Course "Toric Geometry and Applications"

José Ignacio Farrán Martín
ignfar@eis.uva.es
Department of Applied Mathematics
University of Valladolid – Campus Segovia
Escuela Universitaria de Informática

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Basic References

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Introduction and Overview (I)

- Toric Geometry is a topic of increasing interest
- Toric varieties are objects suitable for checking explicit properties and computing invariants from Algebraic Geometry
- This holds for normal toric varieties
  (coming from rational fans in a Euclidean space)
- Non-normal varieties also have interesting and nice applications
- Normal Toric Geometry uses techniques mainly from convex geometry
  - It is based on fans: set of polyhedral cones so that each cones provides an affine chart of the toric variety
  - Thus, the coordinate algebra of such a chart is the semigroup algebra of lattice points inside the dual cone

Introduction and Overview (II)

- In Non-Normal Toric Geometry we need more than cones
  - Consider affine charts whose coordinate algebra is given by a more general type of semigroup
  - Thus, convex geometry should be used just as a tool:
    nice semigroups generate concrete polyhedral cones
- The first part of the course is devoted to show how mathematics involved in Toric Geometry can be regarded as the theory os suitable classes of commutative semigroups with given generators
- This point of view requieres the description of derived ingredients, such as lattices, binomial ideals or polytopes
- Our purpose is to show the mathematical connections between all these ingredients and present some applications, mainly focusing to Integer Programming

Semigroups

- The central objects will be
  finitely generated cancellative commutative semigroups
  - A commutative semigroup S has an internal associative and commutative operation $+$ with a zero element 0
  - Semigroup homomorphisms preserve both the operation $+$ and the zero element 0
- Cancellative means S isomorphic to a subsemigroup of an abelian group. In other words ...
  - The semigroup homomorphism $S \to G(S)$ is injective
  - $G(S)$ denotes the abelian group generated by S, i.e.
    $S \times S$ modulo the relation

$$(m, n) \sim (m', n') \Leftrightarrow m + n' = m' + n$$

Semigroups

- We consider S with a finite number of generators ...
  - This means we can find a finite set $\{n_1, \ldots, n_h\}$
    so that every $m \in S$ can be written as

$$m = \lambda_1 n_1 + \cdots + \lambda_h n_h$$

    for some non-negative integers λ_i
  - This is equivalent to fix a surjective semigroup homomorphism

$$\pi_0 \ : \ \mathbb{N}^h \to S$$

- Toric Mathematics is essentially reduced to understanding of the structure and behavior of the fibers $\pi_0^{-1}(m)$
- This is a difficult problem that becomes the key tool for many purposes in Toric Geometry

Semigroups

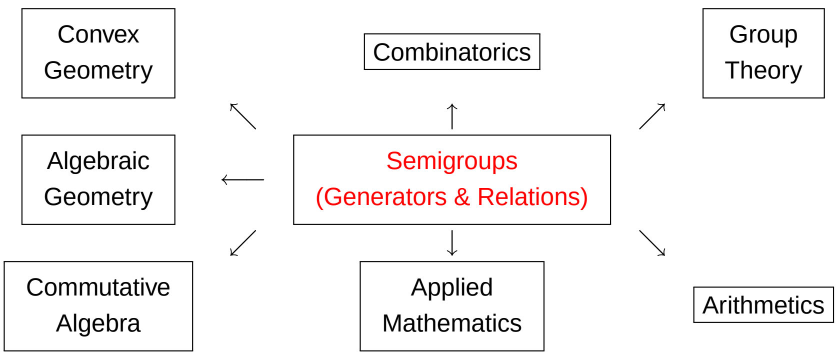
- The first remark is that it would be desirable some kind of finiteness hypothesis, namely all the fibers $\pi_0^{-1}(m)$ to be finite for all $m \in S$
  This finiteness hypothesis is characterized in the following result
- **PROPOSITION**: Let $\pi_0 : \mathbb{N}^h \longrightarrow S$ be a surjective semigroup homomorphism, S being a cancellative commutative semigroup
  The following conditions are equivalent:
  1. $\pi_0^{-1}(m)$ is finite for all $m \in S$
  2. There is no infinite sequence
     $m \in S, m_1, \ldots, m_i, \ldots \in S \setminus \{0\}$ such that
     $m - m_1 - \cdots - m_i \in S$ for all i
  3. $S \cap (-S) = \{0\}$
  4. There exists a semigroup homomorphism $\lambda : S \to \mathbb{N}$
     such that $\lambda(m) = 0$ if and only if $m = 0$

Semigroups

- Semigroups satisfying the condition of the previous result are called with different names in the literature, according to which condition is emphasized:
  1. Combinatorially finite: finite fibers
  2. Nakayama: The non-existence of those infinite sequences ...
  3. Strongly convex: $S \cap (-S) = \{0\}$
  4. Positive: Existence of $\lambda : S \to \mathbb{N}$ s.t. $\lambda(m) = 0$ iff $m = 0$
- Describing the fibers of π_0 is related to computing the relations between the fixed generators of S
- Since the "kernel" is not well-defined in the category of semigroups, these relations must be defined through the so-called "congruence" Γ of π_0 ...

Semigroups

- The congruence Γ of $\pi_0\ \mathbb{N}^h \longrightarrow S$ is the binary relation on $\mathbb{N}^h$ given by pairs $(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^h \times \mathbb{N}^h$ such that $\mathbf{u}$ and $\mathbf{v}$ lie on the same fiber $\pi_0^{-1}(m)$ for some $m \in S$
- Congruences give a semigroup structure on the quotient $\mathbb{N}^n/\Gamma$, since $(\mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}) \in \Gamma$ if $(\mathbf{u}, \mathbf{v}) \in \Gamma$ and $\mathbf{w} \in \mathbb{N}^h$
- Since S is finitely generated, so it is Γ as a congruence, i.e. $\Gamma = \Gamma(R)$ for a finite set R of relations
  In other words, we can say that S is finitely presented
- Toric Mathematics exploit the information in a semigroup concerning generators and relations
- This involves different fields in Mathematics, giving different perspectives and techniques . . .



Groups and Lattices

- Consider again the above map $\pi_0 : \mathbb{N}^h \to S$
- Since the assignment
$$S \mapsto G(S)$$
is functorial, the following exact sequence of abelian groups
$$0 \to L \to G(\mathbb{N}^h) \equiv \mathbb{Z}^h \to G(S) \to 0$$
is induced, where L is a subgroup of $\mathbb{Z}^h$ and so it is
finitely generated and torsion free
- L is the lattice associated to π_0
- It is just the kernel of the map $\pi : \mathbb{Z}^h \to G(S)$
associated to π_0
- Thus, L keeps the information about the group theoretical relations between the semigroup generators $n_1, \ldots, n_h$

Groups and Lattices

- The relation between the congruence Γ and the lattice L is a follows
  - If $(\mathbf{u}, \mathbf{v}) \in \Gamma$ there is a unique $\mathbf{w} \in \mathbb{N}^h$ such that $(\mathbf{u} - \mathbf{w}, \mathbf{v} - \mathbf{w}) \in \Gamma$ and the supports of $\mathbf{u} - \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ are disjoint, where the support of a vector in $\mathbb{Z}^h$ is defined by the set of indices whose coordinates are nonzero.
  - Notice that if $\leq$ denotes the componentwise product ordering on $\mathbb{Z}^h$ then
$$\mathbf{w} = \inf_{\leq}\{\mathbf{u}, \mathbf{v}\}$$
  - Thus, the map
$$\begin{array}{cccc} b & : & \Gamma & \to & \mathbb{N}^h \times L \\ & & (\mathbf{u}, \mathbf{v}) & \mapsto & (\mathbf{w}, \mathbf{u} - \mathbf{v}) \end{array}$$
is well-defined

Groups and Lattices

- **PROOF**POSITION: The map b is a bijection

  - **PROOF**: If $(\mathbf{w}, \mathbf{l}) \in \mathbb{N}^h \times L$ we set
  $$\mathbf{l} = \mathbf{l}^+ - \mathbf{l}^-$$
  where $\mathbf{l}^+ := \sup(\mathbf{l}, \mathbf{0})$ and $\mathbf{l}^- := \sup(-\mathbf{l}, \mathbf{0})$
  - Then, the assignment
  $$(\mathbf{w}, \mathbf{l}) \mapsto (\mathbf{l}^+ + \mathbf{w}, \mathbf{l}^- + \mathbf{w})$$
  is, by construction, the inverse of b

- It follows from the above Proposition that Γ and L contain the same information about S, and it tells how to get such information from one to the other

Groups and Lattices

- Moreover, from abelian groups and lattices we can obtain the semigroups we are interested in
- In fact, if $L \subseteq \mathbb{Z}^h$ is a lattice, from the exact sequence
$$0 \to L \to \mathbb{Z}^h \xrightarrow{\pi} \mathbb{Z}^h/L \to 0$$
we can consider the subsemigroup S of $\mathbb{Z}^h/L$ given by the image of the semigroup $\mathbb{N}^h$, and generators given by the images of the canonical basis $e_1, \ldots, e_h$
- Note that the condition of S being positive is equivalent to
$$L \cap \mathbb{N}^h = (\mathbf{0})$$

Groups and Lattices

- In general $G(S) = \mathbf{Z}^h/L$ may have torsion and so S
- It means that there may be $m, n \in S$, $m \neq n$ and $a \in \mathbf{Z}$ such that

$$am = an$$

- If $T \subseteq G(S)$ is the torsion subgroup, the image of S in $G(S)/T$ is a new semigroup $\overline{S}$ of the same type as S
- Notice that S is positive iff so is $\overline{S}$, since

$$L \cap \mathbf{N}^h = (\mathbf{0}) \Leftrightarrow \overline{L} \cap \mathbf{N}^h = (\mathbf{0})$$

$\overline{L}$ being the lattice for the induced map $\overline{\pi_0} : \mathbf{N}^h \to \overline{S}$

