SEDANO 2005: AN INTRODUCTION TO CONSTRUCTIVE DESINGULARIZATION.

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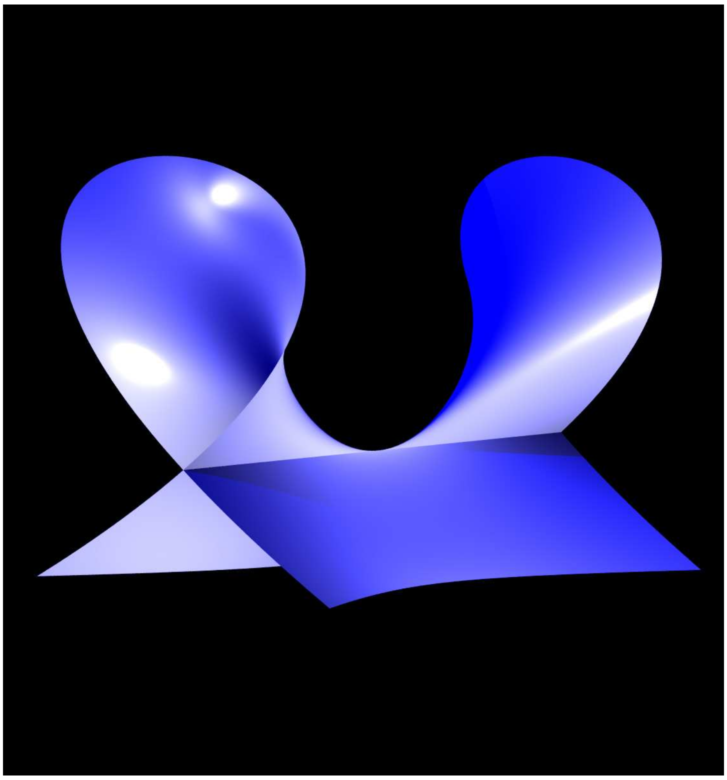

In these notes we prove two important Theorems of algebraic geometry over fields of characteristic zero:

1) Desingularization (or Resolution of singularities).

2) Embedded Principalization or Log-Resolution of ideals.

Both results, stated in Theorems 1.2 and 1.3, are due to Hironaka. We focus here on the proof in [15], which is more elementary than that of Hironaka. In fact, it avoids the use of Hilbert Samuel functions, and of normal flatness.

Hironaka's proof of both theorems is *existential*; he proves that every singular variety, over a field of characteristic zero, can be desingularized. Our proof of the theorems is *constructive*, in the sense that we provide an algorithm to achieve such desingularization. We refer to [5]

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and to [16] for two computer implementations. Bodnár-Schicho's implementation available at

http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup

There are several other proofs of these two theorems, which also provide an algorithm: [3], [10], [12], [25], and [27].

These notes are written as an introduction to the subject. Resolution of singularities is based on a peculiar form of induction. In the case of resolution of hypersurfaces this form of induction was stated clear and explicitly by Abhyankar, in what is called a Tschirnhausen transformation.

We will focus on this point in Part 1, where we discuss examples of this form of induction, with some indication on how it provides inductive invariants. These invariants are gathered in our *resolution functions*, and we prove the two Main Theorems 1.2 and 1.3 by extracting natural properties from these functions. In Part II we prove results which were motivated through examples in the first Part. In Part III we introduce the resolution functions. A mild technical aspect appears in Part II, where the behavior of derivations and monoidal transformations are discussed. But essentially the first three parts are intended to provide a conceptual (non-technical) and self-contained introduction to desingularization.

The algorithm in these notes is equivariant, and it also extends to étale topology. However we do not study these properties in these introductional notes, and we refer to [8] and [14] for the study of these and of further properties of this proof. Among these further properties discussed in those cited papers, there is a new and remarkable formulation of embedded desingularization, with a strong algebraic flavor, obtain in [10] (see 4.4 in these notes).

We finally refer to the notes of D. Cutkosky [11], H. Hauser [19], and K. Matsuki [23], for other introductions to desingularization theorems.

The picture in the front page, of the surface $x^2 - z^3 = y^2z^2$, was produced by Sebastian Gann, University of Innsbruck, Austria (FWF project P15551).

1. First definitions and formulation of Main Theorem.

The set of regular points, of a reduced scheme of finite type over a field, is a dense open set.

Definition 1.1. We say that a birational morphism of reduced irreducible schemes

(1.1.1) $$X \xleftarrow{\pi} X'$$

is a desingularization of X if:

i) π defines an isomorphism over the open set $U = Reg(X)$ of regular points.

ii) π is proper, and X' is regular.

We will prove the existence of desingularizations, over fields of characteristic zero, by proving a theorem of *embedded desingularization* in Theorem 1.2. There we view an irreducible scheme as a closed subscheme in a smooth scheme W.

Let $W_1 \xleftarrow{\pi} W_2$ be a proper birational morphism of smooth schemes of dimension n. If a closed point $x_2 \in W_2$ maps to $x_1 \in W_1$, there is a linear transformation of n-dimensional tangent spaces, say $T_{W_2,x_2} \to T_{W_1,x_1}$. The set of points $x_2 \in W_2$ for which $T_{W_2,x_2} \to T_{W_1,x_1}$ is not an isomorphism defines a hypersurface H in W_2, called the jacobian or exceptional hypersurface. It turns out that there is an open set $U \subset W_1$ such that $U \stackrel{\pi}{\longleftarrow} \pi^{-1}(U)$ is an isomorphism, and $\pi^{-1}(U) = W_2 - H$. Examples of proper birational morphisms of this kind are the monoidal transformations, defined by blowing up a closed and smooth subscheme Y in a smooth scheme W_1. In such case $H = \pi^{-1}(Y)$ is a smooth hypersurface. Let

$$\begin{array}{ccccccc} W_0 & \longleftarrow & (W_1, E_1 = \{H_1\}) & \longleftarrow & (W_2, E_2 = \{H_1, H_2\}) \quad \cdots & \longleftarrow & (W_r, E_r = \{H_1, H_2, .., H_r\}) \\ Y & & Y_1 & & Y_2 & & \end{array} \tag{1.1.2}$$

be a composition of monoidal transformations, where each $Y_j \subset W_j$ is closed and smooth, $H_j \subset W_j$ is the exceptional hypersurface of $W_{j-1} \leftarrow W_j$ (the blow up at Y_{j-1}), and where $\{H_1, H_2, .., H_r\}$ denote the strict transforms of the $H_i's$ in W_r. The composite $W_0 \leftarrow W_r$ is a proper birational morphism of smooth schemes, and $H = \cup_{1 \leq i \leq r} H_i$ is the exceptional hypersurface.

Theorem 1.2 (Embedded Resolution of Singularities). *Given W_0 smooth over a field k of characteristic zero, and $X_0 \subset W_0$ closed and reduced, there is a sequence (1.1.2) such that*

(i) $\cup_{i=1}^r H_i$ *have normal crossings in* W_r.

(ii) $W_0 - \mathrm{Sing}(X_0) \simeq W_r \setminus \cup_{i=1}^r H_i$, *and hence it induces a square diagram*

$$\begin{array}{ccc} W_0 & \stackrel{\Pi_r}{\longleftarrow} & W_r \\ \cup & & \cup \\ X_0 & \stackrel{\overline{\Pi}_r}{\longleftarrow} & X_r \end{array}$$

of proper birational morphisms, where X_r denotes the strict transform of X_0.

(iii) X_r *is regular and has normal crossings with* $E_r = \cup_{i=1}^r H_i$.

In particular $\mathrm{Reg}(X_0) \cong \overline{\Pi}_r^{-1}(\mathrm{Reg}(X_0)) \subset X_r$ and $X_0 \stackrel{\overline{\Pi}_r}{\longleftarrow} X_r$ is a desingularization (1.1).

Theorem 1.3 (Embedded Principalization of ideals). *Given $I \subset \mathcal{O}_{W_0}$, a non-zero sheaf of ideals, there is a sequence (1.1.2) such that:*

(i) *The morphism $W_0 \leftarrow W_r$ defines an isomorphism over $W_0 \setminus V(I)$.*

(ii) *The sheaf $I\mathcal{O}_{W_r}$ is invertible and supported on a divisor with normal crossings, i.e.,*

$$\mathcal{L} = I\mathcal{O}_{W_r} = \mathcal{I}(H_1)^{c_1} \cdot \ldots \cdot \mathcal{I}(H_s)^{c_s}, \tag{1.3.1}$$

where $E' = \{H_1, H_2, \ldots, H_s\}$ are regular hypersurfaces with normal crossings, $c_i \geq 1$ for $i = 1, \ldots, s$, and $E' = E_r$ if $V(I)$ has no components of codimension 1.

Part I

Throughout these notes W will denote a smooth scheme of finite type over a field k of characteristic zero. We first recall here some definitions used in the formulation of the previous theorems.

Definition 1.4. Fix $y \in W$, and let $\{x_1, \ldots, x_d\}$ be a regular system of parameters (r.s. of p.) in the local regular ring $\mathcal{O}_{W,y}$.

1) $Y(\subset W)$, defined by $I(Y) \subset \mathcal{O}_W$, is **regular at** $y \in Y$, if there is a r. s. of p. such that $I(Y)_y =< x_1, ..., x_s >$ in $\mathcal{O}_{W,y}$.

2)A set $\{H_1, \ldots, H_r\}$ of hypersurfaces in W has *normal crossings at* y if there is a r.s. of p. such that $\cup H_i = V(\langle x_{j_1} \cdot x_{j_2} \cdots x_{j_s} \rangle)$ locally at y, for some $j_i \in \{1, \ldots r\}$.

3) A closed subscheme Y has normal crossings with E at y, if there is a r.s. of p. such that, locally at y:

$$I(Y)_y = < x_1, ..., x_s > \text{ and } \cup H_i = V(\langle x_{j_1} \cdot x_{j_2} \cdots x_{j_s} \rangle).$$

Y is said to be *regular* if it is regular at any point; and $E = \{H_1, \dots, H_r\}$ is said to have normal crossings if the condition holds at any point.

Remark 1.5. If

$$\underset{Y}{W_0} \xleftarrow{\pi} W_1 \supset H = \pi^{-1}(Y),$$

denotes a monoidal transformation with a closed and regular center $Y (\subset W_0)$, then:
1) π is proper and W_1 smooth.
2) $H = \pi^{-1}(Y)$ is a smooth hypersurface in W_1.
3) $W_0 - Y \cong W_1 - H$ (i.e. π is birational).

Definition 1.6. The *order* of an non-zero ideal J in a local regular ring (R, M) is the biggest integer $b \geq 0$ such that $J \subset M^b$.

