrt{I} \subseteq R$ be a radical ideal such that I contains a nonzerodivisor of R and I is contained in all prime ideals p of R for which R_p is not normal. Then R is normal if and only if the canonical inclusion $R \subseteq Hom_R(I, I)$ is an equality.*

Proof. de Jong-Pfister, chap. 1.5, p. 38.

The proof uses the following lemma.

Lemma. *Let R be a Noetherian reduced ring, $\tilde{R}$ its normalization. Let $I \subseteq R$ be an ideal containing a nonzerodivisor of R. Then $R \subseteq \mathrm{Hom}_R(I, I) \subseteq \overline{R}$. If, moreover, I is radical, then $\mathrm{Hom}_R(I, I) = \overline{R} \cap \mathrm{Hom}_R(I, R)$.*

Proof. de Jong-Pfister, chap. 1.5, p. 38, Decker-Greuel-de Jong-Pfister, p. 2.

Example 4. Let $R = K[x, y]/\langle x \cdot y\rangle$ and $I = \langle x, y\rangle$ and set $u = \frac{x}{x+y}$. Then $R[u] = \mathrm{Hom}_R(I, I)$. Cf. de Jong-Pfister, p. 40.

Algorithms

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Geometric side

In this section, we assume that K is an algebraically closed field. Let X be an affine variety over K (i.e., an integral scheme of finite type over K) and let $K[X] = K[x_1, \ldots, x_n]/I$ be the affine coordinate ring of X ($I \leq K[x_1, \ldots, x_n]$ prime ideal). We denote the local ring of X at $a \in X$ by $\mathcal{O}_a$, say $\mathcal{O}_a = K[X]_{\mathcal{M}_a}$.

A point $a \in X$ is called a *normal point of* X, or X is said to be *normal at a*, if the ring $\mathcal{O}_a$ is integrally closed in its quotient field $K(X)$. X is called *normal* if it is normal at every point. Note that a variety X is normal if and only if its affine coordinate ring $K[X]$ is integrally closed in the function field $K(X)$. In fact, an intersection of integrally closed rings is integrally closed again and for any domain R, $R = \cap_p R_p$, the intersection running over all $p \subseteq R$ prime.

Example 5. Any regular point of X is a normal point.

Example 6. The cusp $x^2 = y^3$ is not normal at 0. (Compare with example 3.)

Example 7. The node $x^2 = y^2 + y^3$ is not normal at 0. In fact, $u := \frac{x}{y}$ does not belong to the ring $\mathcal{O}_0$, but u is integral over $K(X)$ since $u^2 = 1 + y$. Shafarevich, Basic Algebraic Geometry, p. 109, Mumford, chap. III, sec. 8, p. 279.

Theorem 7. *Let X be a normal variety and let $S = \mathrm{Sing}(X) \subseteq X$ be the singular locus of X. Then* $\mathrm{codim}_X(S) \geq 2$.

Proof. Mumford, chap. III, sec. 8, p. 273; Shafarevich, p. 111.

Corollary. *Let X be a curve. Then X is non-singular if and only if X is factorial if and only if X is normal.*

A variety X is said to be *non-singular in codimension 1* if $\mathrm{codim}_X(S) \geq 2$.

Theorem 8. *Let $X \subseteq \mathbb{A}^n$ be an irreducible affine hypersurface, or, more generally, a complete intersection. Then X is non-singular in codimension* 1 *if and only if X is normal.*

Proof. Mumford, chap. III, sec. 8, p. 274; de Jong-Pfister, chap. 6.5, Cor. 9, p. 263.

Example 8. The cone $x^2 + y^2 = z^2$ is singular at 0, not factorial, but normal, by Theorem 9. Mumford, chap. III, sec. 8, p. 277.

Example 9. The surface in 4-space of equations $x^2y - z^2 = y^3 - w^2 = 0$ is regular in codimension 1, but not normal at 0. Mumford, chap. III, sec. 8, p. 275. See Shafarevich, p. 112, for another example.

Example 10. $x^2 + y^2 + z^2 + v^2 + w^2 = 0$ is factorial and singular at 0. Mumford, chap. III, sec. 7, p. 277.

Example 11. $x^2 + y^3 + z^5 = 0$ has an isolated singularity in 0, it is factorial hence normal, Shafarevich, p. 112.

Example 12. $xy - zw = 0$ is a normal at 0, but not factorial. Mumford, chap. III, sec. 9, p. 291.

Example 13. The Whitney-umbrella $x^2 = y^2z$ has the z-axis as singular locus and is hence not normal.

For an example of integral closures in ring extensions different from the quotient field, see Mumford, p. 279, and Shafarevich, p. 126.

