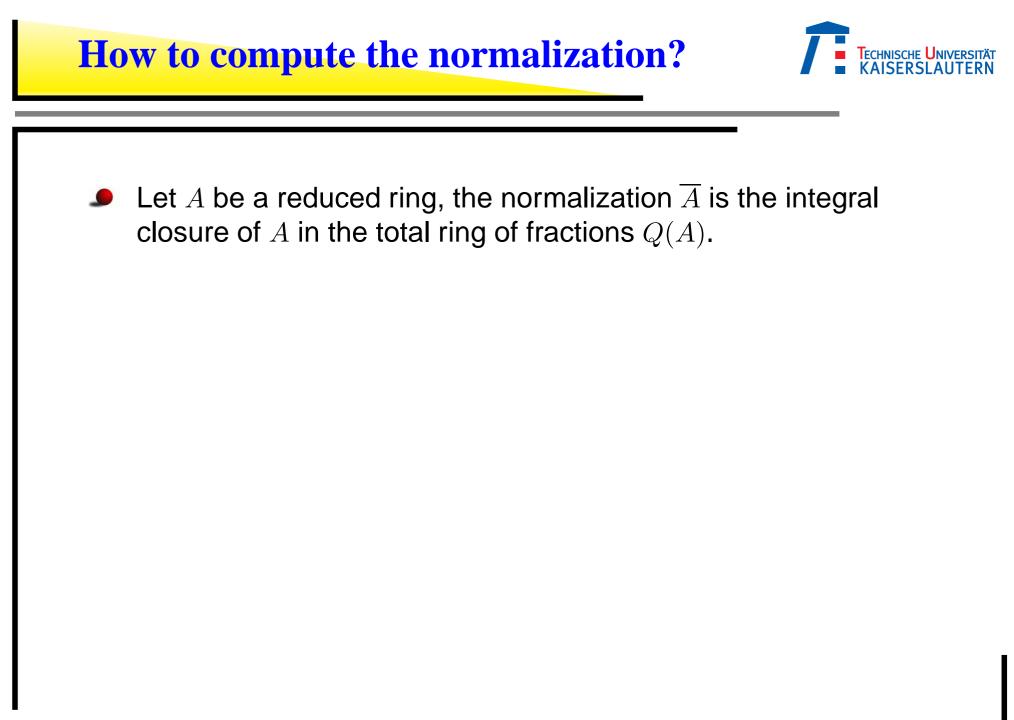


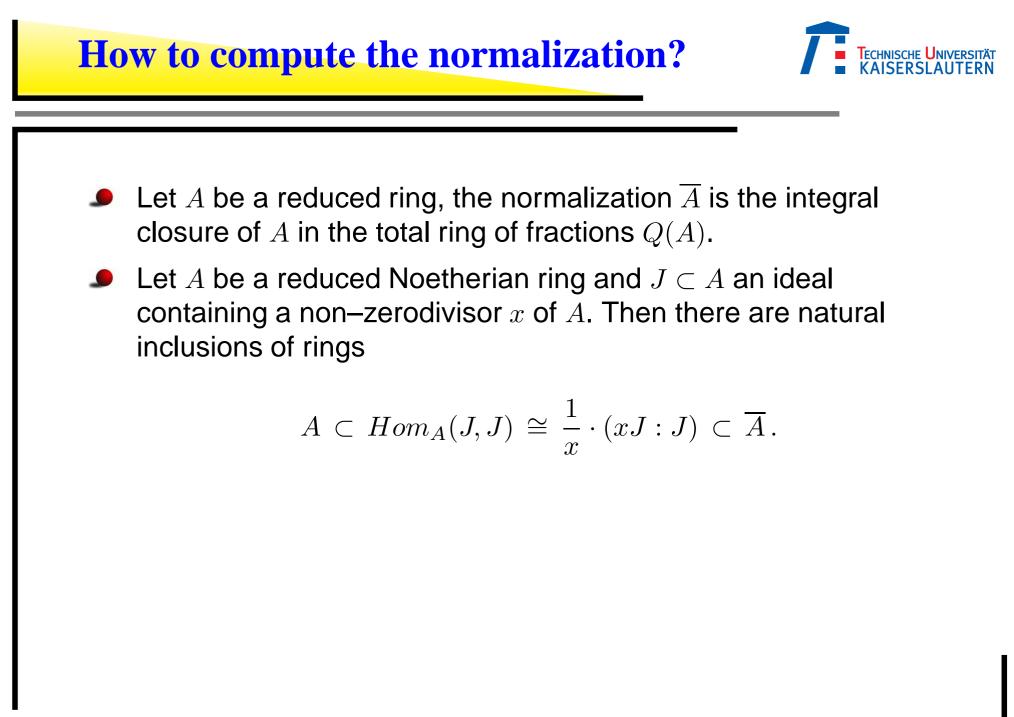
Normalization

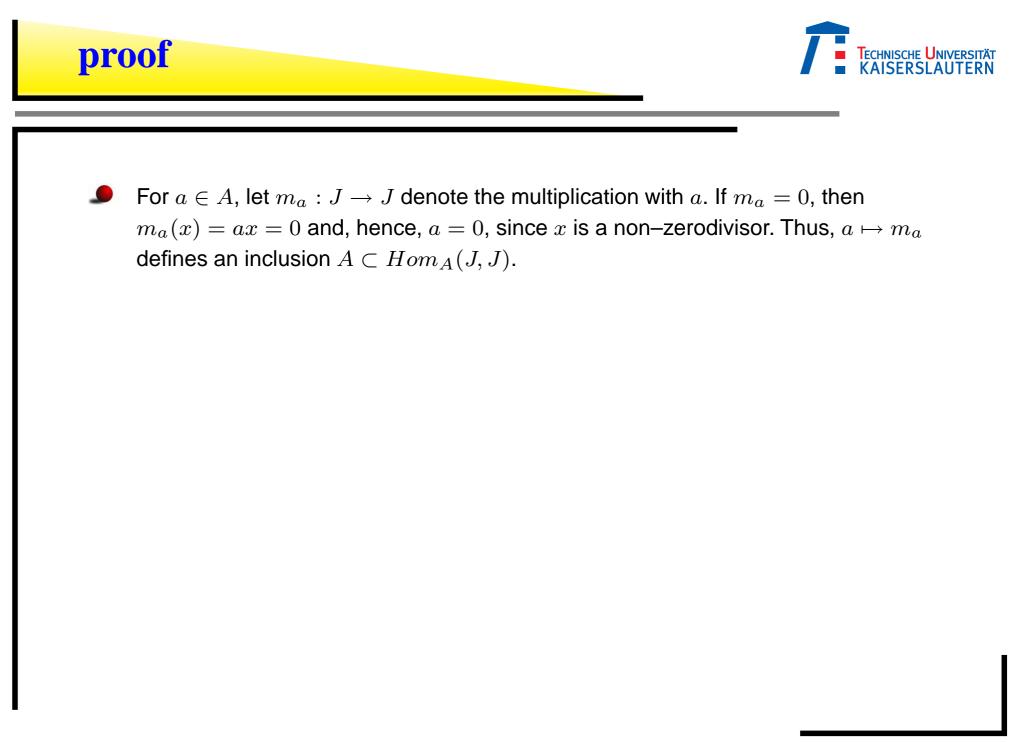
Gerhard Pfister

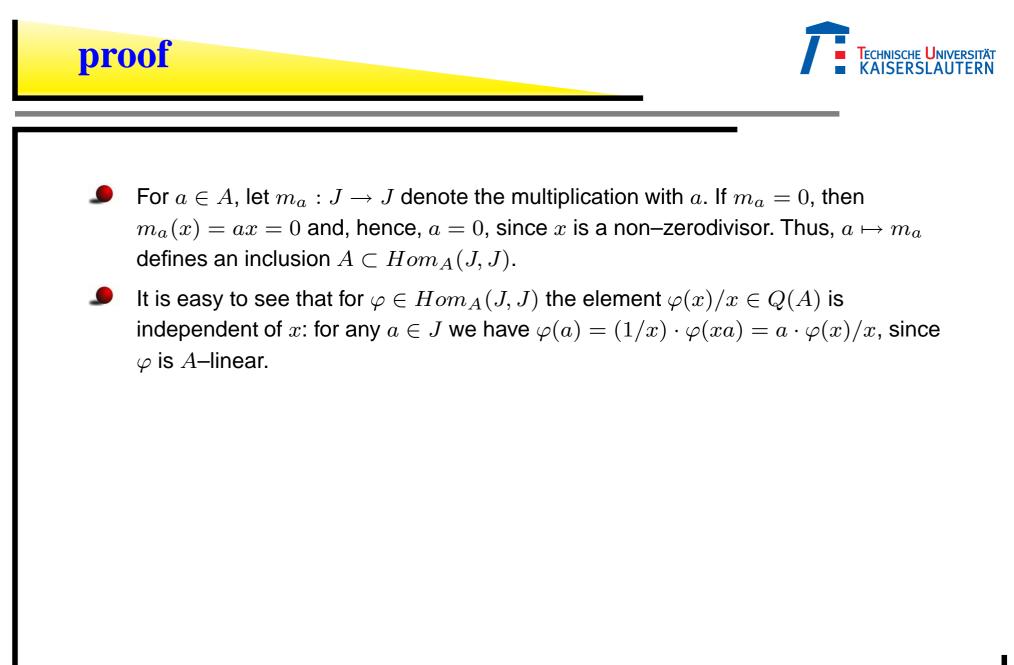
pfister@mathematik.uni-kl.de

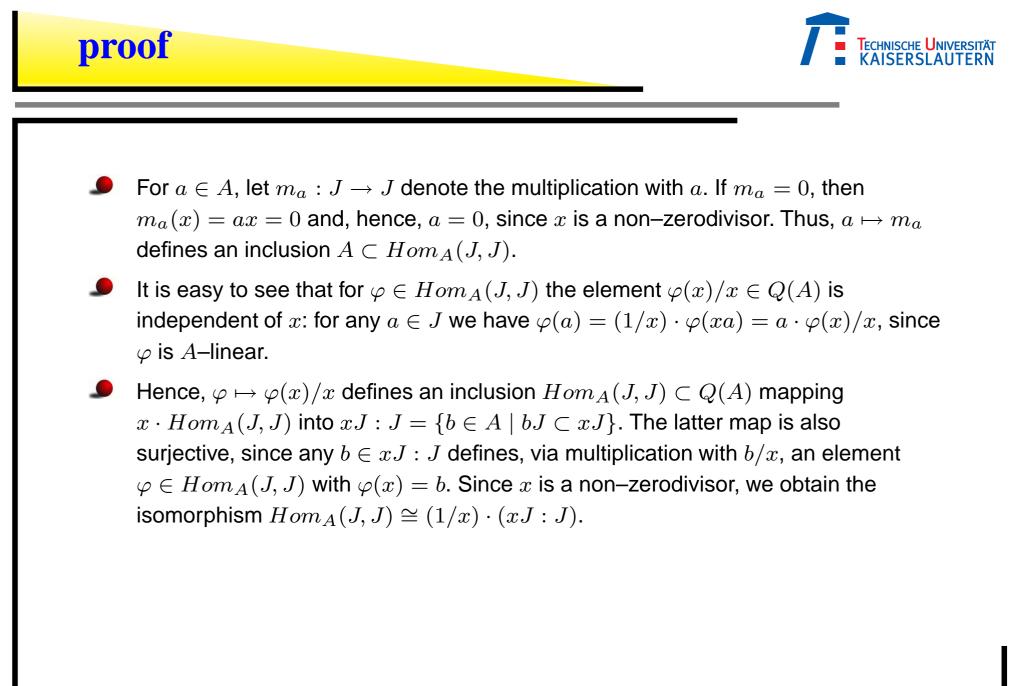
Departement of Mathematics University of Kaiserslautern