Remark 1.7. Assume that Y in 1.5 is irreducible with generic point $y \in W$, and let $h \in W_1$ be the generic point of H. Note that $\mathcal{O}_{W,y}$ is a local regular ring, and that $\mathcal{O}_{W_1,h}$ is a discrete valuation ring. Let M_y denote the maximal ideal of $\mathcal{O}_{W,y}$.

Set $W_0 \longleftarrow W_1$ and $H \subset W_1$ as above. Then, for an ideal $J \subset \mathcal{O}_W$, the following are equivalent:

a) $J_y \subset M_y^b$ (i.e. the order of J at $\mathcal{O}_{W,y}$ is $\geq b$)

b)$J\mathcal{O}_{W_1} = I(H)^b \cdot J_1$ for some J_1 in $\mathcal{O}_{W_1}$.

c) $J\mathcal{O}_{W_1}$ has order $\geq b$ at $\mathcal{O}_{W_1,h}$.

Definition 1.8. Given a sheaf of ideals $J \subset \mathcal{O}_X$ and a morphism of schemes, $X \leftarrow Y$, the sheaf of ideals $J\mathcal{O}_Y$ is called the *total transform* of J in Y. In the previous remark we considered the total transform by a monoidal transformation, and we do not assume b to be the order of J at the generic point of Y. When such condition holds, then b is the highest integer for which an expression $J\mathcal{O}_{W_1} = I(H)^b \cdot J_1$ can be defined; and in such case J_1 is called the *proper transform* of J.

The following result will be used to ensure that E_r has normal crossings in a sequence of monoidal transformations (1.1.2).

Proposition 1.9. *Let W be smooth over k, and let $E = \{H_1, \dots, H_s\}$ be a set of smooth hypersurfaces with normal crossings. Assume that $Y (\subset W)$ is closed, regular, and has normal crossings with $E = \{H_1, \dots, H_s\}$, and set the monoidal transformation*

$$\underset{Y}{(W, E = \{H_1, \dots, H_s\})} \xleftarrow{\pi} (W_1, E_1 = \{H'_1, \dots, H'_s, H_{s+1} = \pi^{-1}(Y)\})$$

where H'_i denotes the strict transform of H_i. Then E_1 has normal crossings in W_1.

2. Examples: Tschirnhausen and a form of induction on resolution problems.

A variety, or an ideal, is usually presented by equations in a certain number of variables. A key point in resolution problems is to argue by induction on the number of variables involved. In order to illustrate the precise meaning of this form of induction we first consider the polynomial $f = Z^2 + 2 \cdot X \cdot Z + X^2 + X \cdot Y^2 \in k[Z, X, Y]$, defining a hypersurface $\mathbb{X} \subset \mathbb{A}^3_k$, where k denotes here an algebraically closed field of characteristic zero. We will see that all points in this hypersurface are of multiplicity at most two.

Question: How to describe the closed set of points of multiplicity 2?, say $\mathcal{F}_2 \subset \mathbb{X}$.

Recall first two definitions:

Definition 2.1. Set $p \in \mathbb{X} = V(\langle f \rangle) \subset \mathrm{Spec}(k[Z, X, Y])$. We say that the hypersurface $\mathbb{X}$ has *multiplicity* b at p, or that p is a *b-fold point* of the hypersurface, if $\langle f \rangle$ has order b at the local regular ring $k[Z, X, Y]_p$ (1.6). We will denote by $\mathcal{F}_b$ the set of points in $\mathbb{X}$ with multiplicity b.

There are now two ways in which we can address our question.

Approach 1): Consider the extension of the ideal $J = \langle f \rangle$, say:

$$J(1) = \langle f, \frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z} \rangle.$$

Clearly $V(J(1)) = \mathcal{F}_2$. In fact, by taking Taylor expansions at any closed point q we conclude that $q \in V(J(1))$ if and only if the multiplicity of $\mathbb{X}$ at q is at least 2. Note also that $\mathbb{X}$ has no closed point of multiplicity higher than 2 since $\frac{\partial^2 f}{\partial^2 Z}$ is a unit. So the hypersurface $\mathbb{X}$ has only closed points of multiplicity one and two.

As for the non-closed points of $\mathbb{X}$, recall first that in a polynomial ring any prime ideal is the intersection of all maximal ideals containing it. On the other hand the multiplicity defines an upper-semi-continuous function on the hypersurface. So the multiplicity at a non-closed point, say $y \in \mathbb{X}$, coincides with the multiplicity at closed points in an non-empty open set of the closure $\overline{y}$. This settles our question.

2.2. Approach 2) (linked to the previous): Set $Z_1 = Z + X$. At $k[Z_1, X, Y] = k[Z, X, Y]$:

$$f = Z_1^2 + X \cdot Y^2. \tag{2.2.1}$$

2i) Note first that $Z_1 \in J(1)$, and hence $\mathcal{F}_2 \subset \overline{W}$, where $\overline{W} = V(Z_1)$ is a smooth hypersurface.

2ii) Set $J^* = \langle X \cdot Y^2 \rangle \subset \mathcal{O}_{\overline{W}}$. We claim that $\mathcal{F}_2 \subset \overline{W}$ is also defined as the set of points $q \in \overline{W}$ where the order of J^*, at the local regular ring $\mathcal{O}_{\overline{W},q}$, is at least 2.

In fact, if $q \in \mathrm{Spec}(k[Z, X, Y])$ is a point (a prime ideal) of order 2, then $J(1) \subset q$, so

$$Z_1 \in q \subset k[Z_1, X, Y].$$

It is clear that among the prime ideals containing Z_1, those where $Z_1^2 + X \cdot Y^2$ has order 2, are those where $X \cdot Y^2$ has order at least 2. So the claim follows by setting $\overline{W} = V(Z_1)$ and $J^* = \langle X \cdot Y^2 \rangle \subset \mathcal{O}_{\overline{W}}$ as before.

2.3. We will see that the answer to our earlier Question, provided in Approach 2, is better adapted to resolution problems, at least over fields of characteristic zero.

We started by asking for those points where the ideal $\langle f \rangle \subset k[Z, X, Y]$ has order at least 2. So we fixed an ideal J ($J = \langle f \rangle$ in this case), and a positive integer b ($b = 2$ in this case), and we considered the closed set $\mathcal{F}_2$ of points where this ideal has order 2. We ended up with a new ideal, $J^* = \langle X \cdot Y^2 \rangle$ in the ring of functions in $\overline{W}$, where

$$\overline{W} = (Spec(k[X,Y]) =)Spec(k[Z_1,X,Y]/\langle Z_1 \rangle) \subset Spec(k[X,Y,Z]),$$

together with an integer $b_1 = 2$, describing the same closed set $\mathcal{F}_2$, but involving one variable less.

Definition 2.4. Fix a scheme W, smooth over a field of characteristic zero. A *couple* will be an ideal $J \subset \mathcal{O}_W$ and an integer b, and will be denoted by (J, b).

The *set described by the couple* will be the set of points $\{x \in W/\nu_x(J) \geq b\}$, where $\nu_x(J)$ denotes the order of J at the local regular ring $\mathcal{O}_{W,x}$.

2.5. The set described by the couple $(J = \langle Z_1^2 + X \cdot Y^2 \rangle, 2)$ in $\mathbb{A}^3_k$ is included in a smooth hypersurface $\overline{W} = V(Z_1)$. The dimension of $\overline{W}$ is of course one less then that of W. This inclusion is called the *local inductive principal.* Note that this closed set is also defined by the couple $(J^*, 2)$ $(J^* = \langle X \cdot Y^2 \rangle \subset \mathcal{O}_{\overline{W}})$.

Example 2.6. The fact that $J^* \subset \mathcal{O}_{\overline{W}}$ is principal just a coincidence of the previous example. Let now $\mathbb{Y} \subset \mathbb{A}^3_k$ be the hypersurface defined by $g = Z^3 + X \cdot Y^2 \cdot Z + X^5 \in k[Z,X,Y]$. Define

$$J(2) = \langle g, \ \frac{\partial g}{\partial x_i}, \ \frac{\partial^2 g}{\partial x_i \partial x_j} \ / \text{ where } x_1 = X, x_2 = Y, x_3 = Z \rangle$$

so $V(J(2)) = F_3$ is the set of points of multiplicity at least 3. The pattern of this equation is

$$Z^3 + a_2 \cdot Z + a_3 \text{ with } a_2, a_3 \text{ in } k[X,Y].$$

One can check that $Z \in J(2)$, and that $\mathbb{Y}$ has at most points of multiplicity 3 since $\frac{\partial^3 g}{\partial^3 Z}$ is a unit.

We can argue as in Approach 2 to show that if $q \in \operatorname{Spec}(k[Z,X,Y])$ is a point (a prime ideal) of multiplicity 3, then $J(2) \subset q$. So

$$Z \in q \subset k[Z,X,Y],$$

and among all prime ideals q containing Z, the polynomial $Z^3 + X \cdot Y^2 \cdot Z + X^5$ has order 3 at $k[Z,X,Y]_q$ if and only if $X \cdot Y^2$ has order at least 2, and X^5 has order at least 3. In fact Z has order one at $k[Z,X,Y]_q$, and Z, X, and Y are independent variables.

Set now $\overline{W} = V(Z)$, $\overline{a_2} = \overline{X \cdot Y^2}$, $\overline{a_3} = \overline{X^5}$ (the class of a_2 and a_3 in $\mathcal{O}_{\overline{W}}$), and note that

$$F_3 = \{x \in \overline{W}/\nu_x(\overline{a_2}) \geq 2; \nu_x(\overline{a_3}) \geq 3\};$$

where $\nu_x(\overline{a_i})$ denotes the order of $\overline{a_i}$ at the local regular ring $\mathcal{O}_{\overline{W},x}$.

Set

$$(J^*, 6), \text{ where } J^* = \langle (\overline{a_2})^3, (\overline{a_3})^2 \rangle \subset \mathcal{O}_{\overline{W}}. \tag{2.6.1}$$

Finally check that $F_3 \subset \overline{W}$ (local inductive principal (2.5)), and note that we use this fact to show that the closed set F_3 is also defined by the couple $(J^*, 6)$.

Remark 2.7. Transformations of couples and stability of inductive principal.

