Gradient-like decoding of binary linear codes.

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Part I, *Gröbner bases associated with binary linear codes*

Part II, *Gradient-like decoding of binary linear codes. Examples*

Part I

Gröbner bases associated with binary linear codes

Outline

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Gröbner Bases and linear codes in the literature

To find a Gröbner basis of the error locator ideal.
A systematic method for encoding and decoding m-dimensional cyclic codes.
The key equation and Berlekam-Massey Algorithm (BCH codes).

Gröbner bases and the integer programing problem related with the soft-decision maximum likelihood decoding of binary linear block codes.

The FGLM algorithm has been connected already to coding theory:
Related with the error locator ideal and a change of ordering.
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Also there are works devoted to extensions of the previous ideas to Algebraic Geometric Codes and codes over rings.

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S. Sakata, T. Mora, M. Sala, E. Orsini, P. Fitzpatrick, J.C. Faugere, J. Fitzgerald, R.F. Lax, D. Ikegami, R. Pellikaan, S. Bulygin, ...

Our approach

(Borges-Quintana, Borges-Trenard, Martínez-Moro): a Gröbner basis is associated with the structure of the linear code.

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Overview of the method

Let $\mathcal{A}$ be a finitely generated algebra (we want to solve a problem in $\mathcal{A}$).

- To find the appropiate morphism

$$\xi : K[X] \to \mathcal{A}.$$

$\mathcal{I}$ will be the ideal such that

$\mathcal{A} \cong K[X]/\mathcal{I} \cong Span_K(N)$ (N: the set of canonical forms).

If $\mathcal{A}$ is a monoid (or group) algebra ($A = K[M]$)

$$\mathcal{I} = \langle \{x^w - x^v \mid \xi(x^w) = \xi(x^v),\ x^w, x^v \in [X]\} \rangle.$$

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- Define a reduction process (s.t. it allows to solve the problem):
  - ⋆ finite numbers of reductions,
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Overview of the method: the instance of linear codes

Having:

$$\xi : K[X] \rightarrow \mathcal{A} = K[M],$$

$$K[M] \cong K[X]/\mathcal{I} \cong Span_K(N) \text{ (N: the set of canonical forms)}$$

The instance of linear codes:

1. $M = \mathbb{F}_2^{n-k}$ (the monoids of the syndromes).
2. ξ: gives the syndrome of each $x^w \in [X]$.
3. $N \Leftrightarrow$ the coset leaders.
4. $\mathcal{I}$: we call it the ideal associated with the code.

Then, we compute a Gröbner basis of $\mathcal{I}$ for a convenient ordering $\prec$ such that:

$$Can(x^w, I, \prec) \Leftrightarrow \text{the coset leader with syndrome } \xi(x^w).$$

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Buchberger's Algorithm

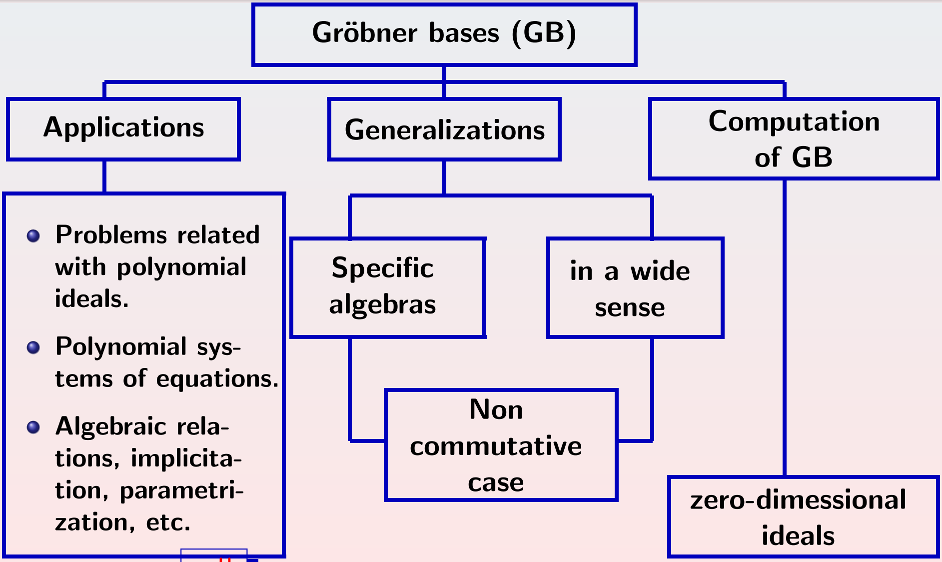
Input: A set of polynomials F s.t.

$$I = Ideal(F),$$

$\prec$ an **admissible** ordering on $[X]$.

