Sheaves.

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Definition 1. Let X be a topological space. A **presheaf** over X is a functor $\mathcal{F} : \mathrm{Op}(X)^{op} \to \mathbf{Sets}$, such that $\mathcal{F}(\emptyset) = \{*\}$. Where **Sets** is the category of sets, $\{*\}$ denotes a set with one element and $\mathrm{Op}(X)$ is the category with objects the open sets of X and arrows the inclusions. $\mathrm{Op}(X)^{op}$ denotes the oposite category: same objects and reversed arrows.
A morphism of presheaves is a morphism in the functor category.

$$\begin{array}{rcl} \mathcal{F} : \mathrm{Op}(X)^{op} & \to & \mathbf{Sets} \\ U & \to & \mathcal{F}(U) = \Gamma(U, \mathcal{F}) \end{array}$$

Every element of $s \in \mathcal{F}(U)$ is called a section of $\mathcal{F}$ in U. In the literature is used also the notation $\Gamma(\mathcal{F}, U) = \mathcal{F}(U)$.
If $V \subset U$ are two open sets, the restriction morphism is:

$$\begin{array}{ccc} U & \supset & V \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \to & \mathcal{F}(V) \\ s & \to & s|_V \end{array}$$

Definition 2. A presheaf $\mathcal{F}$ is a **sheaf** if the following condition holds for every open set U:

- Given any open covering $\{U_i\}_{i \in I}$ of U and given sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

One may define in a similar way presheave and sheaves of Groups, Rings, Modules, just by changing the category of sets by the corresponding category.

Exercise 3. Let X be an algebriac variety over a field k. We define a sheaf $\mathcal{O}$ on X: For every open U, $\mathcal{O}(U)$ is the set of regular functions defined in $U \to k$. Recall that regular functions are the functions which are locally a quotient of polynomials (of the same degree if $X \subset \mathbb{P}^n$).
It is obvious that $\mathcal{O}$ is a presheaf if we consider the usual restrictions of functions.
Prove that $\mathcal{O}$ is a sheaf.
Hint: By patching one obtains a function, which is a regular function by definition.

Example 4. A presheaf which is not a sheaf.
Let X be a topological space and let A be a fixed set. Define $\mathcal{F}(U) = A$ for every open set U. Is is trivial that $\mathcal{F}$ is a presheaf, the constant presheaf.
But $\mathcal{F}$ is not a sheaf. Let $U = U_1 \cup U_2$ be an open set with two connected components, $U_1 \cap U_2 = \emptyset$, Consider two sections $s_i \in A = \mathcal{F}(U_i)$, $i = 1, 2$. If $s_1 \neq s_2$ there is not any section $s \in A = \mathcal{F}(U)$ with $s|_{U_i} = s_i$.

Definition 5. We recall the definitions of limit and colimit in a category $\mathcal{C}$.
Given objects $\{C_i\}_{i\in I}$ and morphisms $\{f_{ij} : C_i \to C_j\}_{(i,j)\in\Lambda}$, with $\Lambda \subset I \times I$. One may define the limit and the colomit of the C_i's (w.r.t the morphisms f_{ij}):
The limit $\varprojlim_{i\in I} C_i$, if it exists, is an object C together with morphisms $f_i : C \to C_i$ such that:

- For every object A and morphisms $g_i : A \to C_i$ such that $f_{ij} g_i = g_j$ for every $(i,j) \in \Lambda$, there exists a unique morphism $g : A \to C$ with $g_i = f_i g$, for any $i \in I$.

$$\begin{tikzcd}
 & & C_i \arrow[dd, "f_{ij}"] \\
A \arrow[urr, "g_i"] \arrow[r, dashed, "g"] \arrow[drr, "g_j"'] & C \arrow[ur, "f_i"] \arrow[dr, "f_j"'] & \\
 & & C_j
\end{tikzcd}$$

The colimit $\varinjlim_{i\in I} C_i$, if it exsits, is an object C together with morphisms $f_i : C_i \to C$ such that:

- For every object A and morphisms $g_i : C_i \to A$ such that $g_j f_{ij} = g_i$ for every $(i,j) \in \Lambda$, there exists a unique morphism $g : A \to C$ with $g_i = g f_i$, for any $i \in I$.

$$\begin{tikzcd}
C_i \arrow[dd, "f_{ij}"'] \arrow[dr, "f_i"] \arrow[drr, "g_i"] & & \\
 & C \arrow[r, dashed, "g"] & A \\
C_j \arrow[ur, "f_j"] \arrow[urr, "g_j"'] & &
\end{tikzcd}$$

We say that the limit is filtered if for every $(i,j) \in \Lambda$ there is a $k \in I$ with $(k,i),(k,j) \in \Lambda$.
The colimit is cofiltered if for every $(i,j) \in \Lambda$ there is a $k \in I$ with $(i,k),(j,k) \in \Lambda$.