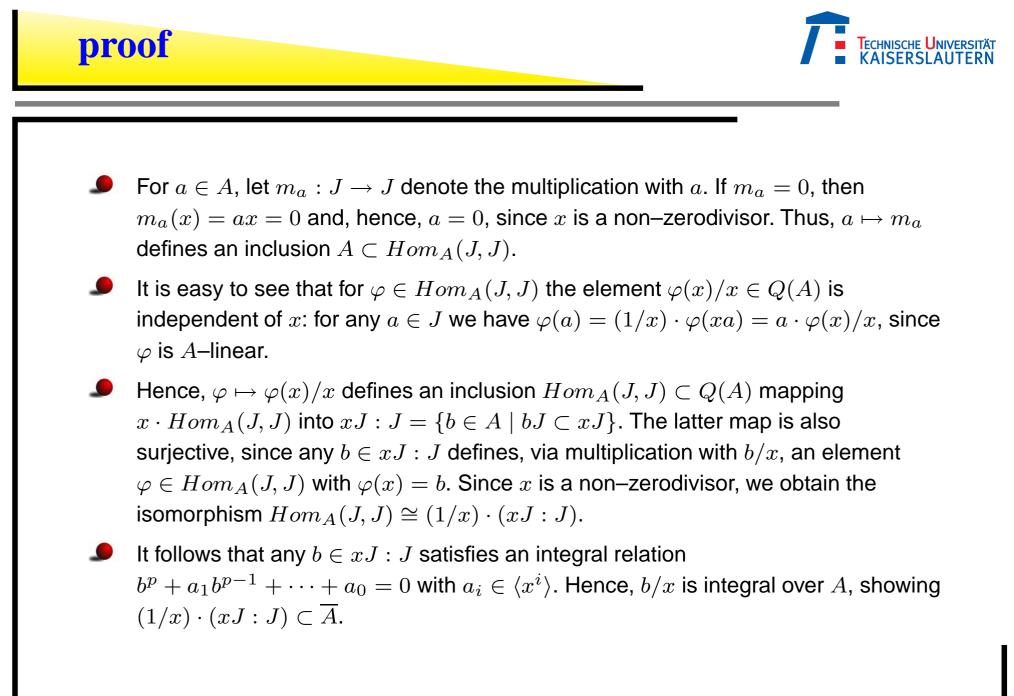


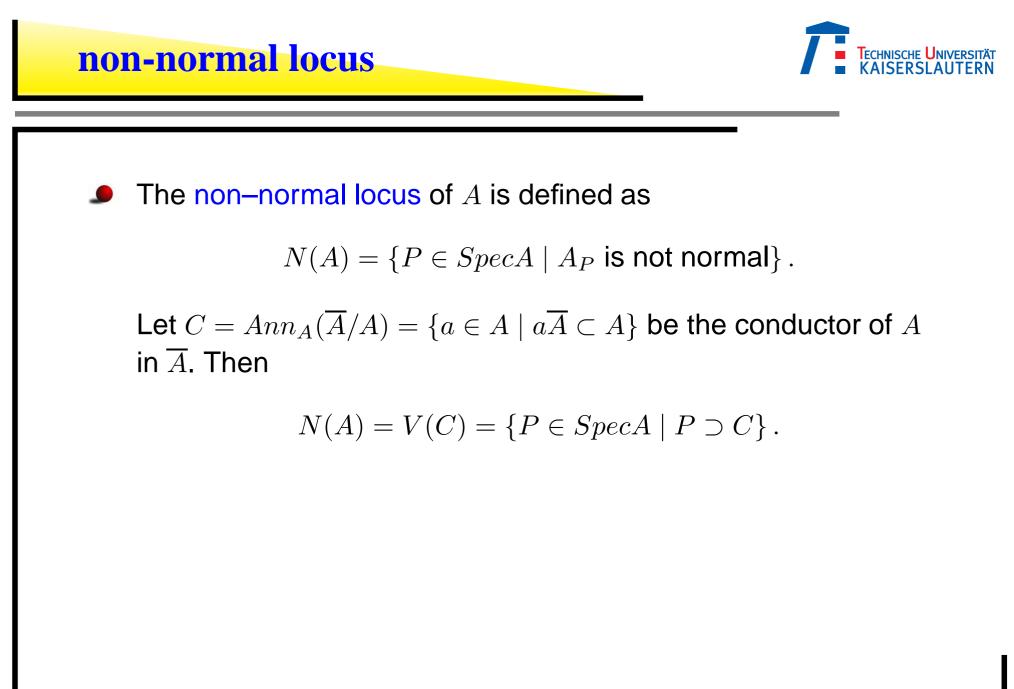


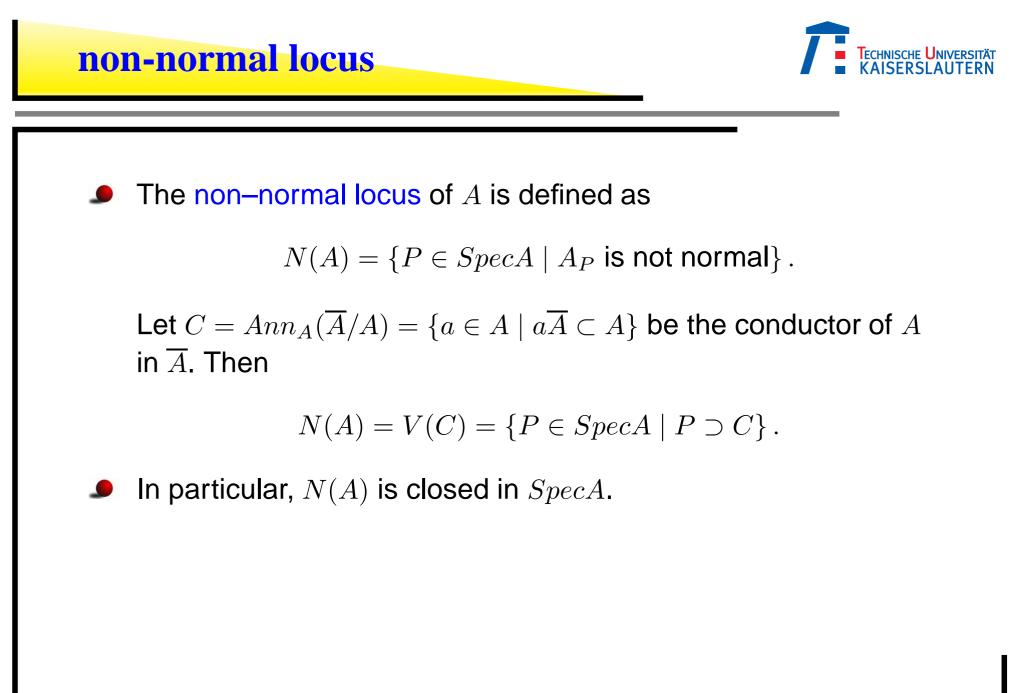


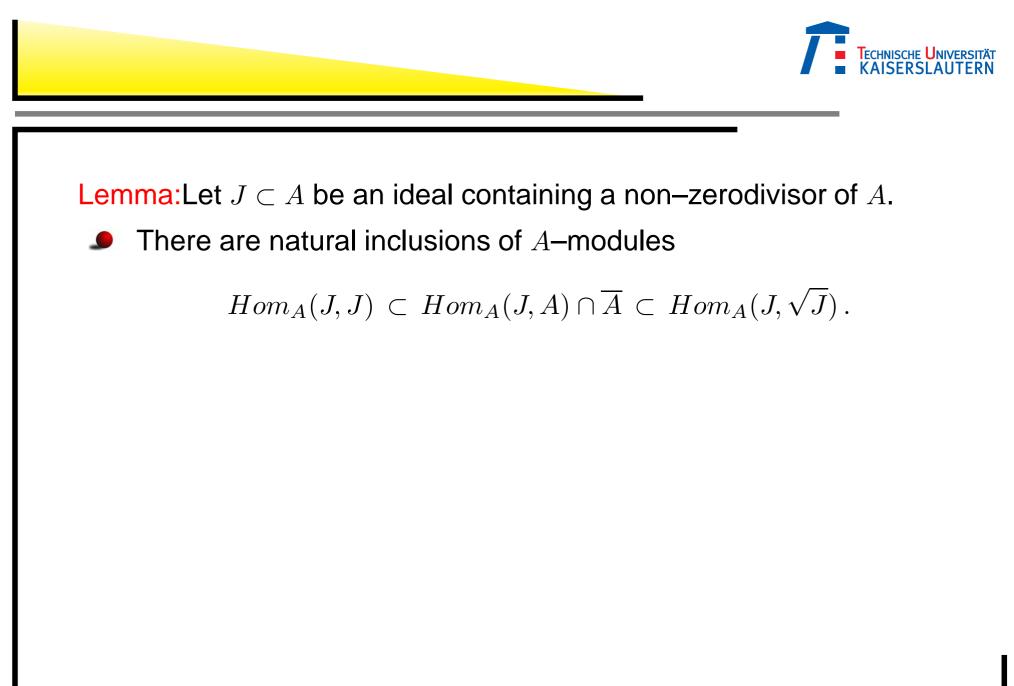


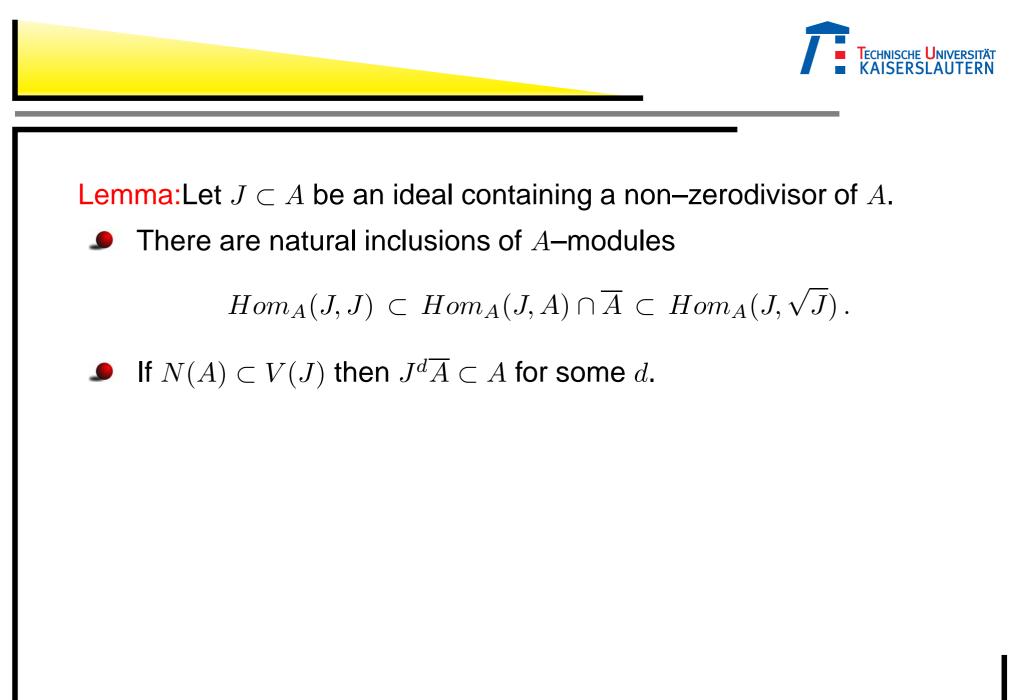














$Hom_A(J,J) \subset Hom_A(J,A) \cap \overline{A}$:

The embedding of $Hom_A(J, A)$ in Q(A) is given by $\varphi \mapsto \varphi(x)/x$, where x is a non-zerodivisor of J. With this identification we obtain

$$Hom_A(J,A) = A :_{Q(A)} J = \{h \in Q(A) \mid hJ \subset A\}$$

and $Hom_A(J, J)$, respectively $Hom_A(J, \sqrt{J})$, is identified with those $h \in Q(A)$ such that $hJ \subset J$, respectively $hJ \subset \sqrt{J}$.