Output: Gröbner basis of I w.r.t. $\prec$.

Ejemplo


Gröbner bases (GB)
Applications
Generalizations
Computation of GB
Problems related with polynomial ideals.
Polynomial systems of equations.
Algebraic relations, implicitation, parametrization, etc.
Specific algebras
in a wide sense
Non commutative case
zero-dimessional ideals

Outline

Computation of GB

Specialization to specific algebras | **Using other inputs**

zero-dimenssional ideals

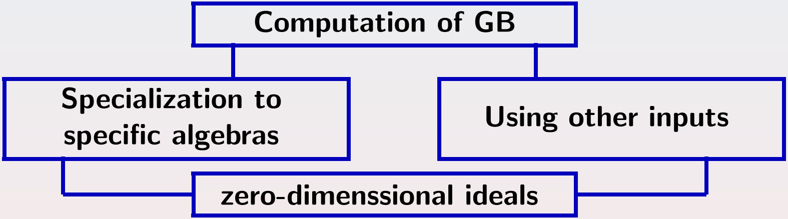
zero-dimenssional: $\dim K\langle X\rangle/I < \infty$

Linear Algebra in $K\langle X\rangle/I$:

$$\forall c_i \in K,\ s_i \in \langle X\rangle \qquad \sum_{i=1}^{r} c_i s_i \in I \setminus \{0\} \Leftrightarrow$$

$\{s_1, \ldots, s_r\}$, is linear dependent module I.

More . . .



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Outline

1. Gröbner Bases
2. The zero-dimenssional ideal case
3. Binary Codes
   - Introduction
   - A monoid representation of $\mathbb{F}_2^n$
4. Some comments about specialized algorithms
   - Considering the code as a monoid
   - Starting from a generator matrix

A primer on binary codes

Let $\mathbb{F}_2$ be the finite field of two elements. A binary code $\mathcal{C}$ of dimension k and length n is the image of a linear (injective) mapping:

$$L : \mathbb{F}_2^k \longrightarrow \mathbb{F}_2^n \quad k \leq n.$$

There exists an $n \times (n-k)$ matrix H called parity check matrix such that for each word (element) we have: $\mathbf{c} \cdot H = \mathbf{0} \Leftrightarrow c \in \mathcal{C}$.

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A primer on binary codes (Cont.)

The weight of a word is its Hamming distance to the word $\mathbf{0}$, i.e. the number of non-zero coordinates of the word. The minimun distance d of the code $\mathcal{C}$ is the minimum weight among all the non-zero codewords.

The error-correcting capacity of a code is $t = \lfloor \frac{d-1}{2} \rfloor$. If we let $B(\mathcal{C}, t) := \{y \in \mathbb{F}_2^n \mid \exists c \in \mathcal{C} \text{ s.t. } d(y, c) \leq t\}$, it is well known that the equation $e \cdot H = y \cdot H$ has a unique solution e with $\mathrm{weight}(e) \leq t$ if $y \in B(\mathcal{C}, t)$. Then $y - e \in \mathcal{C}$ (syndrome decoding).

A primer on binary codes (Cont.)

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A monoid representation

Let us consider the free commutative monoid $[X]$ generated by the n variables $X := \{x_1, \ldots, x_n\}$.

We have the following map from X to $\mathbb{F}_2^n$:

$$\psi : X \to \mathbb{F}_2^n, \text{ where } x_i \mapsto e_i \text{ (the } i\text{-th coordinate vector).}$$

The map ψ can be extended in a natural way to a morphism from $[X]$ onto $\mathbb{F}_2^n$, where $\psi(\prod_{i=1}^n x_i^{\beta_i}) = (\beta_1 \bmod 2, \ldots, \beta_n \bmod 2)$.