A special case of limit and colimit is when $\Lambda = \emptyset$, we do not have morphisms f_{ij}. In this case the limit is called product and the colimit is the coproduct.

Definition 6. Let $\mathcal{F}$ be a presheaf and let $x \in X$. The stalk of $\mathcal{F}$ at the point x is

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$$

The stalk $\mathcal{F}_x$ contains all the *germs* of sections at the point x. Every element $\bar{s} \in \mathcal{F}_x$ is an equivalence class, a representant is a pair (s,U) where $s \in \mathcal{F}(U)$ and $x \in U$. Two representants (s_1,U_1), (s_2,U_2) are equivalent if $s_1|_{U_1\cap U_2} = s_2|_{U_1\cap U_2}$ in $\mathcal{F}(U_1 \cap U_2)$.
If $s \in \mathcal{F}(U)$ is a section and $x \in U$, the natural map $\mathcal{F}(U) \to \mathcal{F}_x$ sends s to the germ, which we will denote by s_x.

Exercise 7. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X. If $\varphi : \mathcal{F} \to \mathcal{G}$ is morphism of sheaves then prove that for any $x \in X$ it induces naturally a morphism on the stalks $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$.

Proposition 8. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X. A morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if for every $x \in X$ the induced morphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism.

Proof. If φ is isomorphism, there is an inverse φ^{-1} and then for every $x \in X$ φ_x^{-1} is the inverse of φ_x. Conversely, assume now that φ_x is an isomorphism for all $x \in X$.
Let $U \subset X$ be an open set. First we prove that $\varphi(U)$ is injective. Consider two sections $s_1, s_2 \in \mathcal{F}(U)$ such that $t = \varphi(U)(s_1) = \varphi(U)(s_2)$. For any point x we have $\varphi_x(s_{1,x}) = \varphi_x(s_{2,x}) = t_x$ so that by hypothesis $s_{1,x} = s_{2,x}$ for all $x \in U$. For every $x \in U$ there is an open set $x \in V_x \subset U$ such that $s_1|_{V_x} = s_2|_{V_x}$. Now $\{V_x\}_{x\in U}$ is a covering of U and by the sheaf property $s_1 = s_2$.
Now we construct the inverse $\varphi^{-1}(U) : \mathcal{G}(\mathcal{U}) \to \mathcal{F}(U)$ for any open set U.
Let $t \in \mathcal{G}(U)$ and consider the germ $t_x \in \mathcal{G}_x$ at every $x \in U$. Let $s_x = \varphi_x^{-1}(t_x)$, the germ s_x has a representant $s(x) \in \mathcal{F}(V_x)$ for some open $x \in V_x \subset U$. The sections $\varphi(V_x)(s(x))$ and $t|_{V_x}$ represent the same germ t_x at x, so that we may assume that V_x is small enough such that $\varphi(V_x)(s(x)) = t|_{V_x}$.
We have section $s(x) \in \mathcal{F}(V_x)$, where $\{V_x\}_{x\in U}$ is a covering of U. We want to prove that they patch to a section $s \in \mathcal{F}(U)$. If $x, y \in U$ the restrictions $s(x)|_{V_x\cap V_y}$ and $s(y)|_{V_x\cap V_y}$ are such that

$$\varphi(V_x \cap V_y)\left(s(x)|_{V_x\cap V_y}\right) = \varphi(V_x \cap V_y)\left(s(y)|_{V_x\cap V_y}\right)$$

We have already prove that $\varphi(V_x \cap V_y)$ is injective, so that $s(x)|_{V_x\cap V_y} = s(y)|_{V_x\cap V_y}$ and the sections $\{s(x)\}_{x\in U}$ patch to a unique section $s \in \mathcal{F}(U)$.
We have defined maps $\mathcal{G}(U) \to \mathcal{F}(U)$ for $U \in \mathrm{Op}(X)$. Exercise: check that those maps are $(\varphi(U))^{-1}$ and they define a map of sheaves $\mathcal{G} \to \mathcal{F}$ inverse of φ. □

Definition 9. Let $\mathcal{F}$ be a presheaf. The associated sheaf to $\mathcal{F}$ is a sheaf $\mathcal{F}^+$ together with a morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$ such that:

- For any sheaf $\mathcal{G}$ and any morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ with $\varphi = \psi\theta$.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathcal{G} \\ \mathcal{F}^+ \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

Proposition 10. The sheaf associated to a presheaf exists and it is unique up to isomorphism.

Proof. Let $\mathcal{E} = \sqcup_{x\in X}\mathcal{F}_x$. Define $\mathcal{F}^+(U)$ as the set of maps $s : U \to \mathcal{E}$ such that

- $s(x) \in \mathcal{F}_x$, for all $x \in U$.
- s is locally given by sections of $\mathcal{F}$. More precisely for every $x \in U$ there is a neighborhood $x \in V \subset U$ and a section $t \in \mathcal{F}(V)$ such that $s(y) = t_y$ for all $y \in V$.