Let X be an affine variety and let L be a finite algebraic extension of $K(X)$. A *normalization of* X *in* L is a normal variety $\tilde{X}$ with function field $K(\tilde{X}) = L$, plus a finite surjective morphism $\pi : \tilde{X} \to X$ such that the induced map $\pi^* : K(X) \to K(\tilde{X}) = L$ is the given inclusion of $K(X)$ in L. If $L = K(X)$, so that π is birational, $\tilde{X}$ and π are simply called a *normalization of X*.

Theorem 9. (Normalization) *For every variety X and every finite algebraic extension L of $K(X)$, there is one and only one normalization of X in L: if $\pi_i : \tilde{X}_i \to X$ were 2 normalizations of X then there is a unique isomorphism $t : \tilde{X}_1 \to \tilde{X}_2$ such that $\pi_1 = \pi_2 \circ t$ and such that t^* is the identity map from L to L.*

Proof. Pass to th affine case and use the finiteness of $\overline{K[X]}$. Mumford, chap. III, sec. 8, p. 277; Shafarevic, chap. II, sec. 5.2, p. 113.

Example 13. The Whitney-umbrella $x^2 = y^2 z$ has normalization with coordinate ring $K[X][u]$ for $u = \frac{x}{y}$.

Theorem 10. *The set of normal points of a variety is open.*

Proof. Mumford, chap. III, 8, p. 278.

Question: Are there natural equations for the non-normal locus of a variety?

Theorem 11. *If X is a projective variety, then its normalization in any finite algebraic extension $L \supseteq K(X)$ is a projective variety.*

Proof. Via *Segre embeddings* of projective varieties. Mumford, chap. III, sec. 8, pp. 280-284; Shafarevich, pp. 120-122 for curves, Lang, pp. 134-139.

Theorem 12. (Zariski's Main Theorem)
Original form: *Let X be a normal variety over K and let $f : Y \to X$ be a birational morphism with finite fibres from a variety Y to X. Then f is an isomorphism of Y with an open subset $U \subseteq X$.*

Topological form: *Let X be a normal variety over $\mathbb{C}$, and let $a \in X$ be a closed point. Let S be the singular locus of X. Then there is a basis $\{U_i\}$ of complex neighbourhoods of a such that $U_i \setminus U_i \cap S$ is connected, for all i.*

Power series form: *Let X be a normal variety over K and let $a \in X$ be a normal point (not necessarily closed). Then the completion $\hat{O}_{X,a}$ is an integral domain, integrally closed in its quotient field.*

Grothendieck's form: *Let $f : Y \to X$ be a morphism of varieties over K with finite fibres. Then there exists a map $g : Z \to X$ where Z is variety, Y is an open set in Y and g is a* finite *morphism.*

Connectedness Theorem: *Let X be a variety over K, normal at a closed point a. Let $f : Y \to X$ be a birational proper morphism. Then $f^{-1}(a)$ is a connected set (in the Zariski topolgy).*

Proof. Mumford, chap. III, sec. 9, pp. 286-295; Zariski-Samuel, pp. 313-320; EGA, chap. III, 4.3, and IV, Lang, p. 124. The original proof of Zariski appeared in: Theory and applications of holomorphic functions on algebraic varieties over arbitrary ground fields. Memoirs Amer. Math. Soc. 1951.

Analytic Side

Let $X \subset \mathbb{C}^n$ be an analytic variety. A holomorphic function $f : X \setminus \mathrm{Sing}(X) \to \mathbb{C}$ is *weakly holomorphic* on X if it is locally bounded at all points of X (i.e., for all $a \in X$, there is a neighborhood U of a in X so that $f_{|U}$ is bounded). The set $\tilde{\mathcal{O}}_{X,a}$ of germs of weakly holomorphic functions on X at a point $a \in X$ forms an overring of $\mathcal{O}_{X,a}$.

Example 14. For X be the union of the two coordinate axes in $\mathbb{C}^2$, the function $f = \frac{x}{x+y}$ on $X \setminus \mathrm{Sing}(X)$ is weakly holomorphic on X.

Theorem 13. *The ring $\tilde{\mathcal{O}}_{X,a}$ equals the integral closure $\overline{\mathcal{O}}_{X,a}$ of $\mathcal{O}_{X,a}$, and X is normal at a if and only if $\tilde{\mathcal{O}}_{X,a} = \mathcal{O}_{X,a}$.*

Proof. De Jong-Pfister, chap. 4.4, p. 167. Compare the assertion with the Riemann Extension Theorem for manifolds, cf. de Jong-Pfister, pp. 84-85:

If $U \subseteq \mathbb{C}^n$ is open and connected, $X \subset U$ analytic, $f : U \setminus X \to \mathbb{C}$ analytic and locally bounded at all points $a \in U$, then f has an analytic extension to U. Similarly, for $n \geq 2$ and $a \in U$, any analytic $f : U \setminus \{a\} \to \mathbb{C}$ has an analytic extension to U.