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The equivalence relation $R_{\mathcal{C}}$ in $\mathbb{F}_2^n$

A binary code $\mathcal{C}$ defines an equivalence relation $R_{\mathcal{C}}$ in $\mathbb{F}_2^n$:

$$(x, y) \in R_{\mathcal{C}} \Leftrightarrow x - y \in \mathcal{C}. \tag{1}$$

Let $\xi(x^u) := \psi(x^u)H$ (note that $x^u \in [X]$). The above congruence can be translated to $[X]$ by the linear morphims ξ as

$$x^u \cong_{\mathcal{C}} x^w \Leftrightarrow (\psi(x^u), \psi(x^w)) \in R_{\mathcal{C}} \Leftrightarrow \xi(x^u) = \xi(x^w). \tag{2}$$

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The binomial ideal associated with the code

Let $I(\mathcal{C})$ be the ideal associated with the relation $R_{\mathcal{C}}$ on $[X]$, that is:

$$\mathbf{I}(\mathcal{C}) := \langle \{ \mathbf{x}^{\mathbf{w}} - \mathbf{x}^{\mathbf{v}} \mid (\psi(\mathbf{x}^{\mathbf{u}}), \psi(\mathbf{x}^{\mathbf{w}})) \in \mathbf{R}_{\mathcal{C}} \} \rangle.$$

Outline

Computing the reduced Gröbner basis

Let G_{T} be the reduced Gröbner basis of the ideal $I(\mathcal{C})$ with respect to $<$ (a total degree compatible ordering).

There are diferent algorithmic ways of computing G_{T} for this setting.

FGLM algorithm for monoid algebras

FGLM algorithm for finite dimension monoid algebras: An algorithm based on the generation of representative elements of the quotient algebra w.r.t. a given admisible ordering (in this case $\mathbb{F}_2^{n-k} \cong K[X]/I(\mathcal{C})$) based on linear algebra techniques. ([4], specialized to linear codes in [1].)

Outputs: G_{T}. It can also give a set N of representative elements corresponding to a set of coset leaders for the code and a matrix structure that allows to multiply the representative elements (to sum the cosets leaders and obtaining the corresponding coset leader).

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Monoid rings

Let M be a finite commutative monoid generated by $g_1, \ldots, g_n$;

$\xi : [X] \longrightarrow M$, the canonical morphism that sends x_i to g_i;

$\sigma \subset [X] \times [X]$, a presentation of M defined by ξ

$$\sigma = \{(x^w, x^v) \mid \xi(x^w) = \xi(x^v)\}.$$

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Then, it is known that the monoid ring $K[M]$ is isomorphic to $K[X]/I(\sigma)$, where $I(\sigma)$ is the ideal generated by $P(\sigma) = \{x^w - x^v \mid (x^w, x^v) \in \sigma\}$

$$I(\sigma) = \langle P(\sigma)\rangle = \langle\{x^w - x^v \mid (x^w, x^v) \in \sigma\}\rangle.$$

Moreover, any Gröbner basis G of $I(\sigma)$ is also formed by binomials of the above form. **In addition, it can be proved that $\{(\mathbf{x^w}, \mathbf{x^v}) \mid \mathbf{x^w} - \mathbf{x^v} \in \mathbf{G}\}$ is another presentation of M.**

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Monoid rings

Note that M is finite if and only if $I = I(\sigma)$ is zero-dimensional.

Specifying the monoid M

The monoid M is set to be $\mathbb{F}_2^{n-k}$ (where the syndromes belong to).