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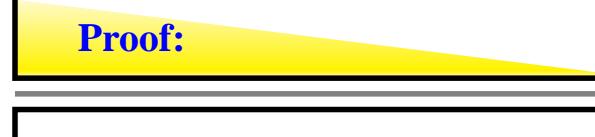
✓ For the second inclusion let $h \in \overline{A}$ satisfy $hJ \subset A$. Consider an integral relation $h^n + a_1 h^{n-1} + \cdots + a_n = 0$ with $a_i \in A$. Let $g \in J$ and multiply the above equation with g^n . Then

$$(hg)^n + ga_1(hg)^{n-1} + \dots + g^n a_n = 0.$$

Since $g \in J$, $hg \in A$ and, therefore, $(hg)^n \in J$ and $hg \in \sqrt{J}$.



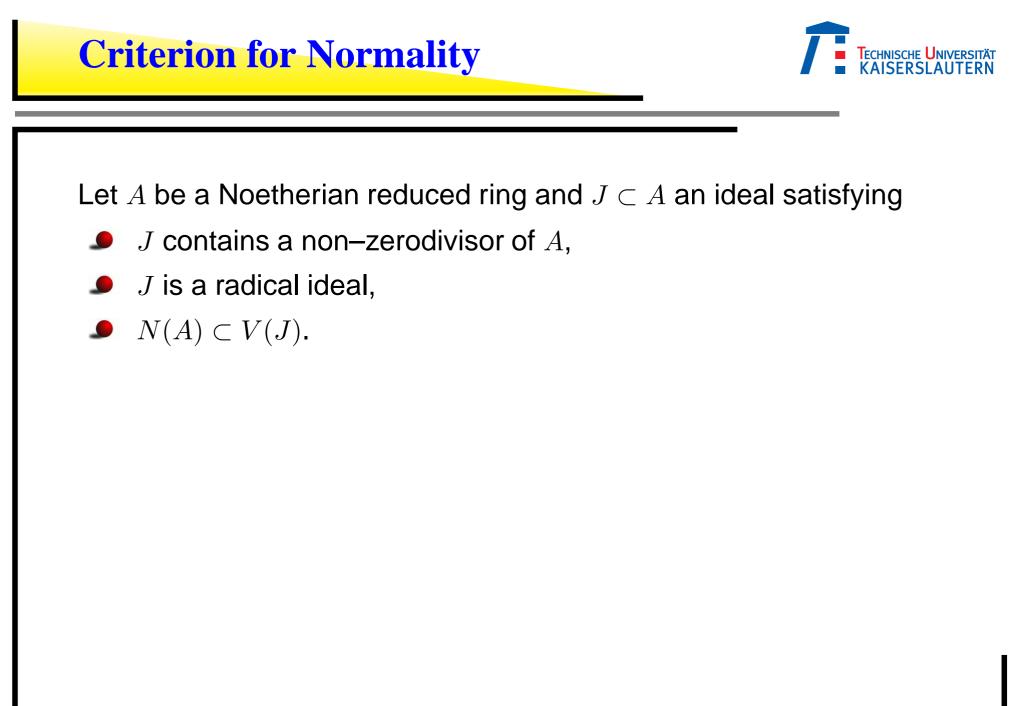
If $N(A) \subset V(J)$ then $J^d\overline{A} \subset A$ for some d.