Groups and Lattices

- Notice that S is not only $\pi(\mathbf{N}^h)$, but also the image of other subsets of $\mathbf{Z}^h$, in particular of $\mathbf{N}^h + L$
- $\mathbf{N}^h + L$ is also a semigroup
- As a semigroup, it is NOT positive except for the trivial case $L = 0$
- However, if S is positive then $\mathbf{N}^h + L$ has a property analogous to the non-existence of infinite sequences, i.e.

$$m = m_0 > m_1 > \cdots > m_i > \ldots \in \mathbf{N}^h + L$$

- In other words, if S is positive then $\mathbf{N}^h + L$ is generated by its minimal elements with respect to the ordering $\leq$
- Such minimal elements are are just the primitive elements of the set $\mathbf{N}^h + L$, i.e. those which are not sum of a nonzero element of $\mathbf{N}^h$ and another one in $\mathbf{N}^h + L$

Semigroup Ideals and Algebras

- In this section, fix a commutative field k
- Define a functor from the category of semigroups to that of k-algebras, assigning to S its semigroup k-algebra
- For a semigroup S, the semigroup k-algebra $k[S]$ is defined as the k-vector space freely generated by the symbols χ^m for each $m \in S$, with a multiplication for symbols given by the rule

$$\chi^m \cdot \chi^n := \chi^{m+n} \qquad m, n \in S$$

- At the homomorphism level, the functor is defined in a natural way (Exercise)

Semigroup Ideals and Algebras

- Consider the map $\pi_0 : \mathbb{N}^h \longrightarrow S$ as in the previous section
- By applying the above functor, one gets the the exact sequence

$$0 \to I \to A = k[\mathbb{N}^h] \xrightarrow{\varphi_0} R = k[S] \to 0$$

where I is the kernel of the k-algebra homomorphism φ_0 associated to π_0

  - I is called the semigroup ideal relative to the generators $n_1, \ldots, n_h$
  - Note that if $X_1, \ldots, X_h$ are variables corresponding to the coordinates in $\mathbb{N}^h$, one has the canonical identification $A \equiv k[X_1, \ldots, X_h]$
  - Moreover, R and A are graded over S (S-graded) ...

Semigroup Ideals and Algebras

- We give degree
  - m to the symbol χ^m
  - n_i to the variable X_i
- We have a decomposition into homogeneous components

$$\begin{aligned} A &= \bigoplus_{m \in S} A_m \\ R &= \bigoplus_{m \in S} k \cdot \chi^m \end{aligned}$$

- A_m is the vector space generated by the monomials of degree m, i.e. $\mathbf{X}^u := X_1^{u_1} \cdots X_h^{u_h}$ with $\sum_{i=1}^{h} u_i n_i = m$

Semigroup Ideals and Algebras

- The homomorphism

$$A = k[\mathbb{N}^h] \xrightarrow{\varphi_0} R = k[S]$$

  becomes graded of degree zero
- Thus, the semigroup ideal I is S-homogeneous, i.e.

$$I = \bigoplus_{m \in S} I_m$$

  with $I_m = I \cap A_m$

Semigroup Ideals and Algebras

- Notice that R is generated, as a k-algebra, by the symbols

$$\chi^{n_1}, \ldots, \chi^{n_h}$$

so that I can be regarded as the ideal of polynomial relations among these symbols

- I is a binomial ideal, since it is generated by $\mathbf{X}^u - \mathbf{X}^v$ for $(\mathbf{u}, \mathbf{v}) \in \Gamma$
- By using the previous bijection b, I is also generated by $\mathbf{X}^{l^+} - \mathbf{X}^{l^-}$ for $\mathbf{l}$ in the lattice L
- Anyway, I is generated by a finite number of binomials, choosing $(\mathbf{u}, \mathbf{v}) \in \Gamma$ generating the congruence Γ

Semigroup Ideals and Algebras

- Assume now that S is positive
- First, by using $S \cap (-S) = \{0\}$ one has
  - $M_R := \bigoplus_{m \neq 0} k \cdot \chi^m$ is an ideal of R
  - $M_A := \bigoplus_{m \neq 0} A_m$ is an ideal of A
- Secondly, by using the finiteness of the fibers, each A_m is a vector space of finite dimension
- Third, by using Nakayama's lemma for S-graded modules, we can speak of minimal systems of generators for I, which are those inducing a basis of the vector space $I/(M_A I)$
- Obviously, we can consider minimal sets of binomial generators for I

Semigroup Ideals and Algebras

- In fact, we can consider S-graded free resolutions of R as an A-module
- If S is positive, Nakayama's lemma shows that one can take the minimal free resolution (unique up to isomorphism), i.e.

$$0 \to F_p \overset{\varphi_p}{\to} \cdots \to F_2 \overset{\varphi_2}{\to} F_1 \overset{\varphi_1}{\to} F_0 = A \overset{\varphi_0}{\to} R \to 0$$

  where

  - F_i is a free S-graded finite A-module
  - φ_i is S-graded of degree 0
  - p is the projective dimension of R, i.e. the largest integer p so that $F_p \neq 0$

Semigroup Ideals and Algebras

- The Auslander-Buchsbaum theorem shows the relation $p + r = h$, where r is the depth of R
- The integer r ranges over the values $0 \leq r \leq d$, d being the Krull dimension of $R = k[S]$
- Note that the Krull dimension of R equals to the rank of the abelian group $G(S)$
  - This fact comes from the computations of dimensions by means of transcendence degrees
  - It implies, in particular, that the dimension d of the k-algebra $k[S]$ does not depend on the base field k
  - This is not true for r, which may depend on the field k

Semigroup Ideals and Algebras

- Commutative Algebra provides interesting particular cases ...
  - If $r = d$ the ring $k[S]$ is called Cohen-Macaulay
  - If moreover F_p has rank 1 as an A-module, $k[S]$ is called Gorenstein
  - In this case the minimal resolution is self-dual, i.e. by applying the functor $\mathrm{Hom}(-, A)$ and considering the natural grading, the induced exact sequence

$$0 \to \mathrm{Hom}(F_0, A) \to \mathrm{Hom}(F_1, A) \to \cdots$$

$$\cdots \to \mathrm{Hom}(F_p, A) \to \mathrm{Coker}(\varphi_p^t) \to 0$$

    is S-graded isomorphic to the minimal resolution of R
  - These two properties depend only on S (and k) but not on the map π_0

Semigroup Ideals and Algebras

- Commutative Algebra provides interesting particular cases ...
  - Finally, if I can be generated by exactly $h - d$ homogeneous polynomials (actually binomials) $k[S]$ is called “complete intersection”
  - Equivalently, complete intersection means that the congruence Γ can be generated by exactly $h - d$ pairs
  - The complete intersection property only depends on the semigroup S (not on the map π_0, but not even on the field k), and implies in particular the Gorenstein property

Cones and Fans

- The next object we can associate to a semigroup S is a cone
  - The cone $C(S)$ generated by S is the cone generated by the image of S in the $\mathbf{Q}$-vector space $V_{\mathbf{Q}} := G(S) \otimes_{\mathbf{Z}} \mathbf{Q}$
  - Since the base ring extension from $\mathbf{Z}$ to $\mathbf{Q}$ kills the torsion, this cone coincides with that of $\overline{S}$ in $G(S)/T$
- If S is not positive, then $C(S)$ is equal to the whole $V_{\mathbf{Q}}$, so that it contains trivial information, and the interesting case is S positive
  - S is positive if and only if $C(S)$ is a strongly convex cone:

$$C(S) \cap -C(S) = 0$$

  - This justifies the terminology “strongly convex semigroup”

Cones and Fans

- Considering generators for S, the cone $C(S)$ is the rational polyhedral cone generated by the images of the generators $n_1, \ldots, n_h$ in $V_{\mathbf{Q}}$
  - That is the cone generated by the convex hull of such images
  - Thus, Convex Geometry becomes a useful technique in Toric Mathematics
- There is a very important case where the cone $C(S)$ determines the semigroup S itself . . .

Cones and Fans

- S is said to be normal if it is torsion free and moreover

$$S = C(S) \cap G(S)$$

  - It is well-known that S is a normal semigroup iff $k[S]$ is an integrally closed domain (i.e. a normal ring)
  - Hochster's theorem shows that S normal actually implies $k[S]$ Cohen-Macaulay

- A trivial example of normal semigroups are free semigroups, i.e. isomorphic to $\mathbb{N}^t$ for some integer t

  - In fact, free semigroups are the only ones such that $k[S]$ is a regular ring

Cones and Fans

- The terminology regular is coherent with that used in Convex Geometry

  - A cone in $\mathbf{Q}^t$ is regular if it is generated by a basis of the lattice $\mathbb{Z}^t$
  - A semigroup S is free if and only if

    1. S is normal, and
    2. $C(S)$ is regular

- Toric Geometry appears initially as the study of normal toric varieties

  - In this way, normal toric varieties are based on Convex Geometry, and Toric Mathematics is equivalent to that of Convex Geometry

Cones and Fans

- Coming back to the general case, the convex cone $C(S)$ provides new interesting invariants for a semigroup S, vgr. the number of edges e of that cone
  - Comparing with the dimension d we have
$$e \geq d$$
    and equality holds if $C(S)$ is a simplicial cone
  - If $e = d$ the semigroups will be called simplicial
  - Free semigroups are a very special case of simplicial semigroups

Cones and Fans

- Toric varieties may be affine or not
- Affine toric varieties are nothing but affine varieties X with coordinate k-algebra $k[X]$ equal to $k[S]$ for some semigroup S, according to the previous terminology
- General toric varieties are algebraic varieties that can be covered by affine toric varieties with overlappings which are also affine toric varieties
- Projective toric varieties are a particular case
- Normal toric varieties are usually given in terms of Convex Geometry . . .