Let $\mathbb{Y} \subset \mathbb{A}^3_k$ be the hypersurface defined by $g = Z^3 + X \cdot Y^2 \cdot Z + X^5 \in k[Z,X,Y]$, as in Example 2.6. The origin $\overline{0} \in \mathbb{A}^3_k$ is clearly a point of the closed set defined by $(J, 3)$. We now define:

$$\mathbb{A}^3_k \longleftarrow W_1 \tag{2.7.1}$$

as the blowup at $\overline{0}$. Let $\overline{W}_1$ be the strict transform of $\overline{W}$, $\mathbb{Y}_1$ the strict transform of $\mathbb{Y}$, and H the exceptional hypersurface. By restriction of the morphism to the subschemes we obtain

$$\overline{W} \longleftarrow \overline{W}_1, \tag{2.7.2}$$

which is also the monoidal transformation at the point $\overline{0} \in \overline{W}$, with exceptional hypersurface $\overline{H} = H \cap \overline{W}_1$.

Note that there is a well defined factorization of the form

$$J\mathcal{O}_{W_1} = I(H)^3 \cdot J_1 \tag{2.7.3}$$

for an ideal $J_1 \subset \mathcal{O}_{W_1}$, defined in terms of (2.7.1); and a factorization

$$J^*\mathcal{O}_{\overline{W}_1} = I(\overline{H})^6 \cdot J_1^* \tag{2.7.4}$$

for $J_1^* \subset \mathcal{O}_{\overline{W}_1}$, defined in terms of (2.7.2). These factorizations hold because $\overline{0}$ is a point of the closed set defined by $(J, 3)$, thus of the closed set in $\overline{W}$ defined by $(J^*, 6)$.

Since $\overline{0}$ is a point of order 3 of J (a point of multiplicity 3 of the hypersurface $\mathbb{Y}$), $J_1 \subset \mathcal{O}_{W_1}$ is the ideal defining the strict transform $\mathbb{Y}_1$.

Claim: The set of 3-fold points of the hypersurface $\mathbb{Y}_1$, or say the closed set of points defined by $(J_1, 3)$, is included in $\overline{W}_1$ and coincides with the closed set defined by $(J_1^*, 6)$.

In other words, we claim that the role played by $\overline{W}$ and $(J^*, 6)$ for the hypersurface $\mathbb{Y}$ (the local inductive principal (2.5)), is now played by $\overline{W}_1$ and $(J_1^*, 6)$ for the hypersurface $\mathbb{Y}_1$. We call this the *stability* of the local inductive principal.

To check this claim note first that W can be covered by three charts:

$$U_X = Spec(k[Z/X, X, Y/X]) = \mathbb{A}_k^3$$
$$U_Y = Spec(k[Z/Y, X/Y, Y]) = \mathbb{A}_k^3$$
$$U_Z = Spec(k[Z, X/Z, Y/Z]) = \mathbb{A}_k^3$$

The morphism: $\mathbb{A}^3 \longleftarrow U_Y = Spec(k[Z/Y, X/Y, Y]) = \mathbb{A}_k^3$, induced by (2.7.1), is defined by the inclusion $k[Z, X, Y] \to k[Z/Y, X/Y, Y]$.

At this chart $I(H) = \langle Y \rangle$, the factorization in (2.7.3) is

$$g = Z^3 + X \cdot Y^2 \cdot Z + X^5 = Y^3 \cdot ((Z/Y)^3 + (X/Y) \cdot Y \cdot (Z/Y) + (X/Y)^5 \cdot Y^2),$$

and $I(\overline{W_1} \cap U_Y) = \langle Z/Y \rangle$.

Note that $g_1 = (Z/Y)^3 + (X/Y) \cdot Y \cdot (Z/Y) + (X/Y)^5 \cdot Y^2 \in k[Z/Y, X/Y, Y]$ has the same general pattern as g, namely: $(Z/Y)^3 + b_2 \cdot (Z/Y) + b_3$, with b_2, b_3 in $k[X/Y, Y]$. So the same argument applied to g asserts that:

1) The set of 3-fold points of $\mathbb{Y}_1 \cap U_Y$ is included in $V(\langle Z/Y \rangle)$, or say in

$$\overline{W}_1 \cap U_Y = \mathrm{Spec}(k[Z/Y, X/Y, Y]/\langle Z/Y \rangle) = \mathrm{Spec}(k[X/Y, Y]).$$

2) The set of 3-fold points $\mathbb{Y}_1$ in U_Y is the closed set in $\overline{W}_1 \cap U_Y$ defined by $(\mathcal{A}, 6)$, where

$$\mathcal{A} = \langle (\overline{b_2})^3, (\overline{b_3})^2 \rangle \subset k[X/Y, Y].$$

We are finally ready to address the main property of our form of induction in the number of variables, namely the compatibility of induction with transformations. To this end note that

$$\overline{W} \longleftarrow \overline{W}_1 \cap U_Y$$

is defined by $k[X,Y] \to k[X/Y,Y]$, and the transform of the couple $(J^*,6)$ in (2.6.1), defined in (2.7.4), is such that

$$J_1^* \mathcal{O}_{(\overline{W}_1 \cap U_Y)} = \mathcal{A}.$$

A similar argument applies for $\mathbb{A}^3 \longleftarrow U_X$. To study our claim for $\mathbb{A}^3 \longleftarrow W_1$ it suffices to check at the charts U_X, U_Y. In fact, $U_X \cup U_Y$ cover all of W_1 except for one point (the origin at $U_Z = \mathbb{A}^3$), which is not a point of $\mathbb{Y}_1$. So U_Z can be ignored for our purpose.

2.8. Summarizing: Stability of inductive principal. Our previous discussion showed that the set of 3-fold points of $\mathbb{Y} \subset \mathbb{A}^3$ (defined by $g = Z^3 + X \cdot Y^2 \cdot Z + X^5 \in k[Z,X,Y]$) is included in a smooth hypersurface $\overline{W}$ (defined by $Z \in k[Z,X,Y]$)(2.5). From this fact we conclude that the set is also defined by $(J^*,6)$, where J^* is an ideal in the surface $\overline{W}$. The property that links $\overline{W}$ with 3-fold points of $\mathbb{Y}$ goes beyond this fact. A transformation at a 3-fold point of $\mathbb{Y}$ defines a strict transform $\mathbb{Y}_1$. It also induces a transformation $\overline{W} \longleftarrow \overline{W}_1$, together with a transformation of $(J^*,6)$, say $(J_1^*,6)$. $\overline{W}_1$ is the strict transform of $\overline{W}$, and the property is that the set of three fold points of $\mathbb{Y}_1$ is included in $\overline{W}_1$. This is what we call the stability of the inductive principal. Furthermore, $(J_1^*,6)$ defines the closed set of 3-fold points of $\mathbb{Y}_1$. In particular, if J_1^* would not have points of order 6 (which is not the case in our example), then $\mathbb{Y}_1$ would not have 3-fold points. Here we have analyzed this stability for one quadratic transformation, but it turns out that the same argument holds for any sequence of monoidal transformations: Defining a sequence of transformations, say

$$\begin{array}{ccccccc} \mathbb{A}^3 & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & W_k, \\ \mathbb{Y} & & \mathbb{Y}_1 & & & & \mathbb{Y}_k \end{array} \tag{2.8.1}$$

where each π_{i+1} is a blow-up at a closed and smooth centers included in the 3-fold points of $\mathbb{Y}_i$, the strict transform of $\mathbb{Y}_{i-1}$, is *equivalent* to the definition of a sequence of transformations

$$\begin{array}{ccccccc} \overline{W} & \xleftarrow{\pi_1} & \overline{W}_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & \overline{W}_k. \\ (J^*,6) & & (J_1^*,6) & & & & (J_k^*,6) \end{array} \tag{2.8.2}$$

where each $J_i^* \subset \mathcal{O}_{\overline{W}_1}$, and $(J_i^*,6)$ is defined in terms of $(J_{i-1}^*,6)$ as in (2.7.4). Moreover, each $\overline{W}_i$ is a smooth hypersurface in W_i, and the closed set defined by $(J_i^*,6)$ in the hypersurface $\overline{W}_i$ is the set of 3-fold points of $\mathbb{Y}_i$. In particular, if the second sequence is defined with the property that J_k^* has no points of order 6 in $\overline{W}_k$, then the hypersurface $\mathbb{Y}_k$ has at most points of multiplicity 2.

This is induction on the dimension of the ambient space, where the lowering of the highest order of an ideal in a smooth scheme of dimension 3 is *equivalent* to a related problem in a smooth scheme of dimension 2. This property of the smooth hypersurface $\overline{W}$ will be discussed in Section 6.

2.9. Tschirnhausen. Set $f = Z^b + a_1 Z^{b-1} + \cdots + a_b \in k[Z, X_1, .., X_n]$, with $a_i \in k[X_1, .., X_n]$ for $i = 1, \dots, b$. If the characteristic of k is zero set $Z_1 = Z + \frac{1}{b}a_1$. Check that $k[Z, X_1, \dots, X_n] = k[Z_1, X_1, \dots, X_n]$, and that $f = Z_1^b + c_2 Z_1^{b-2} + \cdots + c_b$, with $c_i \in k[X_1, \dots, X_n]$ and $c_1 = 0$. One can argue as in Example 2.6, to show that the b-fold points of $\mathbb{Y}$ are included in the hypersurface $\overline{W} =$

$V(Z_1)(\subset \mathbb{A}^{n+1})$(local inductive principle (2.5)). Furthermore, $\overline{W}$ will have the stability property discussed above, where the role of $(J^*, 6)$ in Remark 2.7 (in (2.8.2)) is now played by $(J^*, b!)$, where

$$J^* = \langle c_i^{\frac{b!}{i}}, i = 2, 3, \dots, b\rangle \subset \mathcal{O}_{\overline{W}}.$$

3. Resolution functions and the main resolution theorems.

Our proofs of the two main theorems 1.2 and 1.3 will be constructive, as opposed to the original existential proofs of Hironaka. We introduce here the notion of resolution algorithm, or resolution functions. Constructive resolutions will be defined in terms of these functions, and the main purpose in this Section is to show how both proofs follow easily from natural properties of these functions.

3.1. In 2.6 we study the transform of a hypersurface in $\mathbb{A}^3$ by a monoidal transformation at a 3-fold point. Note that (2.7.3) is an example of a *proper transform* of an ideal, as defined in 1.8. However the ideal J^* has order 9 at the center of the monoidal transformation, so J_1^* in (2.7.4) is not a proper transform. This shows that our form of induction will lead us to transformations, defined by expressions of the form $J\mathcal{O}_{W_1} = I(H)^b \cdot J_1$, even when b is not the highest possible integer in such expression.

We have defined *couples* as pairs (J, b), where $J \subset \mathcal{O}_W$ is a non-zero sheaf of ideals, and $b \in N$ is a positive integer. We introduce now two notions related to couples:

• The **closed set** attached to (J, b):

$$\mathrm{Sing}(J, b) = \{x \in W / \nu_x(J_x) \geq b\},$$

namely the set of points in W where J has order at least b. This is closed in W (see 5.4, ii)).