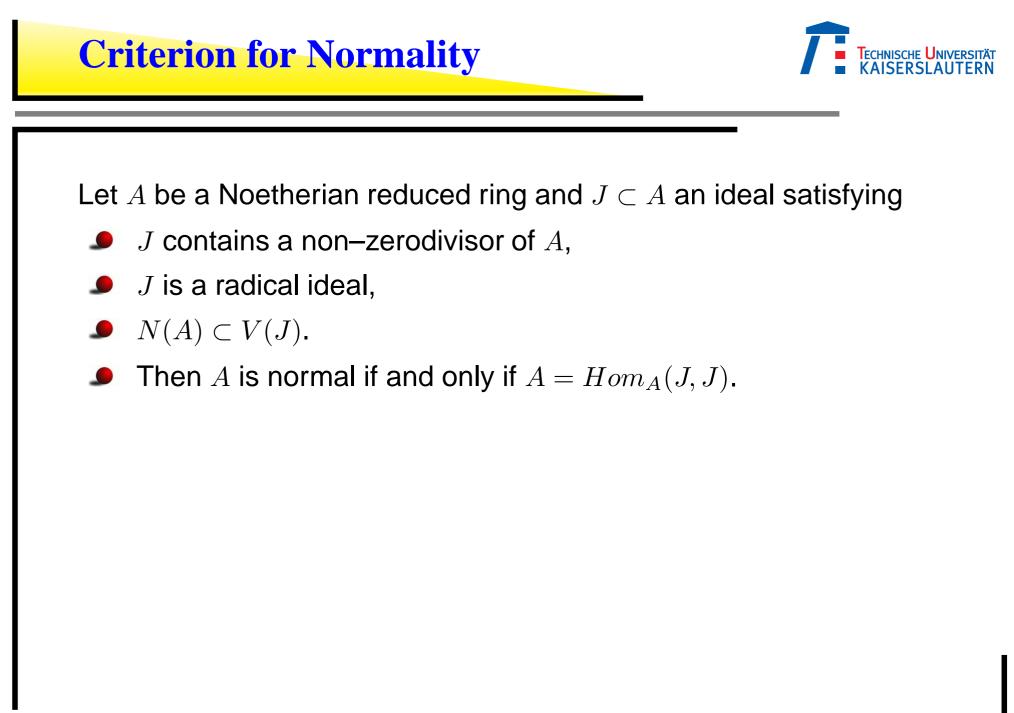


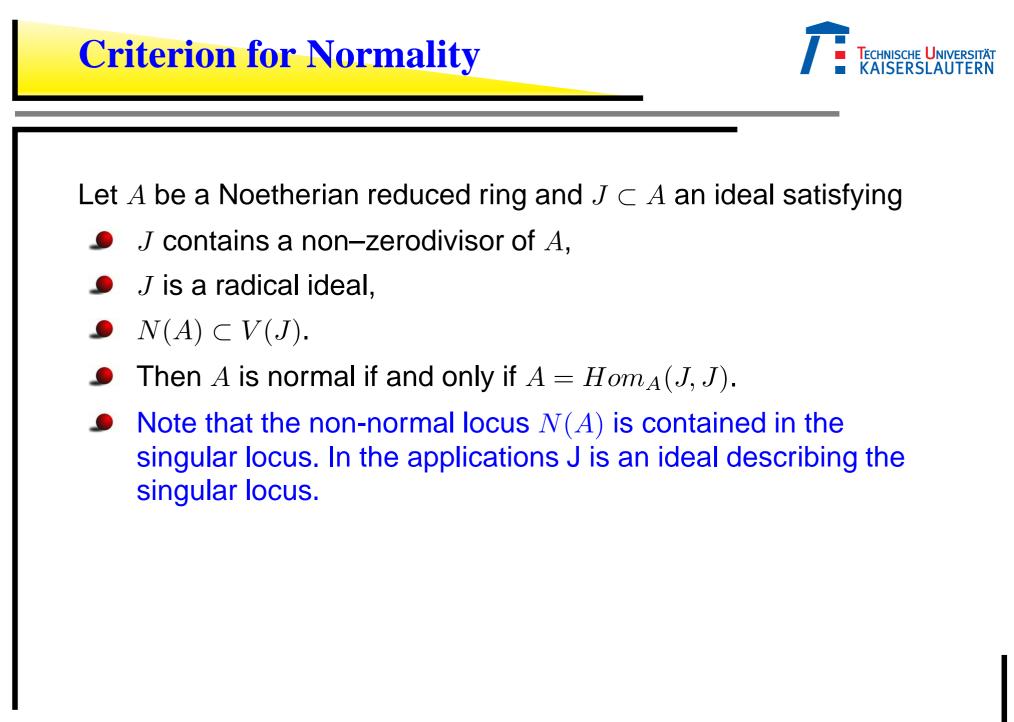


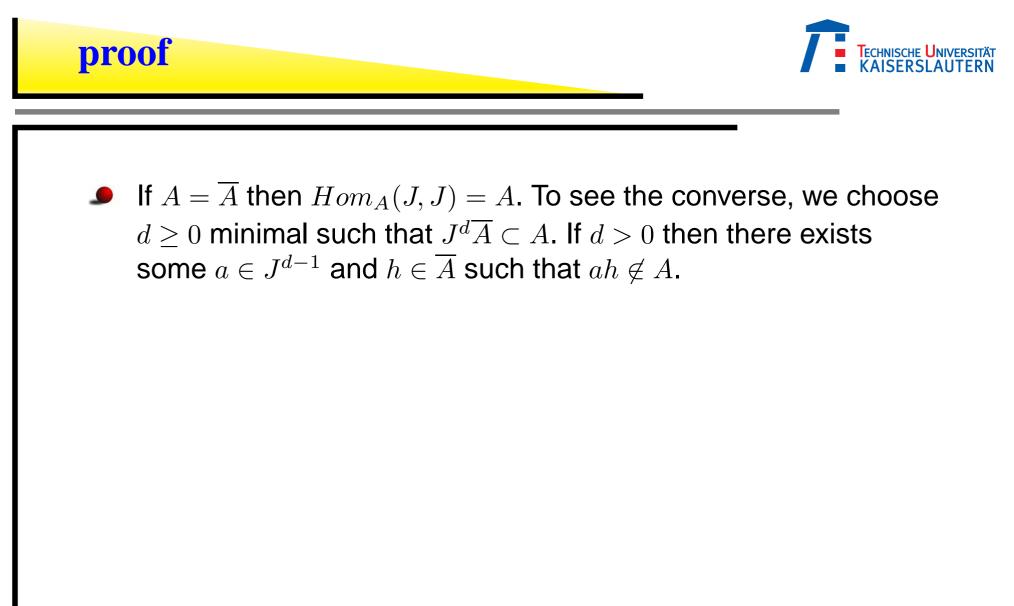
If $N(A) \subset V(J)$ then $J^d \overline{A} \subset A$ for some d. $C = Ann_A(\overline{A}/A) = \{a \in A \mid a\overline{A} \subset A\}$

By assumption, we have V(C) ⊂ V(J) and, hence, $J ⊂ \sqrt{C}$,
 that is, $J^d ⊂ C$ for some d which implies the claim.











- If $A = \overline{A}$ then $Hom_A(J, J) = A$. To see the converse, we choose $d \ge 0$ minimal such that $J^d \overline{A} \subset A$. If d > 0 then there exists some $a \in J^{d-1}$ and $h \in \overline{A}$ such that $ah \notin A$.
- But $ah \in \overline{A}$ and $ah \cdot J \subset hJ^d \subset A$, that is, $ah \in Hom_A(J, A) \cap \overline{A}$, which is equal to $Hom_A(J, J)$, since $J = \sqrt{J}$.



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- By assumption $Hom_A(J, J) = A$ and, hence, $ah \in A$, which is a contradiction. We conclude that d = 0 and $A = \overline{A}$.



Let A be a reduced Noetherian ring, let $J \subset A$ be an ideal and $x \in J$ a non–zerodivisor. Then

$$A = Hom_A(J, J) \text{ if and only if } xJ : J = \langle x \rangle$$



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 if and only if $xJ : J = \langle x \rangle$.

Moreover, let $\{u_0 = x, u_1, \dots, u_s\}$ be a system of generators for the *A*-module xJ: J. Then we can write

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$$u_i \cdot u_j = \sum_{k=0}^{i} x \xi_k^{ij} u_k$$
 with suitable $\xi_k^{ij} \in A$, $1 \le i \le j \le s$.