Cones and Fans

- This is not the usual definition of toric variety, but it is equivalent ...
- A toric variety over the field k is an irreducible variety V such that

  1. The algebraic torus $(k^*)^n$ is a Zariski open subset of V
  2. The action of $(k^*)^n$ on itself extends to an action of $(k^*)^n$ on the whole V

- Examples: $(k^*)^n$, k^n and $\mathbb{P}^n(k)$ are toric varieties

  - Property (1) is obvious; the extended action for the projective case is given by

$$(t_1, \ldots, t_n) \cdot (a_0 : a_1 : \ldots : a_n) = (a_0 : t_1 a_1 : \ldots : t_n a_n)$$

  under the identification $(t_1, \ldots, t_n) \equiv (1 : t_1 : \ldots : t_n)$

Cones and Fans

- Another example: Consider the cuspidal plane cubic $C = V(y^2 - x^3) \subseteq \mathbf{C}^2$

  1. C contains the torus k^* via the map $t \mapsto (t^2, t^3)$
  2. k^* acts on C via the map $t \cdot (u, v) = (t^2 u, t^3 v)$

- This is a non normal variety of dimension one, but gives the idea of the connection between toric varieties and semigroups ...

Cones and Fans

- If we have a semigroup $S \subseteq \mathbf{Z}^d$ generated by $\mathbf{m}_1, \ldots, \mathbf{m}_h$, we obtain a (normal) toric variety as follows:
  - Consider the map $\varphi : (k^*)^d \to k^h$ given by $\varphi(t_1, \ldots, t_d) = (\mathbf{t}^{\mathbf{m}_1}, \ldots, \mathbf{t}^{\mathbf{m}_h})$
  - The affine toric variety is the Zariski closure V of the image of φ
  - φ is injective, so that the torus can be considered as an open set of V
  - The extended action on V is left as an exercise

Cones and Fans

- Another example is $V = V(xy - zw)$ in $\mathbf{C}^4$
  - It is a 3-dimensional toric variety containing the torus $(\mathbf{C}^*)^3$ via

$$(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$$

  - The generators of the semigroup are

$$\begin{array}{ll} \mathbf{m}_1 = (1,0,0) & \mathbf{m}_2 = (0,1,0) \\ \mathbf{m}_3 = (0,0,1) & \mathbf{m}_4 = (1,1,-1) \end{array}$$

  - The above map corresponds to

$$(t_1, t_2, t_3) \mapsto (\mathbf{t}^{\mathbf{m}_1}, \mathbf{t}^{\mathbf{m}_2}, \mathbf{t}^{\mathbf{m}_3}, \mathbf{t}^{\mathbf{m}_4}) = (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$$

  - The implicit equation comes from the equality $w = t_1 t_2 t_3^{-1}$

Cones and Fans

- The given data for a normal variety consist of a fan Φ of rational polyhedral cones in $\mathbf{Q}^n$, i.e.
  - A finite set $\Phi = \{\sigma\}$ where each σ is a strongly convex polyhedral cone in $\mathbf{Q}^n$
  - The faces of each $\sigma \in \Phi$ is also a are also in the set Φ
  - The intersection of any two cones in Φ is a common face of both of them

Cones and Fans

- The normal variety is constructed in the following way
  - For each $\sigma \in \Phi$ consider S_σ the semigroup of integer coordinate points lying inside the dual cone of σ

$$\sigma^\vee := \{\mathbf{m} \in \mathbf{Q}^h \,|\, \langle \mathbf{m}, \mathbf{u} \rangle \geq 0, \;\; \forall \mathbf{u} \in \sigma\}$$

  - Let X_σ be the affine toric variety given by S_σ
  - The variety X is the join of such X_σ's, and $X_\sigma \cap X_\tau = X_{\sigma\cap\tau}$

Cones and Fans

- In this way, for a normal toric variety the fan Φ not only determines the variety, but also shows its geometry
  - In fact, cones correspond to affine charts in such a way that intersections of cones correspond to overlappings of the corresponding charts
- For non normal toric varieties one can proceed in a similar way, but taking a further precision on the semigroups
  - Together with the fan Φ we need, for each σ, a subsemigroup S'_σ of S_σ so that the intersection of any two such charts with coordinate algebras $k[S'_\sigma]$ and $k[S'_\tau]$ is the affine chart given by $k[S'_{\sigma\cap\tau}]$
  - This involves not only Convex Geometry, but also finitely generated cancellative semigroups

Cones and Fans

- The support of a fan Φ is defined by the union of the supports of its cones.
- The fan is called complete if its support is $\mathbf{Q}^n$
- Complete toric varieties are those built from complete fans
- The next section is devoted to projective toric varieties, which are a subclass of the complete toric case

Toric Varieties

- Toric varieties (affine and projective, in particular) are algebraic varieties, so that Algebraic Geometry is a natural frame for Toric Mathematics
- When the data $\pi_0 : \mathbb{N}^h \to S$ is given, the semigroup S leads to the (abstract affine) toric variety $X = Spec(k[S])$
- The choice of generators provided by π_0 corresponds to an embedding of X into some affine space $\mathbf{A}^h$
- The dimension of the variety X is just the rank d of $G(S)$
- We describe later how abstract and embedded projective toric varieties can also be described in nice terms

Toric Varieties

- Let S be a finitely generated cancellative commutative semigroup
- Assume that S is provided with a semigroup morphism $\lambda : S \to \mathbb{N}$ such that S is generated by the elements in the set $S_1 := \lambda^{-1}(1)$
  - The elements of S_1 are irreducible, a posteriori
  - Then for any choice of field k, the couple (S, λ) gives rise to an abstract $(d-1)$-dimensional projective algebraic scheme, namely $Z = Proj(k[S])$, where $k[S]$ is now regarded as an $\mathbb{N}$-graded algebra by relaxing its natural S-grading via the map λ, i.e. homogeneous elements of degree $i \in \mathbb{N}$ are the sum of homogeneous elements of S-degrees in $\lambda^{-1}(i)$
  - Couples (S, λ) are referred as polarized semigroups

Toric Varieties

- For a polarized semigroup (S, λ), $m \in S$ is a sum of $i \geq 0$ elements of S_1 iff $\lambda(m) = i$
- This fact has two consequences:
  - The set S_1 is finite (and so any fiber $\lambda^{-1}(i)$), since S_1 is nothing but the set of irreducible elements in S
  - S is positive (a posteriori), since $\lambda^{-1}(0) = 0$

Toric Varieties

- **PROPOSITION**: Let (S, λ) be a polarized semigroup such that S is torsion free. Then the projective algebraic scheme $Z = Proj(k[S])$ is a projective toric variety
  - **Proof (sketch)**: Notice first that $k[S]$ is a domain
  - Since S is torsion free, it can be viewed as a subset of $V_{\mathbb{Q}}$
  - On the other hand, the map λ can be extended to a group homomorphism

$$\lambda_{\mathbb{Z}} \ : \ G(S) \to \mathbb{Z}$$

and to an $\mathbb{R}$-linear map

$$\lambda_{\mathbb{R}} \ : \ V_{\mathbb{R}} \to \mathbb{R}$$

where $V_{\mathbb{R}} := G(S) \oplus_{\mathbb{Z}} \mathbb{R}$

Toric Varieties

- **PROPOSITION**: Let (S, λ) be a polarized semigroup such that S is torsion free. Then the projective algebraic scheme $Z = Proj(k[S])$ is a projective toric variety

  - **Proof (continued)**: Let Ω_1 be the convex hull of S_1 in $V_{\mathbf{R}}$
  - Let $S_1^0 \subset S_1$ be the set of vertices of S_1
  - Notice that S_1^0, S_1 and Ω_1 lie on the affine hyperplane in $V_{\mathbf{R}}$ given by $\lambda_{\mathbf{R}}^{-1}(1)$
  - Fix $m^0 \in S_1^0$. Then the semigroup $S(m^0)$ generated by the set of elements of the type $m - m^0$ with $m \in S_1$ is a new finitely generated semigroup whose associated group is $\lambda_{\mathbf{Z}}^{-1}(0)$

Toric Varieties

- **PROPOSITION**: Let (S, λ) be a polarized semigroup such that S is torsion free. Then the projective algebraic scheme $Z = Proj(k[S])$ is a projective toric variety

  - **Proof (end)**: It follows that the dimension of the affine toric variety $X(m^0)$ given by the semigroup $S(m^0)$ is $d - 1$, where
  $$d = rank\, G(S)$$
  i.e. the dimension of the projective variety Z
  - Moreover, since $X = Spec(k[S])$ is the projecting cone of Z, the construction shows that the affine toric varieties
  $$X(m^0) \text{ for } m^0 \in S_1^0$$
  is a covering of Z by affine charts, i.e. Z is a projective variety
  $\square$

Toric Varieties

- For projective normal varieties it is possible to describe conditions for Cartier divisors to be ample or very ample
- A polarization of a projective variety means to pick a very ample Cartier divisor class
  - It provides an embedding of the variety in a projective space
  - When the variety is toric, the polarization gives rise to a polarized semigroup (S, λ), so that the variety is isomorphic to that given by the couple (S, λ) [see Fulton]
- Thus, it is equivalent to give an embedded projective toric variety and a polarized semigroup

Toric Varieties

- Notice that, for a given polarized semigroup (S, λ), the set S_1 is the set of irreducible elements of S, so that there is no other generator system contained in S_1 giving the embedding of the affine toric variety $X = Spec(k[S])$, which is the projecting cone of Z
- Thus, a polarized semigroup provides a canonical embedding of the projective toric variety into $\mathbb{P}^{h-1}$, where $h := \sharp(S_1)$
- We remark that the fan producing the projective variety Z lies in the dual space of the hyperplane $\lambda_{\mathbf{Q}}^{-1}(0)$
  - That is, the cones of the fan are exactly the duals of the cones generated by the semigroups $S(m^0)$
  - By construction, it is easy to see that such a fan is a complete fan corresponding to the algebraic geometric fact that “every projective variety is complete”