•**Transformation** of (J, b):

Let $Y \subset \mathrm{Sing}(J, b)$ be a closed and smooth subscheme, and let

$$\begin{array}{ccc} W & \xleftarrow{\pi} & W_1 \supset H = \pi^{-1}(Y) \\ Y & & \end{array}$$

be the monoidal transformation at Y. Since $Y \subset \mathrm{Sing}(J, b)$, the total transform $J\mathcal{O}_{W_1}$ can be expressed as a product:

$$J\mathcal{O}_{W_1} = I(H)^b J_1 (\subset \mathcal{O}_{W_1})$$

for a uniquely defined J_1 in $\mathcal{O}_{W_1}$. The new couple (J_1, b) is called the *transform* of (J, b), and the transformation is denoted by

$$\begin{array}{ccc} W & \xleftarrow{\pi} & W_1 \\ (J, b) & & (J_1, b) \end{array} \tag{3.1.1}$$

A sequence of transformations will be denoted as

$$\begin{array}{ccccccc} W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_k} & W_k. \\ (J, b) & & (J_1, b) & & & & (J_k, b) \end{array} \tag{3.1.2}$$

Note that in such case

$$J\mathcal{O}_{W_k} = I(H_1)^{c_1} \cdot I(H_2)^{c_2} \cdots I(H_k)^{c_k} \cdot J_k \tag{3.1.3}$$

for suitable exponents $c_2, \dots, c_k$, and $c_1 = b$. Furthermore, all $c_i = b$ if for any index $i < k$ the center Y_i is not included in $\cup_{j \leq i} H_j \subset W_i$ (the exceptional locus of $W \longleftarrow W_i$).

Example 3.2. The ideal $J =< x^2 - y^5 > \subset k[x, y]$ has a unique 2-fold point at the origin $(0, 0) \in \mathbb{A}^2$. Let $W = \mathbb{A}^2 \longleftarrow W_1$ be the blow up at the origin. The strict transform of the curve has a unique 2-fold point, say $q \in W_1$. Set $W_1 \longleftarrow W_2$ by blowing-up q. This defines a sequence,

$$\begin{array}{ccccc} W & \longleftarrow & W_1 & \longleftarrow & W_2 \\ (J, b) & & (J_1, b) & & (J_2, b) \end{array} \tag{3.2.1}$$

Here $J\mathcal{O}_{W_2} = I(H_1)^2 \cdot I(H_2)^4 \cdot J_2$ provides an expression of the total transform of J involving J_2.

Remark 3.3. The ideal J_1 in the previous example is the proper transform of J, and J_2 is the proper transform of J_1 (Def 1.8). In particular J_2 does not vanish along H_1 or H_2. Recall however that this is not a general fact as indicated in 3.1. Set now $K = J$, and note the same sequence as before defines $(K, 1)$; $(K_1, 1)$; $(K_2, 1)$ and $K\mathcal{O}_{W_2} = I(H_1)^1 \cdot I(H_2)^2 \cdot K_2$.

In this case the ideal K_2 does vanish along the exceptional hypersurface H_i, in fact there is a unique and well defined expression, say

$$K_2 = I(H_1)^a \cdot I(H_2)^b \cdot \overline{K}_2 \tag{3.3.1}$$

in $\mathcal{O}_{W_2}$, so that $\overline{K}_2$ does not vanish along the exceptional hypersurfaces. It follows from 3.2 that $a = 1$, $b = 2$ and $\overline{K}_2 = J_2$.

Definition 3.4. Fix $J \subset \mathcal{O}_W$, W smooth over a field of characteristic zero, and a couple (J, b). A sequence of transformations as in (3.1.2) is said to be a *resolution* of (J, b) if:

i) $\operatorname{Sing}(J_k, b) = \emptyset$.

ii) The exceptional locus of $W \longleftarrow W_k$, namely $\cup_{1 \leq i \leq k} H_i$, is a union of hypersurfaces with normal crossings.

3.5. We define a *pair*, denoted by $(W, E = \{H_1, .., H_r\})$, to be a set of smooth hypersurfaces $H_1, .., H_r$ with normal crossings in a smooth scheme W.

Let $W \longleftarrow W_1$ be a monoidal transformation at a closed an d smooth center Y. If Y has normal crossings with $\cup H_i$, we say that Y is permissible for the pair (W, E), and that

$$(W, E = \{H_1, .., H_r\}) \longleftarrow (W_1, E_1 = \{H_1, .., H_r, H_{r+1}\})$$

is a *transformation of pairs* (see Prop 1.9).

We define a *basic object* to be a pair $(W, E = \{H_1, .., H_r\})$ together with a couple (J, b), with the condition that $J_x \neq 0 (\subset \mathcal{O}_{W,x})$ at any point $x \in W$. We indicate this basic object by

$$(W, (J, b), E).$$

If a smooth center Y defines a transformation of the pair (W, E), and in addition $Y \subset \operatorname{Sing}(J, b)$, then a transform of the couple (J, b) is defined. In this case we say that

$$(W, (J, b), E) \longleftarrow (W_1, (J_1, b), E_1)$$

is a *transformation* of the basic object. A sequence of transformations

$$(W, (J, b), E) \longleftarrow (W_1, (J_1, b), E_1) \longleftarrow \cdots \longleftarrow (W_s, (J_s, b), E_s) \tag{3.5.1}$$

is a *resolution* of the basic object if $\operatorname{Sing}(J_s, b) = \emptyset$.

In such case

$$J \cdot \mathcal{O}_{W_s} = I(H_{r+1})^{c_1} \cdot I(H_{r+2})^{c_2} \cdots I(H_{r+s})^{c_s} \cdot J_s \tag{3.5.2}$$

for some integer c_i, where J_s is a sheaf of ideals of order at most $b-1$, and the H_j have normal crossings.

Definition 3.6. Let X be a topological space, and $(T, \geq)$ a totally ordered set. A function $g : X \to T$ is said to be *upper semi-continuous* if: **i)** g takes only finitely many values, and, **ii)** for any $\alpha \in T$ the set

$$\{x \in X \ / g(x) \geq \alpha\}$$

is closed in X.

Then largest value achieved by g will be denoted by

$$\max g.$$

Clearly the set

$$\underline{\mathrm{Max}}\, g = \{x \in X : g(x) = \max g\}$$

is a closed subset of X.

3.7. Resolution functions. We now show why *constructive* resolutions of basic objects will lead us to simple proofs of both Main Theorems 1.2 and 1.3.

In 2.2 we defined an upper semi-continuous function, say $h_3 : Spec(k[Z_1, X, Y] \to \mathbb{Z}$, defined by taking order of the ideal $J =< Z_1^2 + X \cdot Y^2 >$. It was shown that $\max h_3 = 2$, and that $\underline{\mathrm{Max}}\, h_3 (= \mathcal{F}_2) \subset \overline{W}$, where $\overline{W} = V(Z_1)$ is a smooth hypersurface isomorphic to $Spec(k[X, Y])$. Furthermore, an ideal $J^* = \langle X \cdot Y^2 \rangle \subset \mathcal{O}_{\overline{W}}$ was attached to $\underline{\mathrm{Max}}\, h_3$. We may take now $h_2 : Spec(k[X, Y]) \to \mathbb{Z}$, defined by taking order of the ideal J^*, so that $\underline{\mathrm{Max}}\, h_2$ is included in a smooth hypersurface; and ultimately define a function h_1 with values at $\mathbb{Z}$.

In this frame of mind it is conceivable to assign a copy of $\mathbb{Z}$ for each dimension, namely $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, with lexicographic order, and a function, say $h = (h_3, h_2, h_1)$ with values at this ordered set, so that h is upper semi-continuous. This is not exactly the way we will proceed, but we will define a totally ordered set for each dimension, and then take the product of copies of this set, one for each dimension.

We will fix an integer d, and define a totally ordered set $(I^d, \geq)$. Moreover, for any basic object

$$B : (W, (J, b), E),$$

dimension of $W = d$, an upper semi-continuous function $f_B : \mathrm{Sing}(J, b) \to I^d$ will be defined with the property that $\underline{\mathrm{Max}}\, f_B$ is a smooth subscheme of $\mathrm{Sing}(J, b)$, and a permissible center for the pair (W, E). Thus, a transformation of the basic object can be defined with center $\underline{\mathrm{Max}}\, f_B$.

In this way a unique sequence (3.5.1) is defined inductively, by setting centers $\underline{\mathrm{Max}}\, f_{B_i}$. In addition, this sequence defined by the functions will be a resolution of the basic object. In fact, for some index s (depending on B) $\mathrm{Sing}(J_s, b) = \emptyset$.

In other words, the set $(I^d, \geq)$ will be fixed, and the functions on this set defined so as to provide a resolution for any basic object of dimension d. We now state the properties that will hold for such sequence:

Properties:

P1) For each l, $\underline{\mathrm{Max}}\, f_l$ is closed regular and has normal crossings with $\cup_{H_i \in E_l} H_i$.

P2) For some index k_0, depending on the basic object B, $\mathrm{Sing}(J_{k_0}, b) = \emptyset$.

If $p \in \mathrm{Sing}(J_k, b)$, and $p \notin \underline{\mathrm{Max}}\, f_k$, then p can be identified with a point in W_{k+1}. Furthermore, $p \in Sing(J_{k+1}, b)$, and:
P3) $f_k(p) = f_{k+1}(p)$.

Of particular interest will be the case of basic objects with $b = 1$. In such case $\mathrm{Sing}(J_0, 1)$ is the underlying topological space of $V(J_0)$ (the subscheme defined by the sheaf of ideals J_0).
P4) There is fixed value $R \in I^d$, and whenever $p \in \mathrm{Sing}(J_0, 1)$ is a point where the subscheme defined by J_0 is smooth, then $f_0(p) = R$ (where $f_0 : \mathrm{Sing}(J_0, 1) \to I^d$).

The definition of $(I^d, \geq)$, and of the functions f, will be discussed in Part III, and studied exhaustively in Part IV. We now prove our two Main Theorems 1.2 and 1.3 using the the properties of resolution functions.

3.8. Proof of Theorem 1.3. Fix $I \subset \mathcal{O}_W$ as in Theorem 1.3, and consider the basic object

$$(W, (J, 1), E = \emptyset), \tag{3.8.1}$$

with $J = I$, and the resolution defined by the resolution functions. Property **P2)** asserts that $\mathrm{Sing}(J_{k_0}, 1) = \emptyset$ for some index k_0. It follows that $J_{k_0} = \mathcal{O}_{W_{k_0}}$, namely that

$$J\mathcal{O}_{W_{k_0}} = I(H_1)^{c_1} \cdot I(H_2)^{c_2} \cdots I(H_{k_0})^{c_{k_0}}.$$

It is easy to check now that the conditions of the Theorem are fulfilled for $W \leftarrow W_{k_0}$.

