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A characterization of sets realizable by compensation in the SNIEP

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A B S T R A C T

The symmetric nonnegative inverse eigenvalue problem (SNIEP) is the problem of characterizing all possible spectra of entry-wise nonnegative symmetric matrices of given dimension. A list of real numbers is said to be symmetrically realizable if it is the spectrum of some nonnegative symmetric matrix. One of the most general sufficient conditions for realizability is the so-called C-realizability, which amounts to some kind of compensation between the positive and negative entries of the list of real numbers whose realizability one is trying to decide. A combinatorial characterization of C-realizable lists with zero trace was given in [11]. In this paper we make use of a recursive method for constructing simmetrically realizable lists due to Ellard and Šmigoc [3] to extend this combinatorial characterization of C-realizability to general lists with nonnegative trace. One consequence of this characterization is that the set of nonnegative C-realizable lists is a union of polyhedral cones whose faces are described by equations involving only linear combinations with coefficients 1 and −1 of the entries in the list. Another remarkable consequence is the monotonicity of C-realizability, i.e., the

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operation of increasing any positive entry of a C-realizable list preserves C-realizability.

1. Introduction

A self-conjugate list of complex numbers is *realizable* if it is the spectrum of some entrywise nonnegative matrix. The *nonnegative inverse eigenvalue problem* is the problem of characterizing all possible realizable lists. If the list is real we have the *real nonnegative inverse eigenvalue problem* (hereafter RNIEP). A complete solution of these problems is known only for spectra of size $n \leq 4$ (see [10], [22], [8]) and of size $n = 5$ with trace 0 (see [9], [22]).

If in the RNIEP we require that the nonnegative matrix be symmetric, we have the *symmetric nonnegative inverse eigenvalue problem* (hereafter SNIEP). Both problems, RNIEP and SNIEP, are equivalent for $n \leq 4$ and are different (and remain both open) for $n \geq 5$ (see [7], [2], [8]). One of the most general sufficient conditions for the SNIEP follows from the *Soules approach* by means of the so-called Soules matrices, first introduced in [19], and later characterized by Elsner, Nabben and Neumann in [4]. The SNIEP for size $n = 5$ with trace 0 was solved by Spector [20].

Many different points of view have been adopted to find sufficient conditions for both the RNIEP and the SNIEP. In [12–14] the authors construct maps of several sufficient conditions for the RNIEP and SNIEP, respectively, in which they prove inclusion or independence relations between them. Two of the strongest sufficient conditions are (i) the so called *C- realizability* (see Definition 2.1 below), introduced by Borobia, Moro and Soto [1], which might be roughly summarized as *realizability by compensation*, and (ii) the sequence of *Soto p* conditions, due to Soto [17]. According to these maps, C-realizability and the union of all Soto p conditions contain as a particular case most of the sufficient conditions in the literature for the RNIEP.

Recently, Ellard and Šmigoc [3] extended the Soules approach to what they call *piecewise* Soules, and proposed a new recursive method to construct symmetrically realizable lists. The main contribution in [3] is proving the equivalence among the four sufficient conditions mentioned above: C-realizability, the union of all Soto p, piecewise Soules and the Ellard-Šmigoc method. As a consequence, C-realizability is shown to be a criterion of *symmetric* realizability. This turn of events is hardly uncommon in the area: several sufficient conditions which were first obtained for the RNIEP have later turned out to be sufficient conditions for the SNIEP as well.

From all of the above, it seems clear that the set of C-realizable lists is a reasonably large part of the set of all symmetrically realizable lists, and that understanding its structure and properties could shed some light on the problem of characterizing the full set of real realizable lists. The fact that this set can be approached from different points of view is an additional advantage when studying it. The main goal of this paper is to characterize the full set of C-realizable lists with arbitrary, nonnegative trace, using the Ellard-Šmigoc method for constructing symmetrically realizable lists. In particular, this set will be shown to be a union of polyhedral cones, with each of these cones given by an inequality which only involves linear combinations with coefficients 1 and -1 of the entries in the list under study.

The paper is organized as follows: Section 2 collects several concepts and results which will be needed to prove our main result: we define the Ellard-Šmigoc class of C-realizable lists, for instance, and we also concisely describe the characterization of C-realizability with zero trace given in [11]. In Section 3 we state and prove our main result, the characterization of C-realizable lists with arbitrary trace.