Doing $g_i := \xi(x_i)$, note that

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Starting from a generator matrix

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Obtaining the ideal I($\mathcal{C}$): Let be $\{w_1, \ldots, w_k\}$ be the row vectors of a generator matrix for a code (more generally any matrix whose rows span the code $\mathcal{C}$), i.e., a basis (spanning set) of the code as subspace of $\mathbb{F}_2^n$ (see [3]). Let

$$I = \langle \{x^{w_1} - 1, \ldots, x^{w_k} - 1\} \cup \{x_i^2 - 1 \mid i = 1, \ldots, n\} \rangle \qquad (3)$$

be the ideal generated by the set of binomials $\{x^{w_1} - 1, \ldots, x^{w_k} - 1\} \cup \{x_i^2 - 1 \mid i = 1, \ldots, n\} \subset K[X]$. **Since $\{\mathrm{w}_1, \ldots, \mathrm{w}_k\}$ generates $\mathcal{C}$ it is clear that $\mathrm{I} = \mathrm{I}(\mathcal{C})$.**

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Let $F = \{x^{w_1} - 1, \ldots, x^{w_k} - 1\} \cup \{x_i^2 - 1 \mid i = 1, \ldots, n\} \subset K[X]$, $r = k + n$. There are two ways for computing G_T:

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- G_T can be computed by **Buchberger's algorithm** starting with the initial set F.
  However, there are some computational advantages in this case. The coefficient field is $\mathbb{F}_2$ (therefore, there is no coefficient growth), and the maximal length of words in the computation is n (the binomials $x_i^2 - 1$ prevent the length being greater than n).

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Starting from a generator matrix

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- **Using the FGLM basis convertion algorithm** to obtain a basis for the syzygy module M in $K[X]^{r+1}$ of the generator set $F' = \{-1, f_1, f_2, \ldots, f_r\}$. Each of the syzygies corresponds to a solution

$$f = \sum_{i=1}^{r} b_i f_i \qquad b_i \in K[X], i = 1, \ldots, r.$$

  and thus points to an element f in the ideal I generated by F.

Part II

Gradient-like decoding of binary linear codes. Examples

Outline

5. Gradient decoding of binary codes
   - Test sets
6. Gröbner test sets
7. Worked example
   - other structures (Matphi and border basis)
8. Remarks and Complexity

Minimal subsets in codes

Let $\mathbb{F}_2^n$ the n-dimensional coordinate space over the field $\mathbb{F}_2$ and $\mathcal{C} \subseteq \mathbb{F}_2^n$ a linear code. We define the **support of a codeword** $\mathbf{c} = (c_1, \ldots, c_n) \in \mathcal{C}$ as

$$\mathrm{supp}(\mathbf{c}) = \{i \in \{1, \ldots, n\} \mid c_i \neq 0\}\,. \tag{4}$$

If $\mathrm{supp}(\mathbf{c}') \subset \mathrm{supp}(\mathbf{c})$ (respectively $\subseteq$) we will write $\mathbf{c}' \prec \mathbf{c}$ (respectively $\preceq$).

Definition (Minimal codeword)

A nonzero vector $\mathbf{c} \in \mathcal{C}$ is said to be **minimal** if $0 \neq \mathbf{c}' \preceq \mathbf{c}$ and $\mathbf{c}' \in \mathcal{C}$ then it implies that there exists a nonzero constant $\alpha \in \mathbb{F}_2$ such that $\mathbf{c}' = \alpha\mathbf{c}$.

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Test set gradient-like decoding

Definition

Any set $\mathcal{T} \subseteq \mathcal{C}$ of codewords such that for every vector $\mathbf{y} \in \mathbb{F}_2^n$ either $\mathbf{y}$ lies in $\mathcal{C}$ or there exists $\mathbf{z} \in \mathcal{T}$ such that

$$\mathrm{d}(\mathbf{y}+\mathbf{z},\mathbf{0}) < \mathrm{d}(\mathbf{y},\mathbf{0})$$

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A **gradient-like** decoding algorithm is obtained using a test set $\mathcal{T}$ as follows.

Let $\mathbf{y}$ be the received vector

1. $\mathbf{c} \leftarrow \mathbf{0}$,
2. Find $\mathbf{z} \in \mathcal{T}$ such that $\mathrm{d}(\mathbf{y}+\mathbf{z}, \mathbf{0}) < \mathrm{d}(\mathbf{y}, \mathbf{0})$.

$$\mathbf{c} \leftarrow \mathbf{c}+\mathbf{z} \text{ and } \mathbf{y} \leftarrow \mathbf{y}+\mathbf{z}.$$

3. Repeat 2. until no such $\mathbf{z}$ is found in $\mathcal{T}$.
4. Return $\mathbf{c}$.

This decoding algorithm always converges to one of the closest codewords to the received vector.