3.9. Proof of Theorem 1.2. Let $J \subset \mathcal{O}_{W_0}$ be the sheaf of ideals defining $X \subset W_0$ in Theorem 1.2, and consider, as above, the resolution of the basic object (3.8.1) defined by the resolution functions. So again $J_{k_0} = \mathcal{O}_{W_{k_0}}$, and hence $J\mathcal{O}_{W_{k_0}} = I(H_1)^{c_1} \cdot I(H_2)^{c_2} \cdots I(H_{k_0})^{c_{k_0}}$.

Let $V = W_0 - \mathrm{Sing}(X)$ be the complement of the singular locus of X. Note that V is an open set, dense in W_0, and $f_0(p) = R$ for any $p \in V \cap \mathrm{Sing}(J, 1)$. Here $X = \mathrm{Sing}(J, 1)$, and $V \cap \mathrm{Sing}(J, 1)$ is dense in $\mathrm{Sing}(J, 1)$ since X is reduced. Furthermore, $f_0(p) = R$ for any $p \in V \cap \mathrm{Sing}(J, 1)$ (**P4)**). So $\max f_0 \geq R$.

If $\max f_0 = R$, then $\mathrm{Sing}(J, 1) = \underline{\mathrm{Max}}\, f_0$ and X is smooth in W_0 (**P1)**).

If $\max f_0 > R$, then V can be identified with an open set, say V_1, in W_1, and $f_1(p) = R$ for any $p \in V_1 \cap \mathrm{Sing}(J_1, 1)$ (**P3)**).

If $\max f_1 = R$, then the strict transform of X is a union of components of $\underline{\mathrm{Max}}\, f_1$, so the strict transform defines an embedded desingularization (**P1)**).

If $\max f_1 > R$ then V can be identified with an open subset V_2 in W_2.

Note that that there must be an index k, for some $k < k_0$, so that $\max f_k = R$. In fact this follows from **P4)**, **P2)**, and the fact that $\mathrm{Sing}(J_{k_0}, 1) = \emptyset$. Note that V can be identified with an open set, $V_k \subset W_k$, and that the strict transform of X in W_k fulfills the conditions of the Theorem.

4. On the notion of strict transforms of ideals.

4.1. The notion of strict transform of embedded schemes appears in the very formulation of our Main Theorem 1.2. A subscheme of a given schemes is defined by a sheaf of ideals. Given a blow-up at the scheme, there is a notion of strict transform of ideals, corresponding to the notion of strict transform of embedded schemes.

A novel aspect of the proof of Theorem 1.2 given in 3.9, as compared to the proof of Hironaka and from previous constructive proofs ([3], [26]), is that we do not consider, within this algorithmic procedure, the notion of strict transform of ideals. In fact, let $J \subset \mathcal{O}_W$ be the sheaf of ideals defining $X \subset W_0$, and let

$$(W_0, (J, 1), E_0) \leftarrow (W_1, (J_1, 1), E_1)$$

be a transformation with center $Y \subset \mathrm{Sing}(J, 1)$. We show here that, in general, J_1 is not the sheaf of ideals defining the strict transform of X in W_1 (i.e. is not the strict transform of J). Let $H \subset W_1$ denote the exceptional locus of

$$W \leftarrow W_1$$

so that $W - Y = W_1 - H$. The strict transform of X is the *smallest subscheme* of W_1 containing $X - Y$, via the identification $W_1 - H = W - Y$. In other words, it is the closure of $X - Y$ in W_1 by this identification.

Such smallest subscheme is defined by the *biggest sheaf of ideals*, say $\widetilde{J_1} \subset \mathcal{O}_{W_1}$, which coincide with J when restricted to $W_1 - H$. We claim that the biggest sheaf ideal that fulfills this condition is that defined by the increasing union of colon ideals:

$$\widetilde{J_1} = \cup_k (J\mathcal{O}_{W_1} : I(H)^k).$$

To check this, set $U = \mathrm{Spec}(A)$, an open affine set of W_1, so that the hypersurface $H \cap U$ is defined by an element $a \in A$. Let K denote the ideal defined by restriction of $\widetilde{J_1}$ to U. The localization $K \cdot A_a$ is also a restriction of the sheaf of ideals J to $U_a = \mathrm{Spec}(A_a)$.

Note that $K \cdot A_a \cap A$ is the biggest ideal in A defining $K \cdot A_a$ at $U_a = \mathrm{Spec}(A_a)$. On the other hand $K \cdot A_a \cap A = \cup_k (K : a^k)$. Since this holds for an affine covering of W_1, it turns out that $\widetilde{J_1}$ is the biggest sheaf of ideals with the previous condition.

The ideal K (the restriction of $\widetilde{J_1}$ to U), is a finite intersection of p-primary ideals, called the p-primary components. The ideal $K \cdot A_a \cap A$ is obtained from K by neglecting, in the previous intersection, those p-primary components corresponding to prime ideals containing the element $a \in A$ (i.e. with closure of p included in the exceptional hypersurface H).

It is not hard to check that

$$J_1 \subset \widetilde{J_1},$$

in fact $J_1 = (J\mathcal{O}_{W_1} : I(H)^1)$ according to the definition of transformation of basic objects.

If W_1 arises from blowing up $W = A^3_k$ at the origin, and $J = < Z, X^2 - Y^3 > \subset k[X, Y, Z]$, then $V(J_1) \cap H$ is a line, whereas $V(\widetilde{J_1})$ (the strict transform of the curve), intersects H at a unique point. So $J_1 \neq \widetilde{J_1}$ in this case.

4.2. Resolution of singularities is defined by a proper birational morphism, defined in a step by step procedure, each step consisting of a suitably defined monoidal transformation. So given equations defining the ideal J, and a monoidal transformation as above, Hironaka provides equations defining the strict transform ideal $\widetilde{J_1}$. This turns out being, in general, a very difficult task. In fact a major part of the proof of Hironaka is devoted to address this particular point; he introduces the notions of Hilbert-Samuel functions and of normal flatness with this purpose. An important conceptual simplification of constructive desingularization, presented in 3.9, relies on the fact that it provides a proof avoiding all these notions. In fact, we prove resolution by means of elementary transformations (defining J_1), avoiding the use of the strict transform ideal $\widetilde{J}_1$.

Example 4.3. The following example illustrates a situation in which both notions of transformations discussed in 4.1 coincide (i.e. where $J_1 = \widetilde{J}_1$).

Let $X \subset W$ be a closed and smooth subscheme of W. Set $J = I(X)$, and note that $\mathrm{Sing}(J, 1) = X$, and that the order of J at $\mathcal{O}_{W,x}$ is one at any $x \in X$.

Let $W \longleftarrow W_1$ be the monoidal transformation with center Y which defines a transformation, say: $(J, 1)$; $(J_1, 1)$. In other words, assume that $Y \subset \mathrm{Sing}(J, 1)$ (so that $J\mathcal{O}_{W_1} = J_1 \cdot I(H)$, where $H \subset W_1$ denotes the exceptional locus). We claim now the following holds:

(1) $\mathrm{Sing}(J_1, 1)$ $(= V(J_1))$ is the strict transform of X.

(2) The subscheme $X_1 \subset W_1$, defined by J_1, is smooth.

Note that **(2)** follows from **(1)**. In fact the induced morphism $X \leftarrow X_1$ is the blowup of X at Y, and the blowup of a smooth scheme in a smooth subscheme is smooth. To prove **1)** note that at any point $x \in W$, there is a regular system of parameters $\{x_1, \dots, x_n\}$ such that $J_x = \langle x_1, \dots, x_r\rangle$ and $I(Y)_x = \langle x_1, \dots, x_s\rangle$ for $r \leq s$. The fiber over $x \in W$ can be covered by $Spec(\mathcal{O}_W[x_1/x_i, x_2, \dots, x_s/x_i, x_{s+1}, \dots, x_n]$ for $i = 1, 2, \dots, s$. Finally **(1)** can be checked directly at the charts corresponding to indices $r + 1 \leq i \leq s$.

4.4. There is a stronger formulation of embedded desingularization than that in 1.2, which was proved in [10]. That theorem proves that given W_0 smooth over a field k of characteristic zero, and $X_0 \subset W_0$ closed and reduced, there is a sequence of monoidal transformations, say

$$\begin{array}{ccccccc} W_0 & \longleftarrow & (W_1, E_1 = \{H_1\}) & \longleftarrow & (W_2, E_2 = \{H_1, H_2\}) & \cdots \longleftarrow & (W_r, E_r = \{H_1, H_2, .., H_r\}), \\ Y & & Y_1 & & Y_2 & & \end{array}$$

such that, in addition to the three conditions i), ii), and iii) in 1.2, it also holds that:

iv) $I(X_0)\mathcal{O}_{W_r} = I(H_1)^{c_1} \cdot I(H_2)^{c_2} \cdots I(H_r)^{c_r} \cdot I(X_r)$

where X_r denotes the strict transform of X.

Consider the particular case in which X is an irreducible subscheme in $W_0 = \mathrm{Spec}(k[X_1, \cdots, X_n])$ defined by a prime ideal P of height h. In this case the theorem says that at any point $x \in W_r$ there is a regular system of parameters $\{Z_1, \cdots Z_n\}$ at $\mathcal{O}_{W_r,x}$, such that:

i)$P \cdot \mathcal{O}_{W_r,x} =< Z_1, \cdots, Z_h > \cdot Z_{j_1}^{a_1} \cdot Z_{j_2}^{a_2} \cdots Z_{j_s}^{a_s}$ if x is a point of the strict transform X_r, and

ii) $P \cdot \mathcal{O}_{W_r,x} =< Z_{j_1}^{a_1} \cdot Z_{j_2}^{a_2} \cdots Z_{j_s}^{a_s} >$ (is an ideal spanned by a monomial in these coordinates) if x is not in X_r.

This result does not hold, in general, for desingularizations which make use of invariants such as Hilbert Samuel functions (which we avoid in our proof). This algebraic formulation of embedded desingularization is not a consequence of the theorem of desingularization as proved by Hironaka.