2. Preliminaries

2.1. C-realizability

In what follows, a *list* is a collection $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ of real numbers with possible repetitions. C-realizability is a kind of *realizability by compensation* based on the following three known results:

- *Rule 1*: Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a realizable list with $\lambda_1 \geq |\lambda|$ for $\lambda \in \Lambda$ and let $\epsilon > 0$. Then $\{\lambda_1 + \epsilon, \lambda_2, \dots, \lambda_n\}$ is also realizable.
- *Rule 2*: Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a realizable list with $\lambda_1 \geq |\lambda|$ for $\lambda \in \Lambda$ and let $\epsilon > 0$. Then $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \dots, \lambda_n\}$ is also realizable (see [6]).
- *Rule 3*: Let Λ_1 and Λ_2 be realizable lists. Then the list $\Lambda_1 \cup \Lambda_2$ is realizable.

This suggests considering three types of 'moves', transforming realizable lists into other realizable lists. Suppose, as above, that $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{R}$ is a realizable list with $\lambda_1 \geq |\lambda|$ for $\lambda \in \Lambda$, and let $\epsilon > 0$. We consider:

Move of type 1: $\Lambda \mapsto \{\lambda_1 + \epsilon, \lambda_2, \dots, \lambda_n\}$;
Move of type 2: $\Lambda \mapsto \{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \dots, \lambda_n\}$;

And, if Λ_1 and Λ_2 are realizable lists, the third type of move is just the union:

Move of type 3: $(\Lambda_1, \Lambda_2) \mapsto \Lambda_1 \cup \Lambda_2$.

Definition 2.1. (see [1]) Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a list of real numbers. We say that Λ is **C-realizable** if it can be reached starting from the n realizable lists

$$\{0\}\ \{0\}\ \dots\ \{0\} \tag{1}$$

and successively applying, in any order and any number of times, any of the moves of types 1, 2 or 3.

It is clear that any C-realizable list is, in particular, realizable, since the three types of move preserve realizability. However, not all realizable lists are C-realizable (see [1]). Nevertheless, C-realizability turns out to be one of the strongest sufficient conditions for the RNIEP, in the sense that it includes any other known sufficient condition except the ones given by Perfect in [15] (see [1,12,14]). Recall that, as mentioned in the Introduction, any C-realizable list is, in particular, realizable by a symmetric matrix.

2.2. Symmetric realization of Suleǐmanova lists with prescribed diagonal entries

One of the earliest, and most well-known results in the NIEP is the realizability of the so-called Suleǐmanova lists [21], namely, lists $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of real numbers with $\lambda_1 > 0 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\lambda_1 + \ldots + \lambda_n \geq 0$. We shall use the fact that such a list can be *symmetrically* realized with diagonal entries $d_j,\ j = 1, 2, \ldots, n$ if and only if the trace $d_1 + \ldots + d_n$ coincides with the sum $\lambda_1 + \ldots + \lambda_n$ of the eigenvalues. This was already proved in [18], but we include a new proof for its independent interest. It relies on the two following results, due to Fiedler, which give, respectively, necessary and sufficient conditions for a list of real numbers to be the spectrum of a symmetric nonnegative matrix with prescribed diagonal:

Theorem 2.2. (Fiedler [5], 1974) *If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are eigenvalues and $d_1 \geq d_2 \geq \cdots \geq d_n$ diagonal entries of an $n \times n$ nonnegative symmetric matrix then*

$$\lambda_1 \geq d_1, \qquad \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i$$

and

$$\sum_{i=1}^{s} \lambda_i + \lambda_k \geq \sum_{i=1}^{s-1} d_i + d_{k-1} + d_k$$

for all indices s, k such that $1 \leq s < k \leq n$.

Theorem 2.3. (Fiedler [5], 1974) *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ satisfy*

$$\sum_{i=1}^{s} \lambda_i \geq \sum_{i=1}^{s} d_i,\ s = 1, \cdots, n-1,$$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i,$$

$$\lambda_k \leq d_{k-1}, \quad k = 2, \cdots, n-1.$$

Then $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ *is realized by a symmetric nonnegative matrix with diagonal entries* $\{d_1, d_2, \cdots, d_n\}$.

The following definition is also needed:

Definition 2.4. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$ with their entries ordered decreasingly. We say that x **majorizes** y if

$$\sum_{j=1}^{k} x_j \geq \sum_{j=1}^{k} y_j, \quad k = 1, \ldots, n-1 \quad \text{and} \quad \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j.$$

Using this definition and the two results above one can prove the following:

Lemma 2.5. *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, *with* $\lambda_1 > 0 \geq \lambda_2 \geq \ldots \geq \lambda_n$, *and* $\Delta = \{d_1, d_2, \ldots, d_n\}$, *with* $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$. *Then the following statements are equivalent*

i) Λ *is symmetrically realizable with diagonal* Δ.

ii) Λ *majorizes* Δ.

iii) $\displaystyle\sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} d_j$.