Toric Varieties

- In the same way as for affine toric varieties, the main algebraic geometric characteristics of projective toric varieties can be described in terms of the polarized semigroup (S, λ)
  - $Z = Proj(k[S])$ is said to be arithmetically Cohen-Macaulay (resp. Gorenstein) if $k[S]$ is Cohen-Macaulay (resp. Gorenstein)
  - Z is projectively normal if $k[S]$ is normal (i.e. the semigroup S is normal)
  - Finally, Z is normal (resp. regular) if every semigroup $S(m^0)$ is normal (resp. free), for all $m^0 \in S_1^0$
- Notice that projectively normal means $S_i = \overline{S_i}$, where $S_i := \lambda^{-1}(i)$ and $\overline{S_i} := C(S) \cap S_i$, for all i

Toric Varieties

- Normalness can be characterized in a similar way in terms of the Ehrhart and Hilbert functions
  - The Ehrhart function is the map $E : \mathbb{N} \longrightarrow \mathbb{N}$ given by $E(i) := \sharp(\overline{S_i})$
  - The Hilbert function is the map $H : \mathbb{N} \longrightarrow \mathbb{N}$ given by $H(i) := \sharp(S_i)$
  - Both coincide with a polynomial map of degree $d - 1$ and coefficients in $\mathbf{Q}$ for $i >> 0$
- Under very general conditions, the leading terms of both "polynomials" E and H are equal, and the projective variety Z is normal exactly when both polynomials are the same
- In the same way, projective normalness is characterized by the equality of the functions $E = H$

Polytopes and Complexes

- If we have an embedded affine or projective toric variety, we would like to describe or compute (if possible) equations and syzygies for such embedding
  - Most results use combinatorial techniques, such as simplicial and cellular complexes or polytopes
- Note that the projective case is reduced to the affine one, since from a polarized semigroup, the equations and syzygies of the projective variety defined by such a semigroup are the same as those of its projecting cone affine variety
  - Such an affine variety is exactly the affine toric variety given by the semigroup S of the polarization with S_1 as chosen system of generators

Polytopes and Complexes

- In the sequel, consider the map $\pi_0 : \mathbb{N}^h \to S$
  where S is a positive semigroup
  - Λ is the generator system given by π_0
  - Π is the set of primitive elements of $M := \mathbb{N}^h + L$
  - For every $m \in S$, Υ_m is the set of monomials of S-degree equal to m
  - Note that Υ_m can be identified to the fiber $\pi_0^{-1}(m)$
  - Remember that S positive implies M generated by Π and each Υ_m is finite
  - There are several combinatorial objects with vertex set equal to either Λ, either Π or Υ_m, which are naturally associated to π_0 ...

Polytopes and Complexes

- Associated to every fixed $m \in S$ we have the simplicial complexes Δ_m and Θ_m, and the polytope Ω_m, defined as follows

  1. Δ_m is the simplicial subcomplex of parts F of Λ such that $m - n_F \in S$, where $n_F := \sum_{n \in F} n$
  2. Θ_m is the simplicial subcomplex of parts G of Υ_m such that all the monomials of G have a non unit GCD
     (i.e. monomials sharing at least one variable)
  3. Ω_m is the polytope in $V_{\mathbb{R}} := \mathbb{Z}^h \otimes_{\mathbb{Z}} \mathbb{R}$ given by the convex hull of the set $\Upsilon_m = \pi_0^{-1}(m)$

- We can consider on S the ordering defined by $m' \preceq m$ iff $m - m' \in S$
- If $m' \preceq m$ then $\Delta_{m'} \subseteq \Delta_m$, and the translation of $\Omega_{m'}$ by any vector in the fiber $\pi_0^{-1}(m - m')$ is a subset of Ω_m

Polytopes and Complexes

- Associated to S as a whole, one has two useful regular cellular subcomplexes of parts of Π, namely ...
  - The so-called Taylor complex Ξ, that is the (simplicial) complex of all parts of Π
  - The hull complex Σ, whose faces are the subsets of Π corresponding to some bounded face of the convex hull defined by the set of points of $V_{\mathbb{R}}$ of type $t^a = (t^{a_1}, \ldots, t^{a_h})$ for $a = (a_1, \ldots, a_h) \in M$ where $t >> 0$ is a positive real number
    - The mentioned correspondence is the obvious one, by taking into account that any vertex of the above convex hull is necessarily one of type t^b with $b \in \Pi$

Polytopes and Complexes

- Sometimes a subcomplex of Σ is considered, namely the so-called Scarf complex, which is defined as a simplicial complex to be the set of parts H of Π satisfying the property $\mathbf{a}_H \neq \mathbf{a}_{H'}$ for every $H \neq H'$, where $\mathbf{a}_H$ stands for the supremum of the elements in H for the ordering $\leq$ (componentwise product ordering in $\mathbf{Z}^h$)
  - The hull and the Scarf complexes coincide when the data π_0 is "generic", i.e. the congruence Γ can be generated by couples $(\mathbf{u}, \mathbf{v})$ such that the union of the supports of $\mathbf{u}$ and $\mathbf{v}$ is the set $\{1, 2, \ldots, h\}$
- In the sequel, we will often use reduced homology with values in the field k for simplicial and cellular complexes
  - The corresponding i-th reduced homology vector spaces will be denoted by $\tilde{H}_i$

Polytopes and Complexes

- Equations have to do, in practice, with computing sets of binomial generators of the semigroup ideal I
- In fact, we would have to compute either a minimal set of generators or a Gröbner basis
  - For each monomial ordering, i.e. total order on the set of monomials for which the monomial 1 is the minimum and which is compatible with multiplication by monomials, one has a well defined reduced Gröbner basis with respect to such an ordering
  - In our situation, this Gröbner basis consists of binomials
  - Thus, such a reduced Gröbner basis can be regarded as a subset of either the congruence Γ or the lattice L

Polytopes and Complexes

- The union of the reduced Gröbner bases for all possible (monomial) orderings is called a universal Gröbner basis, and it has the property of being simultaneously a Gröbner basis of I for all orderings
- Again a universal Gröbner basis of binomials can be seen as a subset of either Γ or L
- A reduced Gröbner basis w.r.t. a concrete ordering can be computed from any other generator system by means of the well known Buchberger algorithm
  - The computation of a universal Gröbner basis becomes much more difficult ...

Polytopes and Complexes

- Consider the subset U of S consisting of those elements $m \in S$ such that the polytope Ω_m has an edge which is not parallel to some edge of a certain $\Omega_{m'}$, for some $m' \preceq m$
- Then, for each $m \in U$ consider the binomials of type $X^{\mathbf{u}} - X^{\mathbf{v}}$, where the coordinates of $\mathbf{u} - \mathbf{v}$ are relatively prime and the segment $[\mathbf{u}, \mathbf{v}]$ is an edge of Ω_m
  - A result by Sturmfels, Weismantel and Ziegler shows that the set of all such binomials (m ranging over U) is precisely a universal Gröbner basis of the ideal semigroup I
  - Such universal basis is finite, since one can see that it is contained in the so-called Graver basis, which is actually finite
  - The Graver basis consists of the binomials corresponding to the primitive elements of the lattice L, i.e. those elements $\mathbf{l} = \mathbf{l}^+ - \mathbf{l}^-$ in L for which there is no other $\mathbf{l}' = \mathbf{l}'^+ - \mathbf{l}'^-$ in L such that $\mathbf{l} \neq \mathbf{l}'$, $\mathbf{l}'^+ \leq \mathbf{l}^+$ and $\mathbf{l}'^- \leq \mathbf{l}^-$

Polytopes and Complexes

- In order to find minimal sets of homogeneous generators for I one can proceed as follows . . .
  - Consider the set C of elements $m \in S$ s.t. $\tilde{H}_0(\Theta_m) \neq 0$
    i.e. those elements whose complex Θ_m is not connected
  - The set C is finite
  - For each $m \in C$ pick a monomial $X^{\mathbf{u}}$ in each connected component of Θ_m and distinguish the monomial $X^{\mathbf{v}}$ picked for one concrete component
  - Then, the binomials $X^{\mathbf{u}} - X^{\mathbf{v}}$, where $X^{\mathbf{u}}$ ranges over the picked monomials for the other components, are the degree m terms of a minimal system of homogeneous generators of I
  - Thus, when m ranges over the set C, the whole set of binomials obtained in this way is a minimal set of homogeneous generators for I

Polytopes and Complexes

- An alternative way to find homogeneous generators for I, involving the complexes Δ_m, is available for higher order syzygies, and it will be discussed below
  - Maybe it is also possible the same discussion for Θ_m ?
  - Can Θ_m or Δ_m be used to find a universal Gröbner basis ?
- The description of syzygies consists of obtaining either the minimal S-graded resolution, or concrete resolutions with special properties
  - For example, the property of preserving the symmetries relative to the action of the lattice L

Polytopes and Complexes

- Under the above notations

$$0 \to F_p \overset{\varphi_p}{\to} \cdots \to F_2 \overset{\varphi_2}{\to} F_1 \overset{\varphi_1}{\to} F_0 = A \overset{\varphi_0}{\to} R \to 0$$

the i-th order syzygy module is the S-graded module $N_i := ker(\varphi_i)$

  - Notice that $N_0 = I$
  - For each degree $m \in S$ the number of generators of degree m in any minimal set of generators for N_i is equal to the dimension of the k-vector space

$$V_i(m) := (N_i)_m / (M_A N_i)_m$$

(Nakayama's lemma)

Polytopes and Complexes

- A first and key connection between syzygies and toric geometry is a result by Hochster which states that one has an explicit and natural vector space identification

$$V_i \equiv \tilde{H}_i(\Delta_m)$$

$\tilde{H}_i$ means reduced simplicial homology with coefficients in k

- Moreover, such an isomorphism can be explicitly computed for direct and inverse images
  - This results illustrates how combinatorics play a natural role for describing syzygies, and thus one should include combinatorics in the useful techniques in Toric Mathematics
  - As an application, there is an effective algorithm to compute minimal sets of binomial generators for I (Briales, Campillo, Marijuán, Pisón)

Polytopes and Complexes

- The main involved computational problems are:

  1. Find the values $m \in S$ such that $\tilde{H}_i(\Delta_m) \neq 0$
  2. Compute the homology

- If one is able to solve both problems, since the above isomorphisms are explicit one can successively construct minimal sets of generators for the syzygy modules in the minimal resolution of R

  - Concerning the first problem, Briales, Pisón and Vigneron determine suitable finite subsets C_i of S with the property
  $$m \notin C_i \;\Rightarrow\; \tilde{H}_i(\Delta_m) \neq 0$$
  obtaining an algorithm for computing the minimal resolution

Polytopes and Complexes

- The second problem is better understood from a computational point of view, since concrete homologies can be calculated by means of both linear algebra and integer linear programming

- However, also toric geometry itself helps to solve integer programming, so that it is better to try a better understanding of the explicit structure of the homologies $\tilde{H}(\Delta_m)$
(Campillo and Gimenez)

  - Consider a partition $\Lambda = \mathcal{E} \cup \mathcal{C}$ where $\mathcal{E}$ is a subset of generators whose image in $V_{\mathbf{Q}}$ minimally generates the cone $C(S)$
  - This means: for each edge of $C(S)$, $\mathcal{E}$ contains exactly one element whose image generates such an edge
  - Note that $e = \sharp(\mathcal{E})$ equals to the number of edges of $C(S)$, that is an invariant of S

Polytopes and Complexes

- From an algebraic viewpoint, $k[S]$ becomes a finite extension of $k[\mathcal{E}]$
- Thus, the minimal graded resolution of $k[S]$ as an A-module can be compared to its minimal resolution as a B-module
  - Remember that $A := k[X_1, \ldots, X_h]$
  - Now $B := k[\mathbb{N}^e]$ and corresponds to the semigroup generated by the set $\mathcal{E}$
- This puts in evidence two kind of objects . . .

Polytopes and Complexes

- First is the Apéry set Q relative to $\mathcal{E}$
  - Q is the set of elements $q \in S$ such that $q - n \notin S$ for all $n \in \mathcal{E}$
  - In other words, Q is the set of exponents whose corresponding symbols minimally generate $k[S]$ as a $k[\mathcal{E}]$-module
  - In particular, Q is finite
- Secondly, for each $m \in S$ one has the analog of Δ_m for this situation, namely the simplicial subcomplex $\mathcal{T}_m$ of parts J of $\mathcal{E}$ such that $m - n_J \in S$
  - One can see that $dim\, \tilde{H}_i(\mathcal{T}_m)$ is exactly the number of degree m elements in a minimal set of $\mathcal{E}$-homogeneous generators of the i-th order syzygy module in the minimal resolution of $k[S]$ as a B-module

Polytopes and Complexes

- Now for a fixed $m \in S$ one has a key long exact sequence of type

$$\cdots \to H_{i+1}(Q_m) \to K_i \to \tilde{H}_i(\Delta_m) \to H_i(Q_m) \to K_{i-1} \to \cdots$$

  - $H_*(Q_m)$ is the homology of a complex associated to the vertex m of a graph $\mathcal{G}_Q$ (Apéry Graph) with colored edges constructed from the knowledge of Q and $\mathcal{C}$ as color set
  - The vertex set of $\mathcal{G}_Q$ consists of the elements m of type $q + n_I$ where $q \in Q$ and $I \subseteq \mathcal{C}$
  - Edges of color $n \in \mathcal{C}$ joint a vertex m' with another m provided $m - m' = n$
  - The complex associated to m has as i-th chain space the one freely generated by the subsets $I \subseteq \mathcal{C}$ of cardinality $i + 1$ such that $m - n_I \in Q$
  - The boundary map is the projection of the usual simplicial one

Polytopes and Complexes

- Now for a fixed $m \in S$ one has a key long exact sequence of type

$$\cdots \to H_{i+1}(Q_m) \to K_i \to \tilde{H}_i(\Delta_m) \to H_i(Q_m) \to K_{i-1} \to \cdots$$

  - The vector spaces K_* are much more complicate to describe in detail, but they can be stepwise computed in two alternative ways
    - One way is in terms of graphs similar to Apéry's but with some sets different from Q
    - Another in terms of homologies of type $\tilde{H}_*(\mathcal{T}_{m'})$ where $m' = m - n_I$ with $I \subseteq \mathcal{C}$
  - See [Campillo,Gimenez] for further details

Polytopes and Complexes

- As an application of the complexes $\mathcal{T}_m$ we get a characterization for the depth r of the ring $k[S]$
- Recall that the integers $r \leq d \leq e$ are associated to a positive semigroup S
  - d and e are easily obtained from S
  - To obtain r, we have that for an integer $1 \leq r_0 \leq d$, the inequality $r \geq r_0$ is equivalent to $\tilde{H}_{e-r_0}(\mathcal{T}_m) = 0$ for all $m \in S$
  - In particular, for $r_0 = d$, the Cohen-Macaulay property is characterized by

$$\tilde{H}_{e-d}(\mathcal{T}_m) = 0, \ \ \forall m \in S$$

  - For the simplicial case $e = d$, this means that all the complexes $\mathcal{T}_m$ are connected

Polytopes and Complexes

- From this fact, we easily recover the well known characterization for simplicial semigroups that the Cohen-Macaulay property is equivalent to

$$m \in G(S), \ \ n, n' \in \mathcal{E}, \ \ n \neq n', \ \ m + n \in S, \ \ m + n' \in S \Rightarrow m \in S$$

  - There are other characterizations of the Cohen-Macaulay property for non-simplicial cases in the literature

Polytopes and Complexes

- A standard application of the above techniques (long exact sequences) is for the case of simplicial Cohen-Macaulay semigroups ($r = d = e$)
  - In this case, since the complexes $\mathcal{T}_m$ are connected one can deduce that $K_i = 0$ for all i and hence from
$$\cdots \to H_{i+1}(Q_m) \to K_i \to \tilde{H}_i(\Delta_m) \to H_i(Q_m) \to K_{i-1} \to \cdots$$
    we get $\quad \tilde{H}_i(\Delta_m) \equiv H_i(Q_m) \quad$ for every m and i
  - Thus, the minimal resolution for the simplicial Cohen-Macaulay case can be derived from the combinatorial object $\mathcal{G}_Q$ (Apéry graph)
- For the general case, there are other ways to derive free resolutions for R from a unique combinatorial object, namely either the Taylor Ξ or the hull Σ complexes

Multinumerical Semigroups

- In practice, the "toric data" π_0 (S with given generators) is often described in arithmetical terms
- In fact the commutative finitely generated group $G(S)$ is
$$G(S) \cong \mathbf{Z}^d \times \mathbf{Z}/(g_1) \times \cdots \times \mathbf{Z}/(g_l)$$
  for some integers d, l and g_i's
  - If such an isomorphism is given, π_0 becomes equivalent to the specification of the coordinate $(d + l)$-tuples (in the above group product) of the generators $n_1, \ldots, n_h$ of S
  - A semigroup given by such a specification is called a multinumerical semigroup
  - For the simplest case $d = 1$ and $l = 0$, they are referred in the literature as numerical semigroups

Multinumerical Semigroups

- The purpose is then to study toric varieties from arithmetics of multinumerical semigroups, i.e. deduce geometrical properties of toric varieties from arithmetic properties of the generators of S as tuples in the product description of $G(S)$
  - ☐ This becomes difficult, and remains as an open problem except for a few special cases
  - ☐ The reason lies on the connections between combinatorics and toric geometry
  - ☐ In fact, by using polytopes, simplicial or cellular complexes, one avoids the dealing with delicate relations among numbers

Multinumerical Semigroups

- However, provided combinatorial techniques have produced nice results, one can hope to interpret them in the framework of arithmetics
  - ☐ This has been done for concrete cases (affine and projective toric curves, or affine and simplicial projective toric surfaces)
  - ☐ There are also nice results for more general cases
  - ☐ To illustrate this strategy, we explicit now some results for toric curves

Multinumerical Semigroups

- An affine toric curve is given by a numerical semigroup S, generated by a set Λ of h non-negative integers
- Obviously $r = d = 1$ and, since the cone $C(S)$ has an only edge, also $e = 1$ so that this is a simplicial Cohen-Macaulay case
- Fix a partition $\Lambda = \mathcal{E} \cup \mathcal{C}$ where $\mathcal{E}$ has a single element $s \in S$ and $\mathcal{C}$ contains the remaining $h-1$ elements
- Consider the Apéry set Q consisting of those integers in $q \in S$ such that $q - s \notin S$, and construct the colored graph $\mathcal{G}_Q$
  - It is not difficult to translate the graph structure into arithmetical relations, so that the homologies $\tilde{H}_i(\Delta_m) = H_i(Q_m)$ for the vertices $m \in \mathcal{G}_Q$ can be derived from such relations
  - The conclusion is that the minimal resolution for affine toric curves can be computed just in arithmetical terms from the generators of the numerical semigroup S

Multinumerical Semigroups

- A projective toric curve of degree s is given by a subsemigroup S of $\mathbb{N}^2$ generated by a set $\Lambda = \mathcal{E} \cup \mathcal{C}$, where
  - $\mathcal{E}$ consists of the two elements $(s, 0)$ and $(0, s)$
  - $\mathcal{C}$ consists of the elements $(c_1, s - c_1), \ldots, (c_{h-2}, s - c_{h-2})$ for different values $0 < c_i < s$
- The semigroup can be polarized by the function $\lambda(c, c') := (c + c')/s$
- Thus, S defines an embedding of the projective toric curve into $\mathbb{P}^{h-1}$
- Notice that $d = e = 2$ and $r = 1, 2$ depending on whether the projective curve is arithmetically Cohen-Macaulay or not