Part II

In 2.8 we discussed a strong link between the set of 3-fold points of the hypersurface $\mathbb{Y} \subset \mathbb{A}^3$, defined by $g = Z^3 + X \cdot Y^2 \cdot Z + X^5 \in k[Z, X, Y]$, and the smooth hypersurface $\overline{W}$ defined by $Z \in k[Z, X, Y]$. The link showed that the reduction of 3-fold points of $\mathbb{Y}$, by means of monoidal transformations, was equivalent to a related problem for a suitable ideal in the smooth subscheme $\overline{W}$ (see also 2.9).

This is the key for induction in resolution Theorems. In this second Part we justify the discussion in 2.8 (see Example 6.15), and generalize this main property in Section 7 . In section 6 we study an important preliminary: the behavior of derivations with monoidal transformations.

5. Derivations and monoidal transformations on smooth schemes.

In this Section we study behavior of derivations when applying monoidal transformations. This will be used in the next Section 6, where the inductive properties discussed in 2.8 will be clarified.

Fix W smooth over a field k, and $y \in W$ a closed point. Let $\{x_1, \dots, x_n\}$ be a regular system of parameters at $\mathcal{O}_{W,y}$.

We define an operator Δ_y on ideals in $\mathcal{O}_{W,y}$ by setting, for $J_y =< f_1, f_2, \dots, f_s >$ in $\mathcal{O}_{W,y}$:

$$\Delta_y(J_y) =< f_1, f_2, \dots, f_s, \ \frac{\partial f_j}{\partial x_i} \ /1 \leq i \leq n; 1 \leq j \leq s > .$$

Note that $\Delta_y(\Delta_y(J_y)) =< f_1, f_2, \dots, f_s, \ \frac{\partial f_j}{\partial x_i}, \ \frac{\partial^2 f_j}{\partial x_i \partial x_j} \ /1 \leq i \leq n; 1 \leq j \leq s >$. The whole point of restriction to fields of characteristic zero relies on the following property:

5.1. Characteristic zero. If k is a field of characteristic zero and $(b \geq 1)$, then J_y has order b at $\mathcal{O}_{W,y}$ iff $\Delta_y(J_y)$ has order $b-1$.

Example 5.2. Let $\mathcal{O}_{W,y} = k[x_1, x_2, x_3]_{<x_1,x_2,x_3>}$.

$$J_y =< x_1^3 + x_2^4 + x_3^4 > \subset \Delta_y(J_y) =< x_1^2, x_2^3, x_3^3 > \subset \Delta_y^2(J_y) =< x_1, x_2^2, x_3^2 > \subset \Delta_y^3(J_y) = \mathcal{O}_{W,y}$$

Note that, if k is of characteristic zero, the orders of these ideals drop by one : 3,2,1,0.

5.3. Further properties of the operator Δ_y are:

i) $J_y \subseteq \Delta_y(J_y) \subseteq \Delta_y(\Delta_y(J_y)) = \Delta_y^2(J_y) \subseteq \Delta_y^3(J_y) \subseteq \dots$

ii) $J_y \subset \mathcal{O}_{W,y}$ has order $b(\geq 1)$ iff $\Delta_y^{b-1}(J_y)$ has order 1.

iii) The order of $J_y \subset \mathcal{O}_{W,y}$ is $\geq s$ iff $\Delta_y^{s-1}(J_y)$ is a proper ideal in $\mathcal{O}_{W,y}$.

5.4. On the Δ operator. The locally defined operators Δ_y can be globalized in the following sense. Fix W smooth over a field k, there is an operator Δ on the class of all $\mathcal{O}_W$-ideals , such that:

$$J \subseteq \Delta(J)(\subset \mathcal{O}_W),$$

and at any closed point $y \in W$:

$$\Delta(J)_y = \Delta_y(J_y).$$

Furthermore, the following properties hold:

i) $J \subseteq \Delta(J) \subseteq \Delta^2(J) \subseteq \dots$ (hence $V(J) \supseteq V(\Delta(J)) \supseteq V(\Delta^2(J)) \supset \dots$

ii) $V(\Delta^{s-1}(J)) = \mathrm{Sing}(J, s)$. In fact $V(\Delta^{s-1}(J))$ is the closed set of points in W where J has order $\geq s$ (i.e. $(\Delta^{s-1}(J))_y = \Delta_y^{s-1}(J_y) \subsetneq \mathcal{O}_{W,y}$) iff the order of $J_y\mathcal{O}_{W,y}$ is $\geq s$).

iii) If b is the biggest order of J, $V(\Delta^b(J)) = \emptyset$ and $V(\Delta^{b-1}(J))$ is locally included in a smooth hypersurface.

Proof of iii) If b is the biggest order of J, $\Delta^b(J) = \mathcal{O}_W$ and $\Delta^{b-1}(J)$ has order at most 1. So if $y \in V(\Delta^{b-1}(J))$, $\Delta^{b-1}(J)\mathcal{O}_{W,y}$ has order 1 at $\mathcal{O}_{W,y}$. If $g \in \Delta^{b-1}(J)$ has order 1 at $\mathcal{O}_{W,y}$, then:

$$\overline{W} = V(< g >) \supset V(\Delta^{b-1}(J)),$$

and $\overline{W}$ is a smooth hypersurface in a neighborhood of y.

Example 5.5. Set $W = A^3_k = Spec(k[X,Y,Z])$, $F = Z^3 + XY^2Z + X^5$, and $J =< F >$, as in 2.8. Then:

$\Delta(J) =< 3Z^2 + XY^2, Y^2Z + 5X^4, 2XYZ, F >\subset \Delta^2(J) =< Z, XY, Y^2, X^3 >\subset \Delta^3(J) = k[X,Y,Z]$.

So, as indicated in 2.8, the 3-fold points of the hypersurface $\mathbb{Y} \subset \mathbb{A}^3$ defined by $V(< J >)$ are included in smooth hypersurface $\overline{W} = V(< Z >)$.

5.6. We now address the compatibility of Δ operators with monoidal transformations. So fix a couple (J,b), and consider a transformation

$$\begin{array}{ccc} W & \overset{\pi}{\longleftarrow} & W_1 \\ (J,b) & & (J_1,b). \end{array} \tag{5.6.1}$$

Lemma 5.7. *Given (J,b) $(J \subset \mathcal{O}_W)$ and a transformation (5.6.1), then:*
1) If $b \geq 2$, (5.6.1) induces a transformation of $(\Delta(J), b-1)$:

$$\begin{array}{ccc} W & \overset{\pi}{\longleftarrow} & W_1 \\ (\Delta(J), b-1) & & ((\Delta(J))_1, b-1). \end{array}$$

2) $(\Delta(J))_1 \subset \Delta(J_1)$.

Proof: Let $Y \subset W$ be the center of the monoidal transformation, and let $H \subset W_1$ be the exceptional locus. By assumption $Y \subset \mathrm{Sing}(J,b)$, so $J \cdot \mathcal{O}_{W_1} = I(H)^b \cdot J_1$. It follows from 5.4,**ii)** that for general b, $\mathrm{Sing}(J,b) \subset \mathrm{Sing}(\Delta(J), b-1)$. In particular $Y \subset \mathrm{Sing}(\Delta(J), b-1)$, which proves 1).

In order to prove 2) we first note that if $U \subset W$ is open, a sheaf of ideals in W induces a sheaf of ideals in U, and the Δ operators (on W and on U) are compatible with restrictions. On the other hand note that the pull-back of U in W_1, say U_1, is an open set, and the induced morphism $U \longleftarrow U_1$ fulfills the conditions in 1) for the restriction of J to U.

If we can prove that 2) holds over U (at $U \longleftarrow U_1$), for all U in an open covering of W, then it is clear that 2) holds. Therefore we may argue locally.

Let $\xi \in W$ be a closed point and choose a regular system of parameters $\{x_1, \ldots x_n\} \subset \mathcal{O}_{W,\xi}$ so that the center of the monoidal transformation is locally defined by $\langle x_1, \ldots, x_s\rangle$. Now consider an affine neighborhood U of ξ such that $x_1, \ldots, x_s$ are global sections of $\mathcal{O}_U$, and such that J is generated by global sections, say $f_1, \ldots, f_r$. We may also assume that $\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$ are global derivations, and that $\Delta(J)$ is generated by the global sections $\{f_k\}_{k=1}^r \cup \left\{\frac{\partial f_k}{\partial x_j}\right\}_{\substack{k=1,\ldots,r \\ j=1,\ldots,n}}$.

By the previous discussion we may assume that $U = W$. The scheme W_1 is defined by patching the affine rings

$$A_i = \mathcal{O}_W[x_1/x_i, \ldots, x_s/x_i], \quad i \in \{1, \ldots, s\},$$

and $I(H) = \langle x_i \rangle$ at A_i. For each index $k \in \{1, \ldots, r\}$ there is a factorization $f_k = x_i^b {g_i}^{(k)}$, and $\{{g_i}^{(1)}, {g_i}^{(2)}, \ldots, {g_i}^{(r)}\}$ generate the restriction of J_1 to $\mathrm{Spec}(A_i)$, say $J_1^{(i)}$. In order to prove 2) we must show that, for each index $k \in \{1, \ldots, r\}$:

a) $\frac{f_k}{x_i^{b-1}} \in \Delta(J_1^{(i)})$, and

b) $\frac{(\frac{\partial f_k}{\partial x_j})}{x_i^{b-1}} \in \Delta(J_1^{(i)})$.

The assertion in a) is clear since $\frac{f_k}{x_i^{b-1}} = x_i g_i^{(k)} \in J_1^{(i)} \subset \Delta(J_1^{(i)})$. We now address b). In what follows we fix an index $k \in \{1, \ldots, r\}$ and set $f = f_k$. We also fix an index $j \in \{1, \ldots, n\}$ and set $\delta = \frac{\partial}{\partial x_j}$ which is a global derivation on U.

Note that

$$\delta\left(\frac{x_j}{x_i}\right) = \frac{\delta(x_j)}{x_i} - \frac{x_j}{x_i}\frac{\delta(x_i)}{x_i},$$

and that

$$I(H) \cdot \delta|_{\mathrm{Spec}(A_i)} = x_i \cdot \delta : A_i \to A_i,$$

and hence $\mathcal{I}(H) \cdot \delta$ is an invertible sheaf of derivations on W_1.

Now in A_i consider the factorization $f = x_i^b g_i$, so $g_i \in J_1^{(i)} \subset A_i$, and $x_i \cdot \delta$ is a derivation on A_i. Finally check that

$$\frac{\delta(f)}{x_i^{b-1}} = \frac{x_i\delta(x_i^b \cdot g_i)}{x_i^b} = \frac{x_i\delta(x_i^b)}{x_i^b} g_i + x_i^b \frac{(x_i\delta)(g_i)}{x_i^b}) = b \cdot \delta(x_i) \cdot g_i + (x_i\delta)(g_i).$$

This already proves b) since the right hand side is in $\Delta(J_1^{(i)})$. ↺

Our argument also shows that this equality is stable by any *sequence* of transformations (see 5.9).