Outline

Let $[X] = \{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}^n\}$ be the set of terms.

Let G be the reduced Gröbner basis of the ideal $I(\mathcal{C})$ with respect to the term ordering $<$ (a total degree compatible ordering) and let $g \in K[X]$, we denote by $\mathrm{Can}(g, G)$ the **canonical form** of g with respect to the Gröbner basis G.

Theorem (GB's Reduction means decoding)

Let $\mathcal{C}$ be a linear code. Let $x^w \in [X]$ and $x^v \in N$ its corresponding canonical form. If $\mathrm{weight}(\psi(x^v)) \leq t$ then $\psi(x^v)$ is the error vector corresponding to $\psi(x^w)$. Otherwise, if $\mathrm{weight}(\psi(x^v)) > t$, $\psi(x^w)$ contains more than t errors. (t is the error correcting capability)

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If $g = \mathbf{x}^{\mathbf{w}} - \mathbf{x}^{\mathbf{v}} \in I(\mathcal{C})$ denote by $\mathbf{c}_g$ the codeword associated to the binomial g, that is,

$$\mathbf{c}_g = \psi(\mathbf{x}^{\mathbf{w}}) + \psi(\mathbf{x}^{\mathbf{v}})$$

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Let G be a Gröbner basis with respect to the term ordering $<$ of the binomial ideal $I(\mathcal{C})$. The **Gröbner codewords set** w.r.t. G is

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The elements of the set $\mathcal{C}_G$ of Gröbner codewords are minimal codewords of the code $\mathcal{C}$.

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The Gröbner decoding algorithm

The theorem allows us to perform a gradient-like decoding algorithm but according to $<$ instead of the weight of the vectors. Thus we say that the set of Gröbner codewords is a "test set".

Input: $\mathcal{C}_G$ and $\mathbf{y}$ a received vector.
Output: One of the closest codewords to $\mathbf{y}$.

1. $i := 0$; $\mathbf{v}_i = \mathbf{y}$; $\mathbf{c}_i = \mathbf{0}$.
2. Repeat
3. Find $\mathbf{w} \in \mathcal{C}_G$ such that $\mathbf{x}^{\mathbf{v}_i} > \mathbf{x}^{\mathbf{v}_{i+1}}$ and $\mathbf{v}_{i+1} = \mathbf{v}_i + \mathbf{w}$.
4. $\mathbf{c}_{i+1} = \mathbf{c}_i + \mathbf{w}$; $i = i + 1$
5. Until such a $\mathbf{w}$ does not exist.
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Outline

Worked example

Consider the code $\mathcal{C}$ in $\mathbb{F}_2^6$ (a $[6,3,3]$ binary linear code) with generator matrix:

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

The set of codewords is

$$\begin{aligned} \mathcal{C} \;=\; & \{(0,0,0,0,0,0),(1,0,1,1,0,0), \\ & (1,1,0,0,1,0),(0,1,0,1,0,1), \\ & (0,0,1,0,1,1),(1,1,1,0,0,1), \\ & (0,1,1,1,1,0),(1,0,0,1,1,1)\} \end{aligned}$$

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The reduced Gröbner basis of $I(\mathcal{C})$ with respect to the degree reverse Lexicographical ordering $<$ is

$$\begin{aligned} G = \{ & x_1^2 - 1, x_2^2 - 1, x_3^2 - 1, x_4^2 - 1, x_5^2 - 1, x_6^2 - 1, \\ & x_1x_2 - x_5, x_1x_3 - x_4, x_1x_4 - x_3, x_1x_5 - x_2, \\ & x_2x_3 - x_1x_6, x_2x_4 - x_6, x_2x_5 - x_1, x_2x_6 - x_4, \\ & x_3x_4 - x_1, x_3x_5 - x_6, x_3x_6 - x_5, \\ & x_4x_5 - x_1x_6, x_4x_6 - x_2, x_5x_6 - x_3 \}. \end{aligned}$$