Multinumerical Semigroups

- Let S_1 be the numerical semigroup generated by $s, c_1, \ldots, c_{h-2}$
  - For each $c \in S_1$ denote by $\mu(c)$ the smallest number of the above generators of S_1 which are needed to obtain c as a sum
  - Notice that the function μ satisfies the property

$$\mu(c) \leq \mu(c-s) + 1$$

    for every $c \in S_1$ such that $c - s \in S_1$

- One can prove that the projective toric curve is arithmetically Cohen-Macaulay if and only if

$$\mu(c) = \mu(c-s) + 1$$

  for every $c \in S_1$ such that $c - s \in S_1$

Multinumerical Semigroups

- In general, if we know μ it is not difficult to find the Apéry set Q relative to the above partition $\Lambda = \mathcal{E} \cup \mathcal{C}$, as well as the set D consisting of those elements $m \in S$ such that
  - $m - (s, 0) \in S$
  - $m - (0, s) \in S$
  - $m - (s, s) \notin S$
- One can consider a colored graph $\mathcal{G}_D$ similar to the Apéry one but replacing Q by D
  - One can prove that the vector space K_i can be identified to the homology $H_i(D_m)$ (analogous to that with Q)

Multinumerical Semigroups

- Thus, one deduces the long exact sequence

$$\cdots \to H_{i+1}(Q_m) \to H_i(D_m) \to \tilde{H}_i(\Delta_m) \to H_i(Q_m) \to \cdots$$

- The involved homologies, as well as the image maps in this exact sequence, can be computed in arithmetical terms from the given data $s, c_1, \ldots, c_{h-2}$
- The same holds for the reduced homologies $\tilde{H}_*(\Delta_m)$
- Hence, the minimal resolution for a projective toric curve can be obtained just from arithmetics

Applications

Some Applications

- The development of toric geometry has provided applications to many problems in algebra and geometry, since toric varieties describe the main ingredients involved in such problems
- There are also some applications outside geometry and algebra, so that toric geometry is becoming an interesting topic in applied mathematics
- Those external applications are related to applied combinatorics, computational geometry, statistics, operations research or coding theory
- We introduce in this section some illustrating applications of current interest

Coin exchange problem

- This is a classical problem in applied combinatorics (we follow the results of [Campillo and Revilla])
  - PROBLEM: Given a system of coins with values

$$c_1 < c_2 < \cdots < c_{h-1}$$

    and a exchange value c, find an algorithm to achieve the value c the minimum possible number of coins

- Setting $s := c_{h-1}$, we have a projective toric curve Z of degree s given by the (polarized) semigroup S of $\mathbb{N}^2$ generated by

$$(0, s), (c_1, s - c_1), \ldots, (c_{h-1}, s - c_{h-1}), (s, 0)$$

- The problem is then to achieve $c \in S_1$ (generated by the coin values) in a minimal way, i.e. with minimum $\mu(c)$

Coin exchange problem

- Coin systems considered in practice have a strong property: The greedy algorithm for achieving all the possible values c obtains a configuration with $\mu(c)$ coins
- The greedy algorithm works as follows:
  - Input: $c \in S_1$
  - Find the largest coin $c_j \leq c$ and
  - Update $c := c - c_j$ and restart the procedure until $c = 0$
- From the discussion in the previous section, if the greedy algorithm “works” then Z has to be arithmetically Cohen-Macaulay
- Then, the natural thing is to use arithmetically C-M coin systems
- In general, the greedy algorithm does not work, but there is an alternative algorithm

Counting points in polyhedra

- Given a (convex) polytope $\mathcal{P}$ with integer vertices, count the number of integer points in the polytope
- Example of application: coding theory (toric codes)
  Toric (error-correcting) codes are constructed as follows:
  - Fix a (rational) polytope $\mathcal{P}$ defined over a finite field $\mathbb{F}_q$, with dimension $r \geq 2$
  - Consider the $\mathbb{F}_q$-vector space of finite dimension $V_{\mathcal{P}}$, with basis $\{\chi^u \mid u \in \mathcal{P} \cap \mathbf{Z}^r\}$, i.e the monomials whose exponents are inside the polytope $\mathcal{P}$ (and the lattice $\mathbf{Z}^r$)
  - For any $t \in T = (\mathbb{F}_q^*)^r$ in the algebraic torus, evaluate each element of $V_{\mathcal{P}}$ at all the points t obtaining codewords of length $(q-1)^r$ (toric code associated to $\mathcal{P}$)
  - Under certain conditions this evaluation map is injective, so that the dimension of the code is precisely the number of (integer/rational) points inside the polytope

Counting points in polyhedra

- We show how to compute the number of points in a polytope by means of the so-called Brion's formula
- Let's start with a intuitive introduction ...
  - If we want to list all positive integers, we can place them as exponents of an infinite series, and write this series in a compact way in the form of a generating function

$$x^1 + x^2 + x^3 + \cdots = \sum_{k \geq 1} x^k = \frac{x}{1-x}$$

  - In a similar way, we can list all integers less than or equal to 5 as

$$\cdots + x^{-1} + x^0 + x^1 + x^2 + x^3 + x^4 + x^5 = \sum_{k \leq 5} x^k = \frac{x^5}{1-x^{-1}}$$

Counting points in polyhedra

- Adding the two generating rational functions, we get a miraculous cancellation

$$\frac{x}{1-x}+\frac{x^5}{1-x^{-1}}=\frac{x}{1-x}+\frac{x^6}{x-1}=\frac{x-x^6}{1-x}=x+x^2+x^3+x^4+x^5$$

- The sum of two rational functions representing two infinite series collapses into a polynomial representing a finite series
- This is a 1-dimensional version of the Brion's formula
  - We list separately the integer points in the rays $[1,\infty)$ and $(-\infty,5]$
  - By adding both functions, we get the list of integer points in the intersection interval $[1,5]$
- We get not only the number of points, but actually the list of points

Counting points in polyhedra

- Let's move up one dimension: consider the quadrilateral $\mathcal{Q}$ with vertices $(0,0),(2,0),(0,2),(4,2)$
- The analog of the 1-dimensional generating functions of the rays are now the generating functions of the cones at each of the (four) vertices generated by the edges incident to such vertex
- For example, the two edges touching the origin generate the nonnegative quadrant, with generating function

$$\sum_{m,n\geq 0}x^my^n=\sum_{m\geq 0}x^m\cdot\sum_{n\geq 0}y^n=\frac{1}{1-x}\cdot\frac{1}{1-y}$$

- In a similar way, the cone at $(0,2)$ has generating function

$$\sum_{m\geq 0,n\leq 2}x^my^n=\frac{y^2}{(1-x)(1-y^{-1})}$$

Counting points in polyhedra

- The cone at vertex $(2,0)$ has generating function

$$\frac{x^2}{(1-xy)(1-x^{-1})}$$

- And finally the cone at vertex $(4,2)$ corresponds to

$$\frac{x^4y^2}{(1-x^{-1})(1-x^{-1}y^{-1})}$$

- The sum of these four rational functions leads to a polynomial, encoding precisely the list of integer points contained in $\mathcal{Q}$

$$(1+x+x^2)+(y+xy+x^2y+x^3y)+(y^2+xy^2+x^2y^2+x^3y^2+x^4y^2)$$

Counting points in polyhedra

- In the general case, this “magic” happens for any polytope $\mathcal{P}$ in any dimension d provided $\mathcal{P}$ is rational (either vertices have integer coordinates, or edges have rational directions)
  - Let $\mathcal{K}_{\mathbf{v}}$ the cone at vertex $\mathbf{v}$ with directions given by the edges of the polytope $\mathcal{P}$
  - The generating function of $\mathcal{K}_{\mathbf{v}}$ is

$$\sigma_{\mathcal{K}_{\mathbf{v}}}(x) := \sum_{\mathbf{m}\in\mathcal{K}_{\mathbf{v}}\cap\mathbf{Z}^d} x^{\mathbf{m}}$$

    where we abbreviate $x^{\mathbf{m}} \equiv x_1^{m_1}x_2^{m_2}\cdots x_d^{m_d}$
  - This is a rational function, provided $\mathcal{P}$ is rational

Counting points in polyhedra

- Let finally $\sigma_{\mathcal{P}}(x)$ the (polynomial) generating function of the polytope $\mathcal{P}$, that is

$$\sigma_{\mathcal{P}}(x) := \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbf{Z}^d} x^{\mathbf{m}}$$

- Brion's formula states that:

$$\sigma_{\mathcal{P}}(x) = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \sigma_{\mathcal{K}_{\mathbf{v}}}(x)$$

- That is, again the sum of some infinite series collapses into a (finite) polynomial

Counting points in polyhedra

- So the problem is to compute the generating function of a cone
- This is feasible for the so-called simple cones, generated by d directions (in d dimensions)

$$\mathcal{K} := \left\{ \mathbf{v} + \sum_{i=1}^{d} \lambda_i \mathbf{w}_i \,|\, \lambda_i \in \mathbf{R}_{\geq 0} \right\} = \mathbf{v} + \sum_{i=1}^{d} \mathbf{R}_{\geq 0} \mathbf{w}_i$$

where $\mathbf{w}_1, \ldots, \mathbf{w}_d \in \mathbf{Z}^d$ are linearly independent

- This cone is tiled by lattice-translations of the half-open (fundamental) parallelepiped

$$\mathcal{P}_0 := \left\{ \mathbf{v} + \sum_{i=1}^{d} \lambda_i \mathbf{w}_i \,|\, 0 \leq \lambda_i < 1 \right\}$$

Counting points in polyhedra

- The generating function for $\mathcal{P}_0$ is the polynomial

$$\sigma_{\mathcal{P}_0}(x) := \sum_{\mathbf{m} \in \mathcal{P}_0 \cap \mathbf{Z}^d} x^{\mathbf{m}}$$

- Thus, the generating function for $\mathcal{K}$ is

$$\sigma_{\mathcal{K}}(x) = \sum_{\alpha \in \mathbb{N}\mathbf{w}_1 + \cdots + \mathbb{N}\mathbf{w}_d} \alpha \cdot \sigma_{\mathcal{P}_0}(x) = \frac{\sigma_{\mathcal{P}_0}(x)}{(1 - x^{\mathbf{w}_1}) \cdots (1 - x^{\mathbf{w}_d})}$$

  - This is a rational function
  - It is even better is the simple d-cone is unimodular . . .