Remark 5.8. Fix $K \subset J$ two ideals in $\mathcal{O}_W$, and couples (J, b) and (K, b). Then clearly:

a) $\mathrm{Sing}(J, b) \subset \mathrm{Sing}(K, b)$.

b) Any transformation, as in (5.6.1), of (J, b), induces the transformation

$$\begin{array}{ccc} W & \xleftarrow{\pi} & W_1 \\ (K, b) & & (K_1, b) \end{array}$$

and $K_1 \subset J_1$.

5.9. We finally extend the previous result to study the behavior of Δ operators with an arbitrary sequence of transformations.

Corollary 5.10. *Fix a couple (J, b) $(J \subset \mathcal{O}_W)$ and a sequence of transformations*

$$\begin{array}{ccccccc} W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \ldots & \xleftarrow{\pi_r} & W_r \\ (J, b) & & (J_1, b) & & & & (J_k, b). \end{array} \tag{5.10.1}$$

1) If $b \geq 2$, then (5.10.1) induces a sequence of transformations

$$\begin{array}{ccccccc} W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \ldots & \xleftarrow{\pi_r} & W_r, \\ (\Delta(J), b-1) & & ((\Delta(J))_1, b-1) & & & & ((\Delta(J))_r, b-1), \end{array}$$

and

2) $(\Delta(J))_r \subset \Delta(J_r)$.

Proof. The case when $r = 1$ is in 5.7. Consider now the case $r = 2$, namely

$$\begin{array}{ccccc} W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & W_3 \\ (J, b) & & (J_1, b) & & (J_2, b). \end{array}$$

Then 5.7 asserts that π_1 defines a transform of $(\Delta(J), b-1)$, say $((\Delta(J))_1, b-1)$, and that $(\Delta(J))_1 \subset \Delta(J_1)$. The same result says that π_2 defines a transform of $(\Delta(J_1), b-1)$, say $((\Delta(J_1))_1, b-1)$, and that $(\Delta(J_1))_1 \subset \Delta(J_2)$. The statement follows in this case from 5.8.

The general case $r \geq 2$ follows similarly, by induction. $\circlearrowleft$

Corollary 5.11. *Fix a couple (J, b) $(J \subset \mathcal{O}_W)$ and, as before, a sequence of transformations (5.10.1). Assume that $b \geq 2$. Then, for each index $1 \leq j \leq b-1$:*

1) The sequence (5.10.1) induces a sequence of transformations $((\Delta^{(j)}(J)), b-1-(j-1))$, say

$$\begin{array}{ccccccc} W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & W_r \\ (\Delta^{(j)}(J), b-1-(j-1)) & & ((\Delta^{(j)}(J))_1, b-1-(j-1)) & & & & ((\Delta^{(j)}(J))_r, b-1-(j-1)) \end{array}$$

and

2) $(\Delta^{(j)}(J))_r \subset \Delta^{(j)}(J_r)$.

Proof. Note that for $j = 1$, $\Delta^{(j)} = \Delta$ and we obtain the previous corollary. So we prove now the statement for j assuming that it holds $j-1$. Set $J^* = \Delta^{(j-1)}(J)$ and $b^* = b-1-(j-2)$. By induction:

i) The sequence of transformations (5.10.1) induces transformations of (J^*, b^*), say:

$$\begin{array}{ccccccc} W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & W_r, \\ (J^*, b^*) & & (J^*_1, b^*) & & & & (J^*_r, b^*) \end{array}$$

and

ii) $J^*_r \subset \Delta^{(j-1)}(J_r)$.

Applying our previous Corollary 5.10 to i), we get:

i') The sequence in i) induces transformations of $(\Delta(J^*), b^*-1)$:

$$\begin{array}{ccccccc} W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & W_r, \\ (\Delta(J^*), b^*-1) & & ((\Delta(J^*))_1, b^*-1) & & & & ((\Delta(J^*))_r, b^*-1) \end{array}$$

and

ii') $(\Delta(J^*))_r \subset \Delta(J^*_r)$.

Here $\Delta(J^*) = \Delta^{(j)}(J)$ and i') is statement 1). On the other hand, applying Δ to ii) we get

$$\Delta(J^*_r) \subset \Delta^{(j)}(J_r),$$

which together with ii') proves 2).

$\circlearrowleft$

In the next Section we shall apply Corollary 5.11, basically in the case $j = b-1$. The reader might want to look into Example 6.15 to get have an overview of this application of the Corollary.

6. Simple couples and a form of induction on resolution problems.

6.1. The purpose of this Section is the study of *simple couples* (J,b) $(J \subset \mathcal{O}_W)$. Examples of simple couples appear already in Section 2. They will play a central role in our inductive arguments (induction on the dimension of the ambient space). The main results of this Section are Theorem 6.5 and Proposition 6.13, where the notion of stability of induction discussed in 2.8 is formalized.

6.2. Fix $J \subset \mathcal{O}_W$, assume that $J_x \neq 0 (\subset \mathcal{O}_{W,x})$ for any $x \in W$, and define a function

$$ord_J : W \to \mathbb{N}, \tag{6.2.1}$$

where $ord_J(x)$ denotes the order of J_x in the local ring $\mathcal{O}_{W,x}$.

Note that ord_J is upper-semi-continuous (3.6). In fact, for any positive integer s:

$$\{x \in W / ord_J(x) \geq s\} = V(\Delta^{s-1}(J)) \text{ (see 5.4).}$$

Remark 6.3. The following conditions are equivalent:
1) $\max -ord_J = b$ (where, as in 3.6, $\max -ord_J$ denotes the maximum value achieved).
2) $V(\Delta^{b-1}(J)) \neq \emptyset$ and $V(\Delta^b(J)) = \emptyset$.
3) $\max -ord_{\Delta^{b-1}(J)} = 1$.

The equivalence follows from the properties of the Δ operator discussed in 5.4.

Definition 6.4. We say that (J,b) is a *simple couple* if the previous conditions hold for J and b.

The following theorem is a central result in this section.

Theorem 6.5. *If (J,b) $(J \subset \mathcal{O}_W)$ is a simple couple, and*

$$\begin{array}{ccc} W & \stackrel{\pi}{\longleftarrow} & W_1 \\ (J,b) & & (J_1,b) \end{array}$$

is a transformation, then either $\mathrm{Sing}(J_1,b) = \emptyset$ *or* (J_1,b) *is a simple couple.*

The case $b=1$ will be proved in Proposition 6.8, and the case $b \geq 2$ in Proposition 6.9.
We shall first draw attention to the case of simple couples of the form $(J,1)$.

Remark 6.6. The following conditions are equivalent:
1) $\max -ord_J = 1$.
2) $V(J)) \neq \emptyset$ and $V(\Delta(J)) = \emptyset$.
3) There is an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of W, and for each λ a smooth hypersurface $\overline{W}_\lambda$ in U_λ such that $I(\overline{W}_\lambda) \subset J_\lambda$, where J_λ denotes the restriction of J to U_λ.

For the proof of 3), note that an ideal of order one in a local regular ring $\mathcal{O}_{W,x}$ contains an element of order one; and that element defines a smooth hypersurface in some open neighborhood of $x \in W$.

Remark 6.7. Fix, as before, an open covering of W, say $\{U_\lambda\}_{\lambda\in\Lambda}$, and a monoidal transformation with center $Y \subset W$, say $W \longleftarrow W_1$. For each index λ set $U_\lambda^{(1)} \subset W_1$ as the pull-back of U_λ. In this way we get

$$U_\lambda \longleftarrow U_\lambda^{(1)}$$

which is either a monoidal transformation (in case $Y \cap U_\lambda \neq \emptyset$), or the identity map (if $Y \cap U_\lambda = \emptyset$). Note also that $\{U_\lambda^{(1)}\}_{\lambda\in\Lambda}$ is an open cover of W_1.

Proposition 6.8. *Fix $J \subset \mathcal{O}_W$ with maximum order 1, and a sequence of transformations*

$$\begin{array}{ccccccc} W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & W_r \\ (J,1) & & (J_1,1) & & & & (J_r,1) \end{array} \tag{6.8.1}$$

then the maximum order of J_r is either 1 or 0 (i.e. $J_r = \mathcal{O}_{W_r}$ in the last case).

Proof. Define an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of W, and inclusions

$$I(\overline{W}_\lambda) \subset J_\lambda, \tag{6.8.2}$$

where $\overline{W}_\lambda$ is a smooth hypersurface in U_λ, as indicated in Remark 6.6,3).

The sequence (6.8.1) defines, for each index λ, a sequence of transformations:

$$\begin{array}{ccccccc} U_\lambda & \xleftarrow{\pi_1} & U_\lambda^{(1)} & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & U_\lambda^{(r)} \\ (J_\lambda,1) & & ((J_\lambda)_1,1) & & & & ((J_\lambda)_r,1), \end{array}$$

and also

$$\begin{array}{ccccccc} U_\lambda & \xleftarrow{\pi_1} & U_\lambda^{(1)} & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & U_\lambda^{\lambda(r)} \\ (I(\overline{W}_\lambda),1) & & ((I(\overline{W}_\lambda))_1,1) & & & & ((I(\overline{W}_\lambda))_r,1). \end{array}$$

Furthermore

$$(I(\overline{W}_\lambda))_r \subset (J_\lambda)_r$$

by Remark 5.8. Let $\overline{W}_\lambda^{(r)} \subset U_\lambda^r$ denote the strict transform of $\overline{W}_\lambda$. Since $\overline{W}_\lambda$ is smooth in U_λ, Example 4.3 asserts that $\overline{W}_\lambda^{(r)}$ is smooth, and defined by the ideal $(I(\overline{W}_\lambda))_r$. In particular $(I(\overline{W}_\lambda))_r$ has maximum order at most one, and hence the same holds for $(J_\lambda)_r$. Since the open sets $(U_\lambda)^{(r)}$ cover W_r it follows that J_r has order at most 1. ◌

Proposition 6.9. *Fix $J \subset \mathcal{O}_W$ with maximum order $b \geq 2$, and consider a sequence of transformations*

$$\begin{array}{ccccccc} W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & W_r \\ (J,b) & & (J_1,b) & & & & (J_r,b). \end{array} \tag{6.9.1}$$

Then then the maximum order of $J_r(\subset \mathcal{O}_{W_r})$ is at most b.