□

Exercise 11. Check that $\mathcal{F}^+$ is a sheaf and it satisfies the universal property of the definition 9.

Exercise 12. Let $\mathcal{F}$ be a sheaf. Prove that the associated sheaf to $\mathcal{F}$ is $\mathcal{F}^+ = \mathcal{F}$.

Exercise 13. Consider the presheaf of 4. Prove that the associated sheaf $\mathcal{F}^+$ is the following: If U is an open set with r connected components, then

$$\mathcal{F}^+(U) = A \sqcup \overset{r}{\cdots} \sqcup A$$

Definition 14. Let $f : X \to Y$ a continuous map.

- If $\mathcal{F}$ is a sheaf on X, we define the direct image sheaf $f_*\mathcal{F}$:

$$f_*\mathcal{F}(V) = \mathcal{F}\left(f^{-1}(V)\right), \qquad V \in \mathrm{Op}(Y)$$

- If $\mathcal{G}$ is a sheaf on Y, we define the sheaf inverse image $f^{-1}(\mathcal{G})$ as the associated sheaf to the presheaf:

$$U \to \varinjlim_{V \supset f(U)} \mathcal{G}(V), \qquad U \in \mathrm{Op}(X)$$

Example 15. A sheaf $\mathcal{G}$ such that the inverse image presheaf is not a sheaf.
Assume that $U = U_1 \cup U_2$ is a union of two connected components and $W = f(U_1) = f(U_2)$ is an open set of Y.

$$f^{-1}\mathcal{G}(U_i) = \varinjlim_{V \supset f(U_i)} \mathcal{G}(V) = \mathcal{G}(f(U_i)) = \mathcal{G}(W), \qquad i = 1, 2$$

$$\varinjlim_{V \supset f(U)} \mathcal{G}(V) = \mathcal{G}(W)$$

$$f^{-1}\mathcal{G}(U) = \mathcal{G}(W) \sqcup \mathcal{G}(W)$$

Exercise 16. Construct explicitly the previous example.

Proposition 17. Let $f : X \to Y$ be a continuous map.

1. If $\mathcal{F}$ is a sheaf on X, there is a natural morphism of sheaves on X, $f^{-1}f_*\mathcal{F} \to F$.
2. If $\mathcal{G}$ is a sheaf on Y, there is a natural morphism of sheaves on Y, $\mathcal{G} \to f_* f^{-1}\mathcal{G}$.
3. Let $\mathcal{F}$ be a sheaf on X and $\mathcal{G}$ be a sheaf on Y. There is a natural bijection of sets:

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$$

Proof.

1. If $U \in \mathrm{Op}(X)$,

$$f^{-1}f_*\mathcal{F}(U) = \varinjlim_{V \supset f(U)} f_*\mathcal{F}(V) = \varinjlim_{V \supset f(U)} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$$

   where the last arrow comes from $f^{-1}(V) \supset U$. Then we have a map of presheaves and the universal property of 9 gives the result.

2. Let $V \in \mathrm{Op}(Y)$,

$$\mathcal{G}(V) \to \varinjlim_{W \supset f(f^{-1}(V))} \mathcal{G}(W) = f^{-1}\mathcal{G}(f^{-1}(V)) = f_* f^{-1}\mathcal{G}(V)$$

   the first arrow comes from $V \supset f(f^{-1}(V))$. This map of presheaves induces a map of sheaves.

3. It follows from the previous results.

□

Definition 18. An **étalé space** over a topological space X is a topological space E togheter with a morphism $\Pi : E \to X$ such that for any point $\xi \in E$ there exists a neighborhood $W \subset E$ such that $\Pi|_W$ defines an homeomorphism to an open set of X.
A section of the étalé space is a continuous map $s : U \to E$ defined in an open set U of X such that $\Pi \circ s = \mathrm{Id}_U$.

It is easy to check that the sections of the étalé space form a sheaf on X, which we will denoted by $\mathcal{F}(E)$.

Proposition 19. Let $\mathcal{F}$ be a presheaf. Define $\mathcal{E} = \sqcup_{x \in X} \mathcal{F}_x$ and $\Pi : \mathcal{E} \to X$ the natural projection. For every open set U, and every section $s \in \mathcal{F}(U)$, we define $s : U \to \mathcal{E}$ as $s(x) = s_x$. Give to $\mathcal{E}$ the strongest topology such that the maps s are continuous.
Then $\mathcal{E}$ is an étalé space. Moreover the sheaf $\mathfrak{F}(\mathcal{E})$ is the sheaf $\mathcal{F}^+$ associated to the presheaf $\mathcal{F}$.

Proof. Exercise. □

Exercise 20. The category of sheaves on X is equivalent to the category of étalé spaces over X.