Counting points in polyhedra

- We say that a rational d-cone $\mathcal{K} = \mathbf{v} + \sum_{i=1}^{d} \mathbb{R}_{\geq 0} \mathbf{w}_i$ is unimodular if $\mathbf{w}_1, \ldots, \mathbf{w}_d \in \mathbf{Z}^d$ generate the integer lattice $\mathbf{Z}^d$
  - The significance of such an unimodular cone $\mathcal{K}$ for us is that its fundamental (half-open) parallelepiped contains exactly one integer point $\mathbf{p}_0$
  - Thus, the generating function of $\mathcal{K}$ has a very simple and short form

$$\sigma_{\mathcal{K}}(x) = \frac{x^{\mathbf{p}_0}}{(1 - x^{\mathbf{w}_1}) \cdots (1 - x^{\mathbf{w}_d})}$$

- Now the natural question is: Can every cone be efficiently decomposed somehow into simple unimodular cones?

Counting points in polyhedra

- Theorem (Barvinok): For a fixed dimension d, the generating function $\sigma_{\mathcal{K}}$ for any rational cone $\mathcal{K} \subseteq \mathbb{R}^d$ can be decomposed into generating functions of unimodular simple cones in polynomial time

  i.e. A polynomial-time algorithm finds polynomially-many unimodular cones $\mathcal{K}_j$ such that

$$\sigma_{\mathcal{K}}(x) = \sum_j \epsilon_j \sigma_{\mathcal{K}_j}(x)$$

  where $\epsilon = \pm 1$

  - From here, one can see that it is possible to count integer points in a rational polytope in polynomial time (w.r.t. the input length of the description of $\mathcal{K}$)

Counting points in polyhedra

- Let's see an example in dimension $d = 2$
- Consider the cone $\mathcal{K}$ with vertex at the origin and edge directions $(1, 0)$ and $(1, 4)$
  - $\mathcal{K}$ can be either the difference of 2 unimodular cones

$$\begin{cases} \langle(1,0),(0,1)\rangle \\ \langle(1,4),(0,1)\rangle \end{cases}$$

  - or a sum of 4 unimodular cones

$$\begin{cases} \langle(1,0),(1,1)\rangle \\ \langle(1,1),(1,2)\rangle \\ \langle(1,2),(1,3)\rangle \\ \langle(1,3),(1,4)\rangle \end{cases}$$

Algebraic Statistics and Models

- In mathematical terms, a statistical model is frequently thought as a parameterized set of probability distributions

$$\begin{array}{ccc} \Theta & \rightarrow & \mathcal{P} \\ \theta & \mapsto & P_\theta \end{array}$$

- Θ is called a parameter space, that is usually a subset of $\mathbb{R}^n$
- Consider discrete data and suppose that both the parameter space and the parametrization map are described by polynomials
- Let's precise this with three-way contingency tables ...

Algebraic Statistics and Models

- Let X, Y, Z be random variables having a, b, c states respectively
- A probability distribution P is an $a \times b \times c$-table of non-negative real numbers that sum to one

  The entries of such a table are the probabilities

$$P_{ijk} := \mathrm{Prob}(X = i, Y = j, Z = k)$$

- The set of all distributions is a simplex Δ of dimension $abc - 1$
- A statistical model is a subset $\mathcal{M} \subseteq \Delta$ which can be described by polynomial equations and inequalities in the coordinates P_{ijk}
- Typically, $\mathcal{M}$ is presented as the image of a polynomial map

$$P : \Theta \rightarrow \Delta$$

  where $\Theta \subseteq \mathbb{R}^n$ is described by polynomial equations

Algebraic Statistics and Models

- The distribution P is called independent if each probability is the product of the corresponding marginal probabilities
$$P_{ijk} = P_{i++} \cdot P_{+j+} \cdot P_{++k}$$
where
$$P_{i++} := \mathrm{Prob}(X = i) = \sum_{j=1}^{b} \sum_{k=1}^{c} P_{ijk}$$
and so on
- The independence model has a parametric representation
$$\begin{array}{lcr} \Theta = \Delta_{a-1} \times \Delta_{b-1} \times \Delta_{c-1} & \to & \Delta = \Delta_{abc-1} \\ (\alpha, \beta, \gamma) & \mapsto & (P_{ijk}) = (\alpha_i \cdot \beta_j \cdot \gamma_k) \end{array}$$
- The image is known as the Segre variety in Algebraic Geometry

Algebraic Statistics and Models

- For example, if $a = b = c = 2$ the independence model (Segre variety) is the threefold in Δ_7 (or in $\mathbb{P}^7$) having the parametrization
$$\begin{array}{ll} P_{000} = \alpha\beta\gamma & P_{001} = \alpha\beta(1-\gamma) \\ P_{010} = \alpha(1-\beta)\gamma & P_{011} = \alpha(1-\beta)(1-\gamma) \\ P_{100} = (1-\alpha)\beta\gamma & P_{101} = (1-\alpha)\beta(1-\gamma) \\ P_{110} = (1-\alpha)(1-\beta)\gamma & P_{111} = (1-\alpha)(1-\beta)(1-\gamma) \end{array}$$
- This threefold is cut out by the trivial constraint
$P_{000} + P_{001} + P_{010} + P_{011} + P_{100} + P_{101} + P_{110} + P_{111} = 1$
- A Markov basis consists of nine quadratic binomials
$$\begin{array}{lll} P_{100}P_{111} - P_{101}P_{110} & P_{010}P_{111} - P_{011}P_{110} & P_{010}P_{101} - P_{011}P_{100} \\ P_{001}P_{111} - P_{011}P_{101} & P_{011}P_{110} - P_{011}P_{100} & P_{000}P_{111} - P_{011}P_{100} \\ P_{000}P_{110} - P_{010}P_{100} & P_{000}P_{101} - P_{001}P_{100} & P_{000}P_{011} - P_{001}P_{010} \end{array}$$

Algebraic Statistics and Models

- Markov bases make sense for every exponential family (log-linear model)
- They are interesting for graphical models and hierarchical models
  - ☐ They minimally generate the corresponding toric ideal
  - ☐ They give Markov chains for sampling from conditional distributions
  - ☐ They can be effectively computed by using software packages
- Theorem: The Markov basis for the independence model on 3 random variables consists of quadratic binomials as above
  - The number of binomials in such s basis equals

$$\frac{1}{8}abc(3abc - ab - ac - bc - a - b - c + 3)$$

Algebraic Statistics and Models

- Let's define properly what a Markov basis is
- Let A be a $d \times n$ integer matrix
- A finite set of vectors $\mathcal{B} \subseteq \ker_{\mathbf{Z}}(A)$ is called a Markov basis if for any couple of vectors with nonnegative integer coordinates $\mathbf{u}, \mathbf{v}$ such that $A\mathbf{u} = A\mathbf{v}$, there exists a sequence of elements $\{\mathbf{m}_i\}_{i=1}^{l}$ such that
  - ☐ $\mathbf{u} + \sum_{i=1}^{l} \mathbf{m}_i = \mathbf{v}$
  - ☐ $\mathbf{u} + \sum_{i=1}^{j} \mathbf{m}_i \geq \mathbf{0}$ for each $0 \leq j \leq l$
- The elements of a Markov basis are often called moves
- A priori, it is not clear that a finite Markov basis should exist, but one has the following result . . .

Algebraic Statistics and Models

- Theorem: A collection of binomials in the toric ideal of A

$$\{X^{\mathbf{m}^+} - X^{\mathbf{m}^-} | \mathbf{m} \in \mathcal{B}\} \subseteq I_A$$

is a generator system for I_A iff $\pm\mathcal{B}$ is a Markov basis for A

  - In particular, since every toric ideal has a finite generating set of binomials, we conclude that Markov bases exist
  - We say that a Markov basis is minimal if the corresponding collection of binomials generating the toric ideal is minimal as generator system
  - Unfortunately, minimal Markov bases are not generally unique
  - We can also consider the universal Markov basis, to be union of all possible minimal Markov bases of A
  - Universal Markov bases can be characterized in terms of primitive elements, in a similar way as in semigroup theory

Algebraic Statistics and Models

- A typical problem is statistical inference
  - Given a point $q \in \Delta$ and a model $\mathcal{M}$, find the point $p \in \mathcal{M}$ which “best agrees” with q
- “Best agrees” usually means maximum likelihood estimate
- Toric models are the “positive part” of a toric variety
- For toric models there is a explicit solution for the MLE problem

Integer Linear Programming

- Another application is the classical problem of applied optimization
- Integer Linear Programming is related to multinumerical subsemigroups of $\mathbf{Z}^d$ which, for simplicity, we assume to be positive
  - Let S be such a semigroup and assume that it is generated by the elements $n_1, \ldots, n_h \in \mathbf{Z}^d$
  - We are interested in finding the “optimal” solution with nonnegative integer coordinates satisfying the constraint
$$\sum_{i=1}^{h} x_i n_i = b$$
optimizing the (objective) linear function
$$\rho(x_1, \ldots, x_h) = \rho_1 x_1 + \cdots + \rho_h x_h$$
where the coefficients are $\rho_i \in \mathbb{R}$

Integer Linear Programming

- An integer (linear) program can be seen as a specification (π_0, ρ) where π_0 gives the data about S and its generators, and ρ the objective function (normally minimizing the cost)
- For each $b \in S$ the feasible solutions of the IP for b are among the elements of the fiber $\pi_0^{-1}(b)$ and, moreover, among the vertices of the polytope Ω_b, and the optimal(s) solution(s) in particular
- Notice now that, once any monomial ordering on the variables $x_1, \ldots, x_h$ is fixed (for instance the reverse lexicographical ordering), the objective function gives rise to another monomial ordering, namely a weighted ordering with weight ρ, as follows:

  1. First compare the monomials by the value of ρ at the exponents
  2. In case of tie, look at the previously fixed monomial ordering

Integer Linear Programming

- In this way, one can prove that the reduced Gröbner basis of the ideal I given by π_0 relative to this new monomial ordering provides a minimal test set for the IP problem, as described in the sequel
  - In fact, such reduced Gröbner basis is generated by binomials, and thus it can be seen as a subset U_ρ of the lattice L
  - On the other hand, we have the property that if $\mathbf{x} = (x_1, \ldots, x_h) \in \mathbb{N}^h$ is in a fiber and $\mathbf{l} \in L$ is a feasible solution then $\mathbf{x} - \mathbf{l}$ is again a feasible solution, provided $\mathbf{x} - \mathbf{l} \in \mathbb{N}^h$
  - Hence U_ρ is a test set for the proposed IP, for it satisfies the two following conditions ...