Proof. From 5.4 we conclude that the maximum order of $\Delta^{b-1}(J)(\subset \mathcal{O}_W)$ is 1. Corollary 5.11 applied for $j = b-1$ says that (6.9.1) defines the sequence of transformations

$$\begin{array}{ccccccc} W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & W_r \\ (\Delta^{b-1}(J),1) & & ((\Delta^{b-1}(J))_1,1) & & & & ((\Delta^{b-1}(J))_r,1), \end{array} \tag{6.9.2}$$

and that $(\Delta^{b-1}(J))_r \subset \Delta^{b-1}(J_r)$. On the other hand Proposition 6.8 asserts that $(\Delta(J))_r$ has order at most 1, and hence $\Delta^{b-1}(J_r)$ has order at most one. From this and 5.4 we conclude that J_r has order at most b. ◌

Remark 6.10. There is a stronger outcome that follows from the proof of Proposition 6.9 that relates to induction in the dimension of the ambient space. Note that J has highest order b, so $\Delta^{b-1}(J)$ has highest order one. We can argue as in the proof of Proposition 6.8, and define an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of W, and for each index λ, a smooth hypersurface $\overline{W}_\lambda \subset U_\lambda$, defined by

$$I(\overline{W}_\lambda) \subset (\Delta^{b-1}(J))_\lambda. \tag{6.10.1}$$

Now use the compatibility of the Δ operator with restriction to open sets and check that $(\Delta^{b-1}(J))_\lambda = (\Delta^{b-1}(J_\lambda))$. Note also that $\mathrm{Sing}(J,b) \cap U_\lambda \subset \overline{W}_\lambda$. Recall that (6.9.2) defines, for each index λ, a sequence of transformations of $((\Delta^{b-1}(J))_\lambda, 1)$, say:

$$\begin{array}{ccccccc} U_\lambda & \xleftarrow{\pi_1} & U_\lambda^{(1)} & \xleftarrow{\pi_2} & \cdots & \xleftarrow{\pi_r} & U_\lambda^{(r)} \\ ((\Delta^{b-1}(J))_\lambda, 1) & & (((\Delta^{b-1}(J))_\lambda)_1, 1) & & & & (((\Delta^{b-1}(J))_\lambda)_r, 1), \end{array}$$

and also

$$\begin{array}{ccccccc} U_\lambda & \xleftarrow{\pi_1} & U_\lambda^{(1)} & \xleftarrow{\pi_2} & \cdots & \xleftarrow{\pi_r} & U_\lambda^{\lambda(r)} \\ (I(\overline{W}_\lambda), 1) & & ((I(\overline{W}_\lambda))_1, 1) & & & & ((I(\overline{W}_\lambda))_r, 1). \end{array}$$

Furthermore, $(I(\overline{W}_\lambda))_r \subset ((\Delta^{b-1}(J))_\lambda)_r$, and $(I(\overline{W}_\lambda))_r$ defines a smooth hypersurface $\overline{W}_\lambda^{(r)} \subset U_\lambda^{(r)}$ which is the strict transform of $\overline{W}_\lambda$. We finally note that $\{U_\lambda^{(r)}\}_{\lambda\in\Lambda}$ is a cover of $W^{(r)}$, and taking restriction of the inclusion $(\Delta^{b-1}(J))_r \subset \Delta^{b-1}(J_r)$, we get that:

$$((\Delta^{b-1}(J))_\lambda)_r = ((\Delta^{b-1}(J))_r)_\lambda \subset (\Delta^{b-1}(J_r))_\lambda,$$

and hence $(I(\overline{W}_\lambda))_r \subset (\Delta^{b-1}(J_r))_\lambda$. In particular

$$(\mathrm{Sing}((J)_r, b) \cap U_\lambda^{(r)} =) \, \mathrm{Sing}((J_\lambda)_r, b) \subset \overline{W}_\lambda^{(r)}.$$

Lemma 6.11. *Fix $J \subset \mathcal{O}_W$ with maximum order b. There is an open covering, say $\{U_\lambda\}_{\lambda\in\Lambda}$ of W, and for each index λ a smooth hypersurface $\overline{W}_\lambda \subset U_\lambda$, such that the following properties hold:*

P1) $\mathrm{Sing}(J_\lambda, b) \subset \overline{W}_\lambda$.

P2) *For* any *sequence*

(6.11.1)
$$\begin{array}{ccccccc} W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \cdots & \xleftarrow{\pi_r} & W_r \\ (J,b) & & (J_1,b) & & & & (J_r,b) \end{array}$$

and setting by restriction, for each λ, say:

(6.11.2)
$$\begin{array}{ccccccc} U_\lambda & \xleftarrow{\pi_1} & U_\lambda^{(1)} & \xleftarrow{\pi_2} & \cdots & \xleftarrow{\pi_r} & U_\lambda^{(r)}, \\ (J_\lambda,b) & & ((J_\lambda)_1,b) & & & & ((J_\lambda)_r,b) \end{array}$$

then $\{U_\lambda^{(r)}\}_{\lambda\in\Lambda}$ is an open covering of W_r, and

(6.11.3)
$$\mathrm{Sing}(J_r, b) \cap U_\lambda^{(r)} = \mathrm{Sing}((J_\lambda)_r, b) \subset \overline{W}_\lambda^{(r)},$$

where $\overline{W}_\lambda^{(r)}$ is the smooth hypersurface defined by the strict transform of $\overline{W}_\lambda$.

Proof. The case $b = 1$ (in which $\mathrm{Sing}(J,1) = V(J)$) is in the proof of Proposition 6.8. The case $b \geq 2$ is in Remark 6.10, and relies entirely on the inclusion (6.10.1).

$\circlearrowleft$

6.12. Let $\overline{W}_\lambda^{(i)}$ denote the strict transform of $\overline{W}_\lambda^{(0)}$ in $U_\lambda^{(i)}$(see (6.11.2)). A consequence of (6.13.1) is that all the centers of monoidal transformations involved in (6.11.2) are included in $\overline{W}_\lambda^{(i)}$; hence (6.11.2) defines a sequence of monoidal transformations

(6.12.1)
$$\overline{W}_\lambda \longleftarrow \overline{W}_\lambda^{(1)} \longleftarrow \cdots \longleftarrow \overline{W}_\lambda^{(r)}.$$

Proposition 6.13. *Fix $J \subset \mathcal{O}_W$ with maximum order b. There is an open covering, say $\{U_\lambda\}_{\lambda\in\Lambda}$ of W, and for each index λ a closed and smooth hypersurface $\overline{W}_\lambda \subset U_\lambda$, and a couple $(K_\lambda^{(0)}, b!)$ with $K_\lambda^{(0)} \subset \mathcal{O}_{\overline{W}_\lambda}$, such that, in addition to* **P1)** *and* **P2)** *(6.11), the following property holds:*

P3) *The sequence (6.12.1) defined by (6.11.1) as above, induces a sequence of transformations*

$$\begin{array}{ccccccc} \overline{W}_\lambda & \xleftarrow{\pi_1} & \overline{W}_\lambda^{(1)} & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & \overline{W}_\lambda^{(r)} \\ (K_\lambda, b!) & & ((K_\lambda)_1, b!) & & & & ((K_\lambda)_r, b!) \end{array} \tag{6.13.1}$$

and

$$\mathrm{Sing}((J_\lambda)_r, b) = \mathrm{Sing}((K_\lambda)_r, b!)(\subset \overline{W}_\lambda^{(r)}). \tag{6.13.2}$$

Remark 6.14. *On the converse.* Set $W = U_\lambda$ so that $(J, b) = (J_\lambda, b)$. The equality in (6.13.2) asserts, by induction on r, that any sequence 6.13.1 induces a sequence (6.11.1). And furthermore, if 6.13.1 is a resolution, so is (6.11.1).

We are interested mainly in this converse, since we will argue by increasing induction on the dimension of the ambient space. If we accept, by induction, that there is a resolution 6.13.1 for each index λ, then we will have defined a resolution (6.11.2) for each λ. We will define these resolutions so that they patch to a resolution (6.11.1).

Full details of the proof of Proposition 6.13 will be given in Part IV, however the following example illustrates the basic idea of the proof.

Example 6.15. In Example 5.5 we considered the case $W = A^3_k = Spec(k[X, Y, Z])$, and

$$J =< Z^3 + XY^2Z + X^5 >,$$

an ideal of maximum order $b = 3$. In such example we noted that $Z \in \Delta^2(J) =< Z, XY, Y^2, X^3 >$, and we considered the smooth hypersurface $\overline{W} = V(< Z >)$. This is a particular example of Lemma 6.11, where there is no need to consider the open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of W. In fact here the Lemma applies globally in W. In this example $b! = 6$, and Proposition 6.13 applies by setting $K = J^*$ as in (2.6.1).

A similar situations holds, more generally, in 2.9, for $K = J^* = \langle c_i^{\frac{b!}{i}}, i = 2, 3, \dots, b\rangle$.

Remark 6.16. The compatibility of the Δ operator with open restrictions has played an important role in the proofs in this section. If the transformation in Theorem 6.5 is defined with center $Y \subset W$, and if $H \subset W_1$ denotes the exceptional locus, then $J\mathcal{O}_{W_1} = I(H)^b \cdot J_1$, and J_1 has at most order b. Suppose now that the highest order of J along points in W is bigger than b, but that we simply know that the order of J is constant and equal to b along any point of the center Y. Since the order of J along points in W defines an upper-semi-continuous function on W, then there is an open neighborhood, say $U \subset W$ of Y, so that b is the highest order of the restriction J_U. In particular there is an open neighborhood U_1 of H in W_1 so that the restriction $(J_1)_{U_1}$ has highest order $\leq b$.

Remark 6.17. The compatibility of the Δ operator with open restrictions will also play a role in our proof of Proposition 6.13, and this will allow us to present the ideals K_λ so that they are also compatible with a restriction of W to an open set U, at least if the restricted ideal J_U is again of highest order b.

There is yet another context in which there is a natural compatibility of the operator Δ, which are not open restrictions, but will also play a role in the proof of Proposition 6.13. In fact, set

$W \longleftarrow W_1 = W \times A^1_k$ where A^1_k denotes the affine line and the map is the projection on the first coordinate. Note that if J is an ideal in $\mathcal{O}_W$, and if Δ_1 denotes the operator on the smooth scheme W_1, then

$$\Delta_1(J\mathcal{O}_{W_1}) = \Delta(J)\mathcal{O}_{W_1}.$$

Note that a covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of W induces by pull-back, a covering of W_1. The setting of Proposition 6.8 and the inclusions (6.8.2) are compatible with pull-backs; and so are the setting of Proposition 6.9 and the inclusions (6.10.1). This will guarantee the compatibility of all our development for this particular kind of projection.

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