Therefore the code is 1-correcting (i.e. $t = 1$) and the set of Gröbner codewords is

$$\mathcal{C}_G = \left\{ \begin{array}{l} (1,1,0,0,1,0), (1,0,1,1,0,0), (0,1,0,1,0,1), \\ (0,0,1,0,1,1), (1,1,1,0,0,1), (1,0,0,1,1,1) \end{array} \right\}$$

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1. If we recive $\mathbf{y} = (1,1,0,1,1,0)$; then

$$\mathbf{c}_1 := (1,1,0,0,1,0) \text{ and } \mathbf{y}_1 = \mathbf{y} + \mathbf{c}_1 = (0,0,0,1,0,0).$$

Since $\mathrm{d}(\mathbf{y}_1, \mathbf{0}) = 1$, i.e. the codeword corresponding to $\mathbf{y}$ is $\mathbf{c}_1$.

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$\mathbf{y}_1$ can not be reduced following the algorithm; thus, $\mathrm{d}(\mathbf{y}_1, \mathbf{0}) > 1$; and in this case $\mathbf{y}$ contains more errors than the error-correcting capability of the code. However, note that $\mathbf{c}_1$ is the closest codeword to $\mathbf{y}$.

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Example: Other outputs

$N = \{1, x_1, x_2, x_3, x_4, x_5, x_6, x_1x_6\}$;
Matphi (Practical representation):

$$
\begin{array}{ccccl}
 & & & & \qquad\qquad\quad \psi(w) \qquad\qquad\qquad {\color{red}\phi(w, x_i)} \\
 & & & & \qquad\qquad\quad \downarrow \qquad\qquad\qquad\quad {\color{red}\downarrow} \\
w & \longrightarrow & 1 & \longrightarrow & [\,[[0,0,0,0,0,0],\ 1,\ {\color{red}[2,3,4,5,6,7]}\,], \\
 & & x_1 & \longrightarrow & \ \ [[1,0,0,0,0,0],\ 1,\ {\color{red}[1,6,5,4,3,8]}\,], \\
 & & x_2 & \longrightarrow & \ \ [[0,1,0,0,0,0],\ 1,\ {\color{red}[6,1,8,7,2,5]}\,], \\
 & & x_3 & \longrightarrow & \ \ [[0,0,1,0,0,0],\ 1,\ {\color{red}[5,8,1,2,7,6]}\,], \\
 & & x_4 & \longrightarrow & \ \ [[0,0,0,1,0,0],\ 1,\ {\color{red}[4,7,2,1,8,3]}\,], \\
 & & x_5 & \longrightarrow & \ \ [[0,0,0,0,1,0],\ 1,\ {\color{red}[3,2,7,8,1,4]}\,], \\
 & & x_6 & \longrightarrow & \ \ [[0,0,0,0,0,1],\ 1,\ {\color{red}[8,5,6,3,4,1]}\,], \\
 & & x_2x_3 & \longrightarrow & \ \ [[0,1,1,0,0,0],\ 0,\ {\color{red}[7,4,3,6,5,2]}\,]\,]
\end{array}
$$

Example: Decoding

(i.) $y \in B(C,t)$: $y = (1,1,0,1,1,0)$; $w_y := x_1x_2x_4x_5$; $\phi(1,x_1) = x_1$; $\phi(x_1,x_2) = x_5$; $\phi(x_5,x_4) = x_2x_3$; $\phi(x_2x_3,x_5) = x_4$, this means $e = \psi(x_4) = (0,0,0,1,0,0)$, $weight(e) = 1$ then, the codeword corresponding to y is $c = y - e = (1,1,0,0,1,0)$.

(ii.) $y \notin B(C,t)$: $y = (0,1,0,0,1,1)$; $w_y := x_2x_5x_6$; $\phi(1,x2) = x_2$; $\phi(x_2,x_5) = x_1$; $\phi(x_1,x_6) = x_2x_3$; $e = (0,1,1,0,0,0)$; $weight(e) > 1$ then, we report an error in the transmission process.