Proof. *i)* $\Longrightarrow$ *ii)* The necessary conditions on Theorem 2.2 give $\lambda_1 \geq d_1$, $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i$. The third condition, for $k = s+1$, from $s = 1$ to $s = n-1$, leads to the remaining majorization conditions, since $\displaystyle\sum_{i=1}^{s} \lambda_i + \lambda_{s+1} \geq \sum_{i=1}^{s-1} d_i + d_s + d_{s+1}$ reads $\displaystyle\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} d_i$ for $k = 2, \ldots, n$.

Obviously *ii)* $\Longrightarrow$ *iii)*.

iii) $\Longrightarrow$ *i)* Since $\lambda_2, \ldots, \lambda_n$ are nonpositive, and $d_1, \ldots, d_n$ are nonnegative, then $\displaystyle\sum_{i=1}^{s} \lambda_i \geq \sum_{i=1}^{s} d_i$, $s = 1, \cdots, n-1$ and $\lambda_k \leq d_{k-1}$, $k = 2, \cdots, n-1$. Then, by the sufficient conditions in Theorem 2.3, Λ is symmetrically realizable with diagonal Δ. □

2.3. The Ellard-Šmigoc realization

Ellard and Šmigoc defined in [3] a class $\mathcal{H}_n$ of symmetrically realizable lists of real numbers by making the use of a list-patching construction previously introduced by Šmigoc in [16]. This class is defined recursively and involves, in principle, the diagonal entries of the realizing symmetric matrices: given a list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of real numbers with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and a collection $d_1, d_2, \ldots, d_n$ of nonnegative numbers, the list Λ is said to be in $\mathcal{H}_n(d_1, d_2, \dots, d_n)$ if it can be obtained through a certain recursive process (of no relevance to our discussion) starting from shorter realizable lists. In particular, Λ is realizable by a symmetric matrix with diagonal entries $d_1, d_2, \dots, d_n$, and we will simply say that Λ is *Ellard-Šmigoc realizable*, in short, *EŠ-realizable*. These EŠ-realizable lists also verify the three Rules that characterize C-realizability (see Observation 6.3.7, Lemma 6.3.9 and Theorem 6.3.9 in [3]). The class $\mathcal{H}_n$ is defined as the union of all classes $\mathcal{H}_n(d_1, d_2, \dots, d_n)$ for all $d_1, d_2, \dots, d_n \geq 0$. This set $\mathcal{H}_n$ is shown in [3] to coincide not only with the set of C-realizable lists of length n, but also with two other sets of symmetrically realizable lists, based on work by, respectively, Soto [17], and Soules [19] (see [3, Theorem 4.1]).

One of the constructions in [3] will be extremely useful for our purposes:

Theorem 2.6 *(Theorem 3.9 in [3]). Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{R}$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and let $d_1, d_2, \dots, d_n \geq 0$. Then $\Lambda \in \mathcal{H}_n(d_1, d_2, \dots, d_n)$ if and only if there exist*

$$0 \leq \epsilon \leq \frac{1}{2}(\lambda_1 - \lambda_2)$$

and two partitions

$$\{3, 4, \dots, n\} = \{p_1, p_2, \dots, p_{l-1}\} \cup \{q_1, q_2, \dots, q_{n-l-1}\}$$
$$\{1, 2, \dots, n\} = \{r_1, r_2, \dots, r_l\} \cup \{s_1, s_2, \dots, s_{n-l}\}$$

such that

$$\{\lambda_1 - \epsilon; \lambda_{p_1}, \lambda_{p_2}, \dots, \lambda_{p_{l-1}}\} \in \mathcal{H}_l(d_{r_1}, d_{r_2}, \dots, d_{r_l})$$