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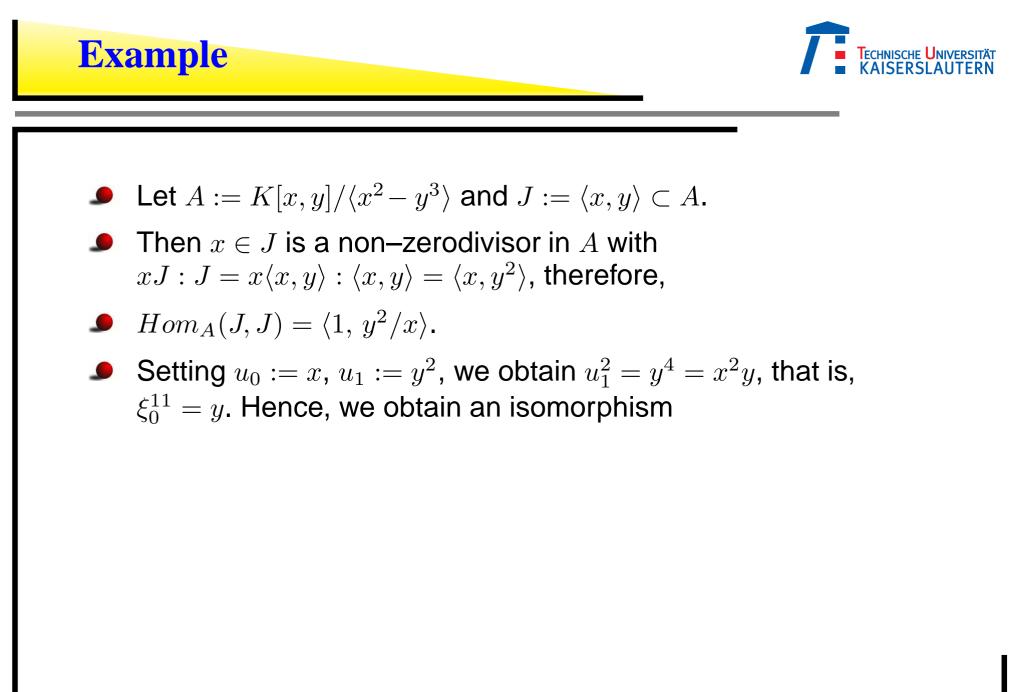
$$u_i \cdot u_j = \sum_{k=0}^{\circ} x \xi_k^{ij} u_k \text{ with suitable } \xi_k^{ij} \in A, 1 \le i \le j \le s.$$

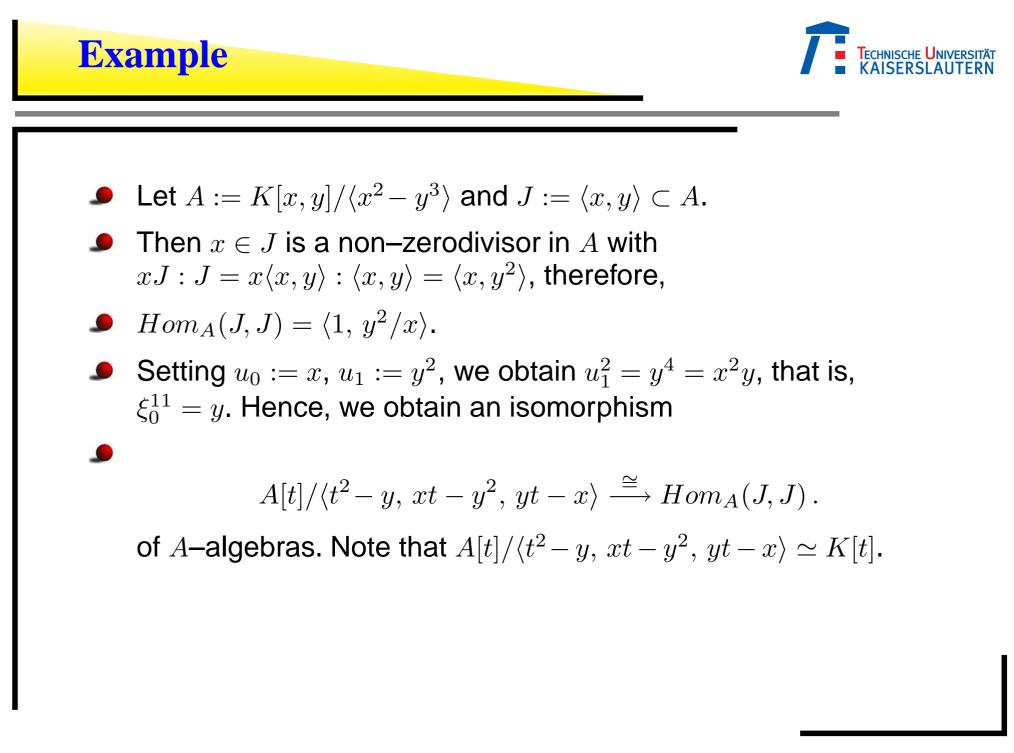
• Let $(\eta_0^{(k)}, \ldots, \eta_s^{(k)}) \in A^{s+1}$, $k = 1, \ldots, m$, generate $syz(u_0, \ldots, u_s)$, and let $I \subset A[t_1, \ldots, t_s]$ be the ideal ($t_0 := 1$)

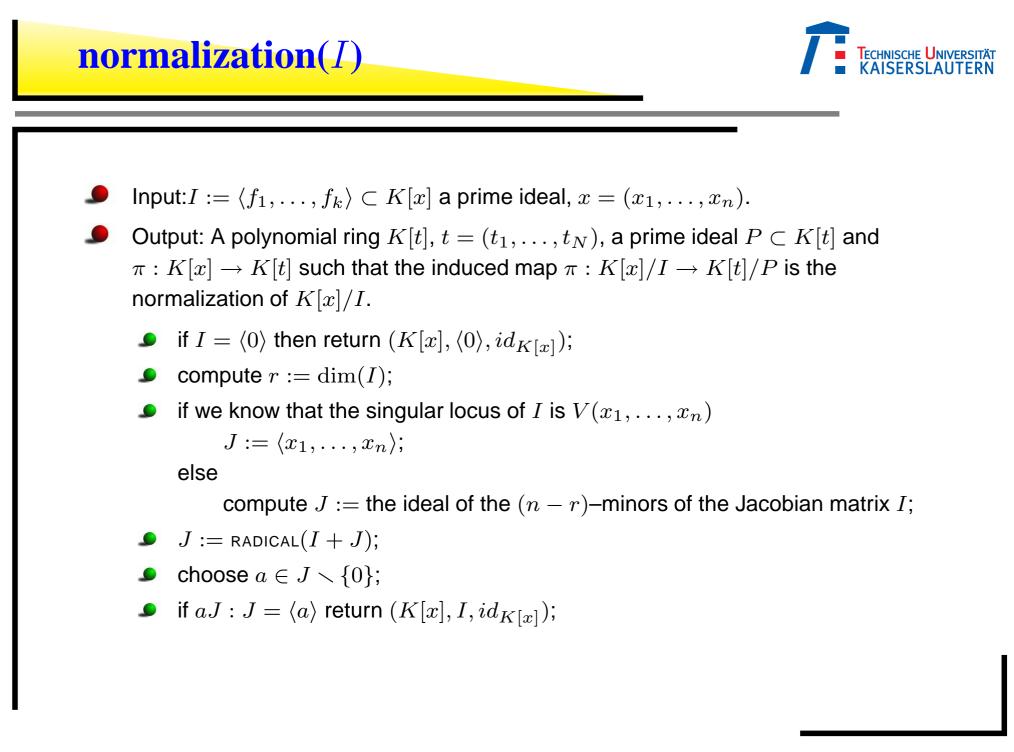
$$I := \left\langle \left\{ t_i t_j - \sum_{k=0}^s \xi_k^{ij} t_k \middle| 1 \le i \le j \le s \right\}, \left\{ \sum_{\nu=0}^s \eta_\nu^{(k)} t_\nu \middle| 1 \le k \le m \right\} \right\rangle,$$

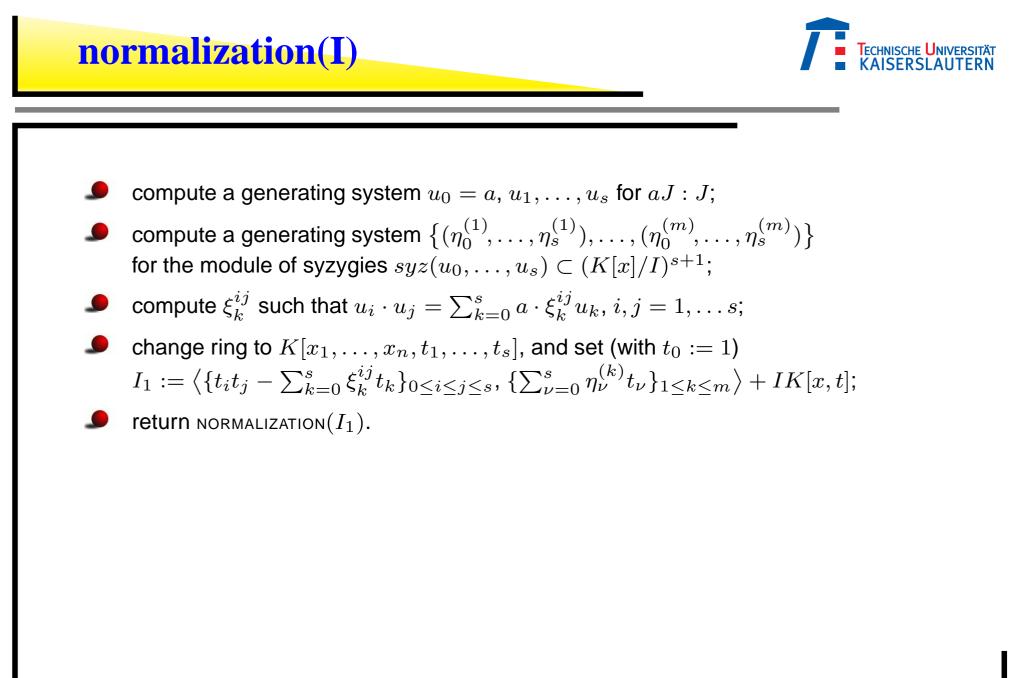
● $t_i \mapsto u_i/x$, i = 1, ..., s, defines an isomorphism

$$A[t_1, \dots, t_s]/I \xrightarrow{\cong} Hom_A(J, J) \cong \frac{1}{x} \cdot (xJ : J).$$