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Example: Decoding

(i.) $y \in B(C,t)$: $y = (1,1,0,1,1,0)$; $w_y := x_1x_2x_4x_5$; $\phi(1,x_1) = x_1$; $\phi(x_1,x_2) = x_5$; $\phi(x_5,x_4) = x_2x_3$; $\phi(x_2x_3,x_5) = x_4$, this means $e = \psi(x_4) = (0,0,0,1,0,\ 0)$, $weight(e) = 1$ then, the codeword corresponding to y is $c = y - e = (1,1,0,0,1,0)$.

(ii.) $y \notin B(C,t)$: $y = (0,1,0,0,1,1)$; $w_y := x_2x_5x_6$; $\phi(1,x2) = x_2$; $\phi(x_2,x_5) = x_1$; $\phi(x_1,x_6) = x_2x_3$; $e = (0,1,1,0,0,0)$; $weight(e) > 1$ then, we report an error in the transmission process.

Outline

Some computations with the binary Golay Code

We use the **GAP** package **GUAVA** to construct a generator matrix of the binary Golay code $[23, 12, 7]$ and **GBLA_LC**: a group of programs made in **GAP** to carry out our approach.

The code has 4096 codewords, 2048 syndromes.

1. Computing the reduced Gröbner basis (**Gr**): 17.42 min.
2. Computing the border basis (**BB**): 13.34 min.
3. $\mid \mathbf{Gr} \mid = \mathbf{8878}$.
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Some computations with the binary Golay Code

Computing the reduced Gröbner basis in some system of Symbolic Computation:

- **Mathematica (4.0)**: was interrupted after 4 hours.
- **Maple (9.0)**: was interrupted after 4 hours.
- **Singular (3-0-0)**: it succeeded in 2 hours.

- The decoding procedure is a complete decoding procedure, that is, it always finds the codeword that is the closest to the received vector.
- Furthermore, with the same procedure it is easy to know whether the result is reliable or not, if $\mathrm{d}(\mathbf{v}_i, \mathbf{0}) \leq t$ then $\mathbf{c}_i$ is the codeword corresponding to $\mathbf{y}$, where t is the error-correcting capability of $\mathcal{C}$. As a byproduct of the computation of $\mathcal{C}_G$ it is possible to obtain t.
- Generalizations to linear codes is possible by using the border basis, the extension of the ordering "the error vector ordering" (not anymore a degree compatible ordering) is not admissible.

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Complexity

Preprocesing Computing the reduced Gröbner basis or the border basis performed $\mathcal{O}(n^2 2^{n-k})$ operations.

Decoding The decoding complexity depends on the size of $\mathcal{C}_G$ (or $\mathcal{C}_{\mathcal{B}}$) and the number of reductions. The number of reductions for $\mathcal{C}_{\mathcal{B}}$ is less than n.

Computing t The error correction capability of an arbitrary linear code (not neccesary binary) can be computed in at most $m \cdot n \cdot S(t+1)$ itererations of the Algorithm showed in B.,B. & M. where

$$S(l) = \sum_{i=0}^{l} \binom{n}{i} (q-1)^i.$$

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Part III

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Gradient-like decoding of binary linear codes.

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Universidad de Valladolid, Spain

July 5, 2008, S^{3}CM, Soria, España.

Outline

Outline

Admissible ordering

$\prec$ is admissible on $\langle \mathbf{X} \rangle$:

If it is a total ordering on $\langle X \rangle$ s.t., for all $s, t, u \in \langle X \rangle$:

i) $1 \preceq s$.

ii) $t \prec u : (st \prec su$ and $ts \prec us)$.

Return

Example of a Gröbner basis

$F := \{ x^2 - 1, x^2y - 1 \},$

$p := x^2y^2 - y^2.$

Reducing p module F:

With $x^2y - 1$: $\quad y - y^2 = x^2y^2 - y^2 - (x^2y - 1)y.$

With $x^2 - 1$: $\quad 0 = x^2y^2 - y^2 - (x^2 - 1)y^2.$

$G := \{ x^2 - 1, y - 1 \}$ is a Gröbner basis for Ideal(F).