Integer Linear Programming

1. If $\mathbf{x}$ is a feasible solution which is not optimal, there exists $\mathbf{l} \in U_\rho$ with $\mathbf{l} \succ_\rho \mathbf{0}$ such that $\mathbf{x} - \mathbf{l}$ is also a feasible solution
2. If $\mathbf{x}$ is an optimal solution, then $\mathbf{x} - \mathbf{l}$ is not feasible, for every $\mathbf{l} \in U_\rho$

   - The condition on the Gröbner basis to be reduced implies that the test set U_ρ is minimal among all the possible test sets

- Test sets provide nice algorithms to solve integer linear programming problems, in the obvious way suggested by the definition of test set
- Non reduced Gröbner bases provide non minimal test sets
- In particular, the set U giving the universal Gröbner basis in one of the previous sections, which is finite and it is the union of all the U_ρ for all possible objective functions ρ, is a test set for all possible IP problems (varying ρ)

Integer Linear Programming

- In the sequel we give some more explicit details about solving integer programming with Gröbner bases
- We start with the Conti-Traverso algorithm ...
  - We first need to characterize the feasible (integer) solutions
  - Our aim is to find an optimal solution of the “standard problem” of minimization (i.e. ρ is a cost function)
- Consider an integer linear program

$$\begin{cases} \text{Minimize} & \rho_1\alpha_1 + \cdots + \rho_h\alpha_h \\ \text{subject to} & A\alpha = \mathbf{b},\ \alpha \in \mathbb{N}^h \end{cases}$$

Integer Linear Programming

- We introduce a variable for each constraint and write such restrictions in polynomial form

$$\prod_{j=1}^{h}\left(\prod_{i=1}^{d} Y_i^{a_{ij}}\right)^{\alpha_j} = \prod_{i=1}^{d} Y_i^{b_i}$$

- Since exponents may be negative, we must interpret this equality in the Laurent Polynomial ring

$$k[Y_1,\ldots,Y_d,Y_1^{-1},\ldots,Y_d^{-1}]/\langle Y_iY_i^{-1}-1\rangle|1 \le i \le d\rangle$$

- We can save variables by using an isomorphic k-algebra

$$k[Y_1,\ldots,Y_d,T]/\langle Y_1\cdot\ldots\cdot Y_d\cdot T-1\rangle$$

Integer Linear Programming

- A monomial like $Y^{\mathbf{a}}$ has a representative like $T^{r(\mathbf{a})}Y^{\tilde{\mathbf{a}}}$ where

$$r(\mathbf{a}) := \max\{|a_i| : a_i < 0\} \text{ and } \tilde{\mathbf{a}} := \mathbf{a} + (r(\mathbf{a}), \dots, r(\mathbf{a}))$$

- Thus, considering the k-algebra homomorphism

$$\begin{array}{rcl} \varphi_A : k[X_1, \dots, X_h] & \to & k[Y_1, \dots, Y_d, T]/\langle Y_1 \cdot \ldots \cdot Y_d \cdot T - 1\rangle \\ X_j & \mapsto & T^{r(A_j)}Y^{\tilde{A}_j} + \langle Y_1 \cdot \ldots \cdot Y_d \cdot T - 1\rangle \end{array}$$

  where A_j stands for the j-th column of A,
  we can characterize feasible solutions as follows

  - Proposition: α is a feasible solution iff $\varphi_A(X^\alpha) \equiv T^{r(\mathbf{b})}Y^{\tilde{\mathbf{b}}}$ modulo $(Y_1 \cdot \ldots \cdot Y_d \cdot T - 1)$
  - This can be solved with the aid of Gröbner bases . . .

Integer Linear Programming

- Algorithm (Conti-Traverso)

Input: $(A, \mathbf{b}, \prec)$, where $\prec$ is a monomial ordering over $k[X_1, \dots, X_h, Y_1, \dots, Y_d, T]$ which is an elimination ordering for $\{Y_1, \dots, Y_d, T\}$

Step 1: Compute $\mathcal{G}$ a Gröbner basis w.r.t. $\prec$ of the ideal

$$J := \langle T^{r(A_1)}Y^{\tilde{A}_1} - X_1, \dots, T^{r(A_h)}Y^{\tilde{A}_h} - X_h, Y_1 \cdot \ldots \cdot Y_d \cdot T - 1\rangle$$

Step 2: Compute the normal form h of $T^{r(\mathbf{b})}Y^{\tilde{\mathbf{b}}}$ modulo $\mathcal{G}$

Output: A feasible solution α if $h = X^\alpha$, or $\emptyset$ otherwise

- If we want moreover α to be an optimal solution, we need a special type of monomial ordering . . .

Integer Linear Programming

- A monomial ordering $\prec$ over $k[X_1, \ldots, X_h, Y_1, \ldots, Y_d, T]$ is said to be adapted to the IP problem given by $(A, \mathbf{b}, \rho)$ if

  1. It is an elimination ordering for $\{Y_1, \ldots, Y_d, T\}$
  2. It is compatible with the objective function ρ w.r.t. the matrix A, i.e.

  $$A\alpha = A\alpha' \text{ and } \langle \rho, \alpha \rangle < \langle \rho, \alpha' \rangle \Rightarrow X^{\alpha} \prec X^{\alpha'}$$

- If such an ordering exists, the IP problem has a solution iff Conti-Traverso algorithm returns a vector α which is, in that case, the optimal solution for the IP problem

- We will see now how to find an adapted ordering, if possible ...

Integer Linear Programming

- Consider the free **Z**-module (lattice) $L_A := \{\mathbf{u} \in \mathbb{Z}^h | A\mathbf{u} = 0\}$

- The feasible set of IP is bounded (or empty) iff $L_A \cap \mathbb{N}^h = \{0\}$

  - This condition implies the existence of a linear combination $\mathbf{d} > \mathbf{0}$ of the rows of A and, in particular, $\mathbf{d}$ is orthogonal to all vectors in L_A
  - This is the key to construct an adapted ordering:

    1. Take any $\prec$ over $k[X_1, \ldots, X_h, Y_1, \ldots, Y_d, T]$
    2. Choose $\mu >> 0$ so that $\rho + \mu \cdot \mathbf{d} > \mathbf{0}$ and define

    $$\mathbf{u}_1 := (0, \ldots, 0, 1, \ldots, 1) \in \mathbb{N}^{h+(d+1)}$$
    $$\mathbf{u}_2 := (\rho + \mu \cdot \mathbf{d}, \mathbf{0}) \in \mathbb{R}^{h+(d+1)}_{\geq 0}$$

    3. Consider (adapted) the monomial ordering $\prec_{\mathbf{u}_1, \mathbf{u}_2}$

Integer Linear Programming

- Now we give details how universal test sets can solve families of integer programming problems, where the cost vector $\mathbf{b}$ and the objective function ρ vary
- Consider an IP problem given by a full-rank matrix A together with $\mathbf{b}$ and ρ, and assume $L_A \cap \mathbb{N}^h = \{0\}$
  - Denote by $\mathcal{A} := \{A_1, \ldots, A_h\}$ the set of columns of A
  - Consider the semigroup in $\mathbb{Z}^d$ generated by $\mathcal{A}$ that is

$$S \equiv S_A := \{A\alpha | \alpha \in \mathbb{N}^h\} \subseteq \mathbb{Z}^d$$

- Denote by $\mathrm{IP}_{A,\rho}$ the family of problems $\mathrm{IP}_{A,\rho}(\mathbf{b})$ where $\mathbf{b}$ varies in S, and by IP_A the family where only A is fixed
- Under these assumptions, all such IP problems are feasible and bounded

TORIC GEOMETRY 117 / 120

Integer Linear Programming

- If we have a test set that works simultaneously for all possible $\mathbf{b}$'s (universal), and we have a concrete feasible solution for a concrete $\mathbf{b}$, we can solve the IP problem with the following test-set algorithm ...
  - Successively subtract to the initial feasible solution elements of the test set to obtain a decreasing chain of vectors
  - The condition $L_A \cap \mathbb{N}^h = \{0\}$ (i.e. S positive) guarantees that this procedure terminates after a finite number of steps, and obtains an optimal solution for such IP
  - Such universal test set can be found from either a universal Gröbner basis or a Graver basis, and only depends on A (i.e. the semigroup S) and not on $\mathbf{b}$ nor ρ
- Alternative: for A and ρ fixed we can compute a Gröbner basis $\mathcal{G}_\rho$ of the toric ideal I_A w.r.t the ordering $\prec_\rho$, and then if we have a feasible $\mathbf{u}$ then the remainder of the division of $X^{\mathbf{u}}$ by $\mathcal{G}_\rho$ is optimal

TORIC GEOMETRY 118 / 120

SINGULAR

- There are some SINGULAR libraries to work with Toric Geometry . . . but not too much
  - `toric.lib` Compute Gröbner bases of toric ideals
  - `lll.lib` Compute a reduced basis for a lattice
  - `intprog.lib` Solve Integer Linear Programming problems using Gröbner bases
  - `polymake.lib` Computations with polytopes and fans
  - `homolog.lib` and other general procedures for Algebraic Geometry and Commutative Algebra: Homological Algebra, Krull dimension, Gröbner bases . . .