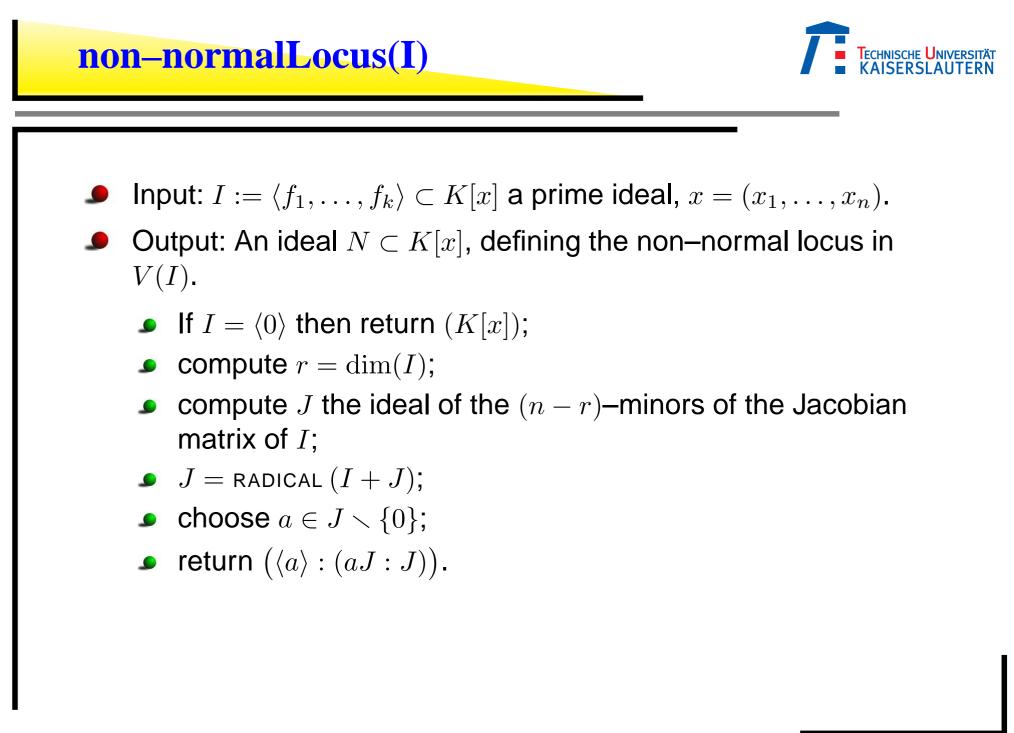




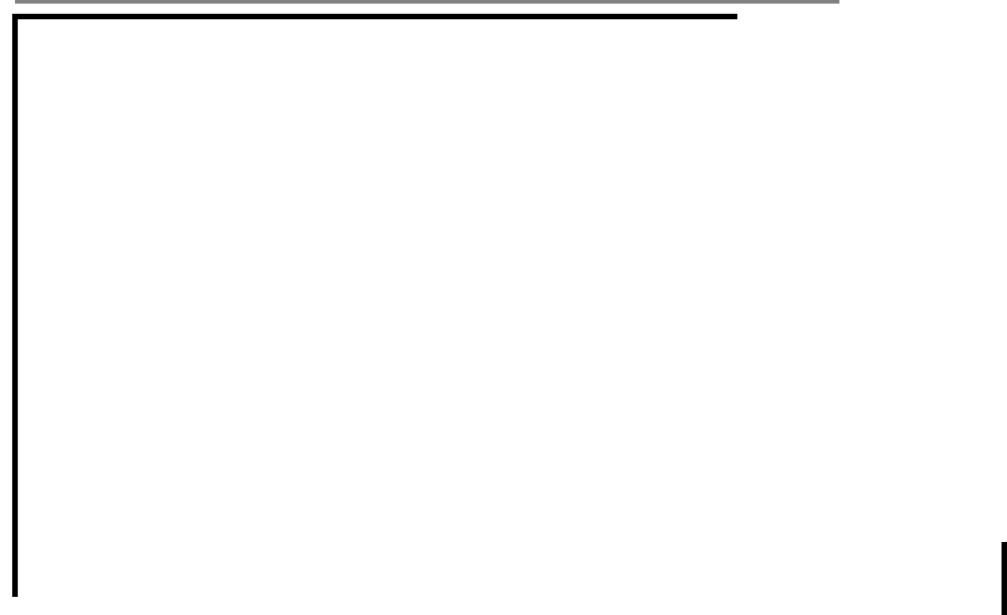
The ideal $Ann_A(Hom_A(J, J)/A) \subset A$ defines the non–normal locus. Moreover,

$$Ann_A(Hom_A(J,J)/A) = \langle x \rangle : (xJ:J)$$

for any non–zerodivisor $x \in J$.









Problem: Characterize the class of finite solvable groups *G* by 2–variable identities.

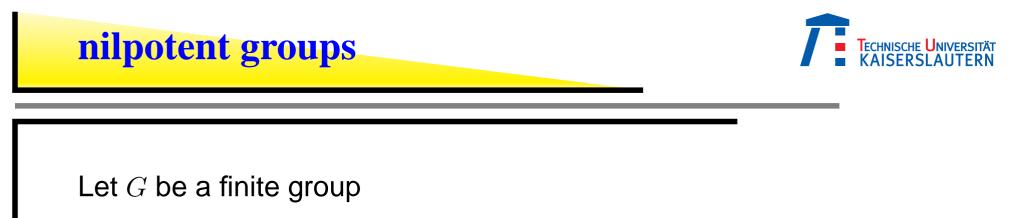


Problem: Characterize the class of finite solvable groups G by 2-variable identities.

Example:

- $G \text{ is abelian} \Leftrightarrow xy = yx \ \forall \ x, y \in G$
- ✓ (Zorn, 1930) A finite group G is nilpotent $\Leftrightarrow \exists n \geq 1$, such that $v_n(x,y) = 1 \forall x, y \in G$ (Engel Identity)

 $v_1 := [x, y] = xyx^{-1}y^{-1}$ (commutator) $v_{n+1} := [v_n, y]$



$$G^{(1)} := [G, G] = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle.$$

Let $G^{(i)} := [G^{(i-1)}, G]$, then G is called nilpotent, if $G^{(m)} = \{e\}$ for a suitable m.

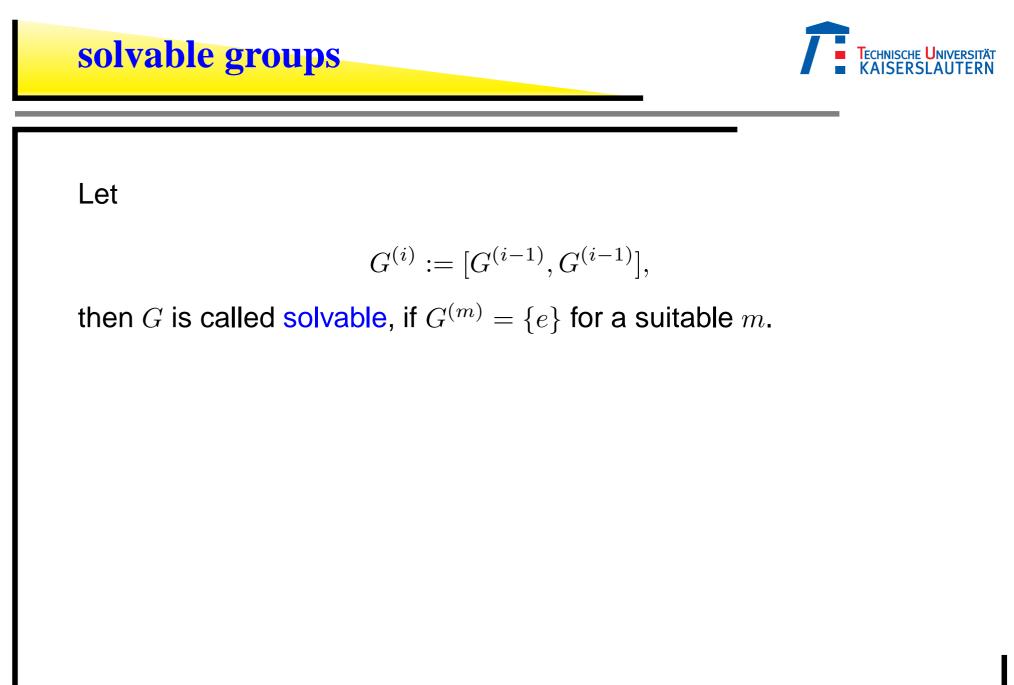


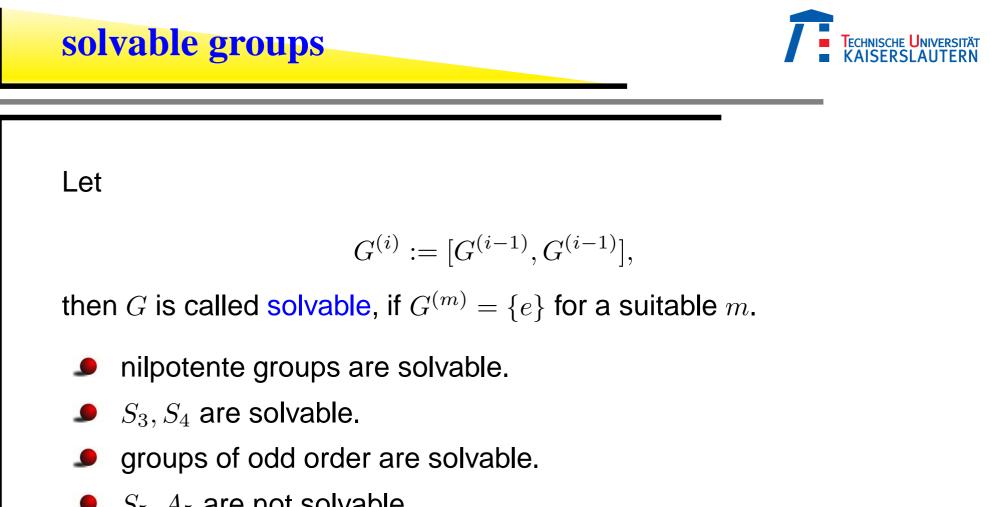
Let G be a finite group

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Let $G^{(i)} := [G^{(i-1)}, G]$, then G is called nilpotent, if $G^{(m)} = \{e\}$ for a suitable m.