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$\mathbf{F} := \{ \mathbf{x}^2 - \mathbf{1}, \mathbf{x}^2\mathbf{y} - \mathbf{1} \},$

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Definition of Gröbner basis

$I = Ideal(F)$, let $T_{\prec}(I) = \{T(f) \mid f \in I\}$ be the semigroup ideal of the maximal terms of I with respect to (w.r.t.) $\prec$.

G is a Gröbner basis of I w.r.t. $\prec$ if and only if

$$\mathbf{T(I)} = \langle \mathbf{T\{G\}} \rangle.$$

The set of maximal terms of I is generated by the set of maximal terms of G.

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Applications of GB

- ⋆ Algebraic Geometry.
- ⋆ Coding Theory.
- ⋆ Criptography.
- ⋆ Differential Equations.
- ⋆ Integer Programming.
- ⋆ Statistics.

Return

Outline

GBLA: Gröbner bases by linear algebra

GBLA $\equiv$ FGLM techniques (in a more general sense).

Let $<$ be a fixed term ordering on $\langle X \rangle$, I a zero-dimenssional ideal.

$Span_K(N_<(I))$ is represented by

- a K-vector space E with an effective function

  LinearDependency$[v, \{v_1, \ldots, v_r\}]$

  $\{v_1, \ldots, v_r\} \subset E$ linear independent vectors

  $$\begin{cases} \{\lambda_1, \ldots, \lambda_r\} \subset K & \text{si } v = \sum_{i=1}^{r} \lambda_i v_i, \\ \textbf{False} & \text{if } v \text{ is not a linear combination of } \{v_1, \ldots, v_r\}. \end{cases}$$

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GBLA pattern algorithm

Input: $\prec$, a term ordering on $\langle X \rangle$;
$\xi : Span_K(N_<(I)) \mapsto E$, a suitable representation of $Span_K(N_<(I))$.

Output: $rGb(I, \prec)$.

$<$ could be equal to $\prec$.

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GBLA algorithm

$G := \emptyset$; $List := \{1\}$; $N := \emptyset$; $r := 0$;
While $List \neq \emptyset$ **do**
  $t :=$ **NextTerm**$[List]$;
  If $t \notin T_{\prec}(G)$ **then**
    $\mathbf{v} := \xi(\mathbf{Can}(\mathbf{t}, \mathbf{I}, <))$; $\Longleftarrow$
    $\mathbf{\Lambda} :=$ **LinearDependency**$[\mathbf{v}, \{\mathbf{v_1}, \ldots, \mathbf{v_r}\}]$; $\Longleftarrow$
    If False $\neq \Lambda$ **then** $G := G \cup \{t - \sum_{i=1}^{r} \lambda_i t_i\}$
     $(where\ \ \Lambda := (\lambda_1, \ldots, \lambda_r))$
    else $r := r + 1$
      $v_r := v$;
      $t_r := t$; $N := N \cup \{t_r\}$;
      $List :=$ **InsertNexts**$[t_r, List]$;
Return$[G]$.

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GBLA: Main objects

N: A set of representative elements for $K\langle X\rangle/I$.

Matphi (ϕ):

- ⋆ Allows to perform multiplication in N.
- ⋆ To reduce an element in $K\langle X\rangle$ to its representative in N **(Allows to solve the Word Problem)**.

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Other possible outputs

Gröbner representation: (N, ϕ), $N := \{h_1, \ldots, h_s\}$ s.t:

$$K\langle X\rangle / I \cong Span_K(N), \text{ and}$$

Matphi structure:

$$\phi(k): \quad N \times X \longrightarrow K$$

$$\forall_{i \in [1,s]} \quad (h_i x_k = \sum_{j=1}^{s} \phi(k)[h_i, x_j] h_j). \quad \text{(in the quotient)}$$

Border basis: $\mathcal{B}(I, \prec) := \{w - Can(w, I, \prec) \mid w \in B_\prec(I)\}$

$$B_\prec(I) = \{w \mid w \in T(I) \text{ and } \exists\, v \in N \text{ and } x \in X \text{ s.t. } w = vx\}$$

(the border of $T(I)$).

$\mathbf{N = N_\prec(I)}$ if $\prec$ is admissible.

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