- abelian groups are nilpotent.
- if the order of the group is a power of a prime it is nilpotent.
- \blacksquare G ist nilpotent \Leftrightarrow it is the direct product of its Sylow groups.
- S_3 is not nilpotent.





 S_5, A_5 are not solvable.



Theorem (T. Bandman, G.-M. Greuel, F. Grunewald, B. Kunyavsky, G. Pfister, E. Plotkin)

$$U_1 = U_1(x, y) := x^2 y^{-1} x,$$
$$U_{n+1} = U_{n+1}(x, y) = [x U_n x^{-1}, y U_n y^{-1}].$$

A finite group G is **solvable** $\Leftrightarrow \exists n$, such that $U_n(x, y) = 1 \forall x, y \in G$.



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•
$$U_1(x,y) = 1 \Leftrightarrow y = x^{-1}$$

• $U_1(x,y) = U_2(x,y)$
 $\Leftrightarrow x^{-1}yx^{-1}y^{-1}x^2 = yx^{-2}y^{-1}xy^{-1}$
• Let $x, y \in G$ such that $y \neq x^{-1}$ and
 $U_1(x,y) = U_2(x,y) \Rightarrow U_n(x,y) \neq 1 \forall n \in \mathbb{N}.$









Idea of \Leftarrow

Theorem (Thompson, 1968)





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Let *G* minimally not solvable. Then *G* is one of the following groups:

PSL $(2, \mathbb{F}_p)$, *p* a prime number ≥ 5





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Theorem (Thompson, 1968)

- **PSL** $(2, \mathbb{F}_p)$, p a prime number ≥ 5
- **PSL** $(2, \mathbb{F}_{2^p})$, *p* a prime number
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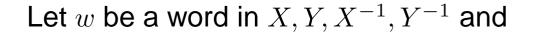
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- PSL $(3, \mathbb{F}_3)$

Sz (2^p) *p* a prime number.

If is enough to prove (for G in Thompson's list): $\exists x, y \in G$, such that $y \neq x^{-1}$ and $U_1(x, y) = U_2(x, y)$.





$$U_1 = w$$
$$U_{n+1} = [XU_n X^{-1}, YU_n Y^{-1}].$$



Let w be a word in X, Y, X^{-1}, Y^{-1} and

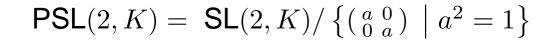
$$U_1 = w$$
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A Computer–search through the 10,000 shortest words in X, X^{-1}, Y, Y^{-1} found the following four words such that the equation $U_1 = U_2$ has a non-trivial solution in PSL(2, *p*) for all p < 1000:

$$w_1 = X^{-2}Y^{-1}X$$
$$w_2 = X^{-1}YXY^{-1}X$$
$$w_3 = Y^{-2}X^{-1}$$
$$w_4 = XY^{-2}X^{-1}YX^{-1}$$











$$\mathsf{PSL}(2,K) = \mathsf{SL}(2,K) / \{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a^2 = 1 \}$$

especially

$$\mathsf{PSL}(2, \mathbb{F}_5) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right\}, \ a_{11}a_{22} - a_{21}a_{12} = 1 \right\}$$
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} 4a_{11} & 4a_{12} \\ 4a_{21} & 4a_{22} \end{pmatrix} \right\}.$$





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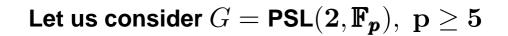
It holds:

$$\mathsf{PSL}(2,\mathbb{F}_5)\cong \mathsf{PSL}(2,\mathbb{F}_4)\cong A_5$$

Normalization - p. 23









Let us consider
$$G = \mathsf{PSL}(2, \mathbb{F}_p), \ p \ge 5$$

Consider the matrices

$$x = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 1 & b \\ c & 1 + bc \end{pmatrix}$$

$$x^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}$$
 implies $y \neq x^{-1}$ for all $(b, c, t) \in \mathbb{F}_p^3$.



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$$U_1(x,y) = U_2(x,y)$$
, i.e.
 $x^{-1}yx^{-1}y^{-1}x^2 = yx^{-2}y^{-1}xy^{-1}$

has a solution $(b, c, t) \in \mathbb{F}_p^3$.



The entries of $U_1(x, y) - U_2(x, y)$ are the following polynomials in $\mathbb{Z}[b, c, t]$ Let $I = \langle p_1, \dots, p_4 \rangle$ and $I^{(p)}$ the induced ideal over \mathbb{Z}/p :

$$p_{1} = b^{3}c^{2}t^{2} + b^{2}c^{2}t^{3} - b^{2}c^{2}t^{2} - bc^{2}t^{3} - b^{3}ct + b^{2}c^{2}t + b^{2}ct^{2} + 2bc^{2}t^{2} + bct^{3} + b^{2}c^{2} + b^{2}ct + bc^{2}t - bct^{2} - c^{2}t^{2} - ct^{3} - b^{2}t + bct + c^{2}t + ct^{2} + 2bc + c^{2} + bt + c^{2}t + ct^{2} + 2bc + c^{2} + bt + c^{2}t + ct^{2} + bt^{2}ct + ct^{2} + bt^{2}ct^{2} + bt^$$

$$p_{2} = -b^{3}ct^{2} - b^{2}ct^{3} + b^{2}c^{2}t + bc^{2}t^{2} + b^{3}t - b^{2}ct - 2bct^{2} - b^{2}c + bct$$
$$+c^{2}t + ct^{2} - bt - ct - b - c - 1$$

$$p_{3} = b^{3}c^{3}t^{2} + b^{2}c^{3}t^{3} - b^{2}c^{2}t^{3} - bc^{2}t^{4} - b^{3}c^{2}t + b^{2}c^{3}t + b^{2}c^{2}t^{2} + 2bc^{3}t^{2} + bc^{2}t^{2} + bc^{2}t^{2} + bc^{2}t^{2} - c^{2}t^{3} - ct^{4} - 2b^{2}ct + bc^{2}t + c^{3}t + bct^{2} + 2c^{2}t^{2} + ct^{3} - b^{2}c - b^{2}t + bct + c^{2}t + bt^{2} + 3ct^{2} + bc - bt - b - c + 1$$

$$p_{4} = -b^{3}c^{2}t^{2} - b^{2}c^{2}t^{3} + b^{2}c^{2}t^{2} + bc^{2}t^{3} + b^{3}ct - b^{2}c^{2}t - b^{2}ct^{2} - 2bc^{2}t^{2}$$
$$-bct^{3} - 2b^{2}ct + c^{2}t^{2} + ct^{3} + b^{2}t - bct - c^{2}t - ct^{2} + b^{2} - bt$$
$$-2ct - b - t + 1$$





Let $C \subseteq \mathbb{A}^n$ be an absolutely irreducible affine curve defined over the finite field \mathbb{F}_q and $\overline{C} \subset \mathbb{P}^n$ its projective closure \Rightarrow

 $\#C(\mathbb{F}_q) \ge q + 1 - 2p_a\sqrt{q} - d$

($d = \text{degree}, p_a = \text{arithmetic genus of } \overline{C}$).



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The Hilbert–polynomial of \overline{C} , $H(t) = d \cdot t - p_a + 1$, can be computed using the ideal I_h of \overline{C} : We obtain $H(t) = 10t - 11 \Rightarrow d = 10$, $p_a = 12$.



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Proposition: $V(I^{(p)})$ is absolutely irreducibel for all primes $p \ge 5$.



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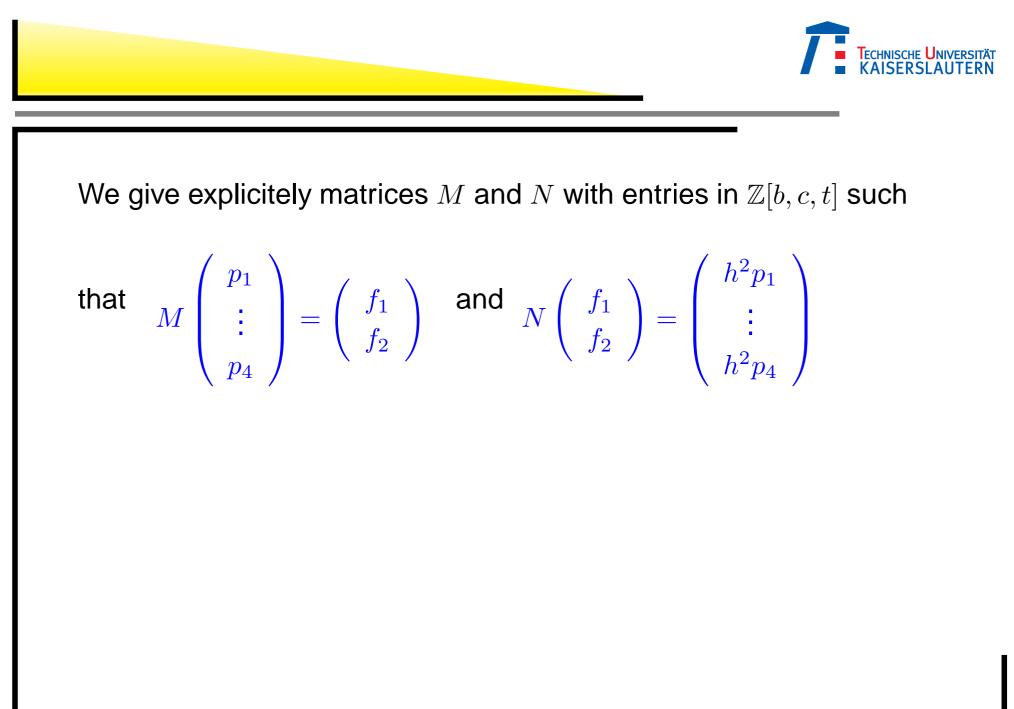
$$f_{1} = t^{2}b^{4} + (t^{4} - 2t^{3} - 2t^{2})b^{3} - (t^{5} - 2t^{4} - t^{2} - 2t - 1)b^{2}$$

$$-(t^{5} - 4t^{4} + t^{3} + 6t^{2} + 2t)b + (t^{4} - 4t^{3} + 2t^{2} + 4t + 1)$$

$$f_{2} = (t^{3} - 2t^{2} - t)c + t^{2}b^{3} + (t^{4} - 2t^{3} - 2t^{2})b^{2}$$

$$-(t^{5} - 2t^{4} - t^{2} - 2t - 1)b - (t^{5} - 4t^{4} + t^{3} + 6t^{2} + 2t)$$

$$h = t^{3} - 2t^{2} - t$$





We give explicitly matrices M and N with entries in $\mathbb{Z}[b, c, t]$ such

that
$$M\begin{pmatrix}p_1\\\vdots\\p_4\end{pmatrix} = \begin{pmatrix}f_1\\f_2\end{pmatrix}$$
 and $N\begin{pmatrix}f_1\\f_2\end{pmatrix} = \begin{pmatrix}h^2p_1\\\vdots\\h^2p_4\end{pmatrix}$

We obtain for all fields K

 $IK[b,c,t] = \left(\langle f_1, f_2 \rangle K[b,c,t] \right) : h^2.$





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geometrically:

Curve V(I) is irreducibel, if the projection to the b, t-plane is irreducibel.



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 $\begin{aligned} x^4 + (t^3 - 2t^2 - 2t)x^3 - (t^5 - 2t^4 - t^2 - 2t - 1)x^2 - \\ (t^6 - 4t^5 + t^4 + 6t^3 + 2t^2)x + (t^6 - 4t^5 + 2t^4 + 4t^3 + t^2). \end{aligned}$

We prove, that the induced polynomial $P \in \mathbb{F}_p[t, x]$ is absolutely irreducibel for all primes $p \ge 2$.

(Using the lemma of Gauß this is equivalent to P being irreducibel in $\overline{\mathbb{F}}_p(t)[x]$.)



Ansatz

(*)
$$P = (x^2 + ax + b)(x^2 + gx + d)$$

a, b, g, d polynomials in t with variable coefficients

 $a(i), \ b(i), \ g(i), \ d(i)$.



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The decomposition (*) with a(i), b(i), g(i), $d(i) \in \overline{\mathbb{F}}_p$ does not exist iff the ideal C generated by the coefficients with respect to x, t of $P - (x^2 + ax + b)(x^2 + gx + d)$ has no solution in $\overline{\mathbb{F}}_p$. This is equivalent to the fact that $1 \in \mathbb{C}$.



The ideal of the coefficients of C:

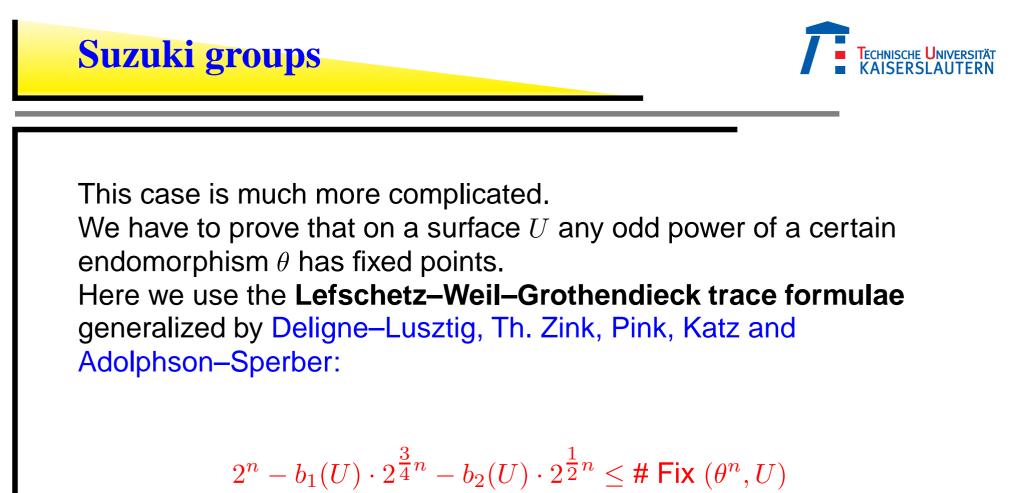
```
C[1] = -b(5) * d(3)
C[2] = -b(5) * g(2)
C[3] = -b(4) * d(3) - b(5) * d(2)
C[4] = -b(4)*g(2)-b(5)*g(1)-d(3)-1
C[5] = -b(3)*d(3)-b(4)*d(2)-b(5)*d(1)+1
C[6] = -b(5) - g(2) - 1
C[7] = a(0) * b(5) - a(2) * d(3) - b(3) * g(2) - b(4) * g(1) - d(2) + 4
C[8] = -a(0)^{2} + b(0) + b(0) + b(0) - b(2) + d(3) - b(3) + d(2) - b(4) + d(1) - b(5) - 4
C[9] = -a(2) * g(2) - b(4) - g(1) + 2
C[10] = a(0) * b(4) - a(1) * d(3) - a(2) * d(2) - b(2) * g(2) - b(3) * g(1) - d(1) - 1
C[11] = -a(0)^{2} + b(0) + b(0) + b(1) + d(3) - b(2) + d(2) - b(3) + d(1) - b(4) + 2
C[12]=a(0)-a(1)*g(2)-a(2)*g(1)-b(3)-d(3)
C[13] = -a(0)^{2}+a(0)*b(3)-a(0)*d(3)-a(1)*d(2)-a(2)*d(1)+b(0)-b(1)*g(2)-b(2)*g(1)-7
C[14] = -a(0)^{2} + b(3) + b(0) + b(3) - b(0) + d(3) - b(1) + d(2) - b(2) + d(1) - b(3) + 4
C[15] = -a(2) - g(2) - 2
C[16]=a(0)*a(2)-a(0)*g(2)-a(1)*g(1)-b(2)-d(2)+1
C[17] = -a(0)^{2}*a(2) + a(0)*b(2) - a(0)*d(2) - a(1)*d(1) + a(2)*b(0) - a(2) - b(0)*g(2) - b(1)*g(1) - 2
C[18] = -a(0)^{2}*b(2)+b(0)*b(2)-b(0)*d(2)-b(1)*d(1)-b(2)+1
C[19] = -a(1) - g(1) - 2
C[20]=a(0)*a(1)-a(0)*g(1)-b(1)-d(1)+2
C[21] = -a(0)^{2}*a(1)+a(0)*b(1)-a(0)*d(1)+a(1)*b(0)-a(1)-b(0)*g(1)
C[22] = -a(0)^{2} + b(1) + b(0) + b(1) - b(0) + d(1) - b(1)
C[23] = -a(0)^{3}+2*a(0)*b(0)-a(0)
C[24] = -a(0)^{2}b(0) + b(0)^{2}-b(0)
```



Using SINGULAR, one shows that over $\mathbb{Z}[\{a(i)\}, \{b(i)\}, \{g(i)\}, \{d(i)\}]$

$$4 = \sum_{i=1}^{24} M_i \, \operatorname{C}[i] \, .$$

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This case is much more complicated. We have to prove that on a surface U any odd power of a certain endomorphism θ has fixed points.	



for n sufficientely large.