and

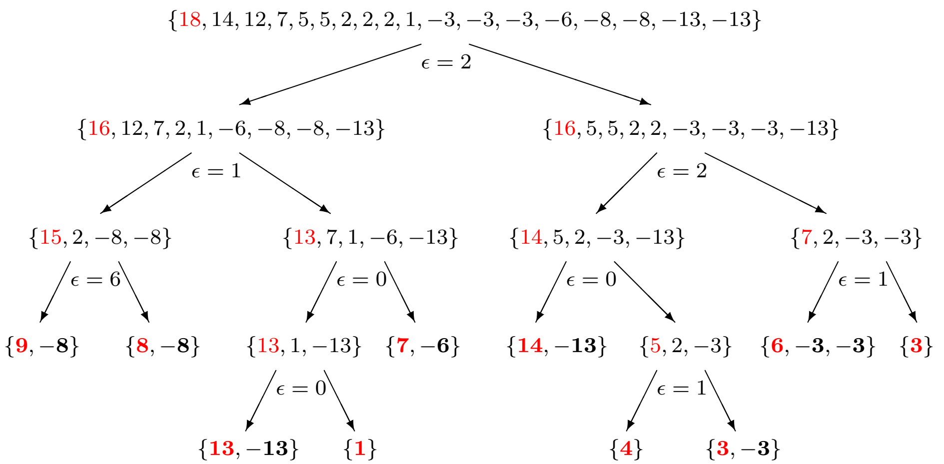
$$\{\lambda_2 + \epsilon; \lambda_{q_1}, \lambda_{q_2}, \dots, \lambda_{q_{n-l-1}}\} \in \mathcal{H}_{n-l}(d_{s_1}, d_{s_2}, \dots, d_{s_{n-l}})$$

We allow $l = 1$, in which case $\{p_1, p_2, \dots, p_{l-1}\}$ is the empty set, and $l = n-1$, in which case $\{q_1, q_2, \dots, q_{n-l-1}\}$ is the empty set.

Of course the splitting of Λ in Theorem 2.6 can be repeated, over and over again, on ever shorter lists as long as there are positive entries in them. This iterative construction gives rise to a binary rooted tree (henceforth called the *Ellard-Šmigoc tree*, in short, *EŠ-tree*), which can be associated to each list $\Lambda \in \mathcal{H}_n$: the vertices are the subsequent sublists into which the procedure splits the starting list Λ (the root vertex), and each pair of arcs at a given level connects an intermediate list in the process with the two sublists into which it is divided according to Theorem 2.6.

Although the original construction in [3] continues until all vertices at the bottom level are nonnegative singletons, we need not go that far for the time being: we shall stop the splitting as soon as either a Sulel̆manova list or a positive singleton is reached (more on this after the figure below). Later on, in Section 3, we will need the full construction, including the last level beyond the Sulel̆manova sets.

As an example, we present the following EŠ-tree, which shows an Ellard-Šmigoc symmetric realization of the spectrum $\Lambda = \{18, 14, 12, 7, 5, 5, 2, 2, 2, 1, -3, -3, -3, -6, -8, -8, -13, -13\}$. The Perron values of each sublist are distinguished in red and the leaves (vertices of outdegree zero) of the tree in boldface. (For interpretation of the colors, the reader is referred to the web version of this article.) In order to avoid excessive clutter, we have chosen not to include the diagonal entries of EŠ-realizing matrices at each vertex of the tree. They shall be discussed, however, whenever needed throughout the analysis that follows.



A few general remarks about EŠ-trees are in order: first notice that, for simplicity, we have chosen every leaf (i.e., terminal vertex) to contain exactly one positive entry: take for instance, the leftmost leaf $\{\mathbf{9}, -\mathbf{8}\}$ in the example above. The construction in [3] would produce still another level by splitting that list into two singletons $\{9 - \epsilon\}$ and $\{-8 + \epsilon\}$ with $8 \le \epsilon \le 17/2$, for example $\{1\}$ and $\{0\}$ using $\epsilon = 8$. Similarly, the $\{\mathbf{8}, -\mathbf{8}\}$ leaf would be split into two $\{0\}$ singletons, again using $\epsilon = 8$. In our case, we do not need to go that far, so we will simply skip these last splitting steps, which have no influence on realizability. In other words, each of the terminal vertices in our EŠ-tree will either be a positive singleton (i.e., a list with a single positive entry), or a Sulel̆manova list if there are additional (i.e., negative) entries. Consequently, in our EŠ-tree of a realizable list of the form $\{\lambda_1, \ldots, \lambda_n, -\mu_m, \ldots, -\mu_1\}$ with $n > m$ and all λ_i and μ_j positive, there will be at least $n - m$ singleton leaves, and at most m Sulel̆manova ones.

Notice also that one can easily follow the evolution of every individual entry of Λ throughout the splitting process described by the EŠ-tree: every time a new level is created, each entry is either transcribed identically, or shifted by a quantity $\pm\epsilon$, which makes predecessors immediately recognizable, even in the presence of coincident entries. More specifically, given any positive entry at one of the terminal vertices, we can backtrack along the tree, identifying the whole sequence of entries, one at each level of the tree, which ultimately leads to a positive original entry at the root vertex Λ. The negative elements of the original list Λ are scattered throughout the leaves of the EŠ-tree, remain unchanged throughout the process, and can be identified by simple backtracking.

Recall that, although not shown explicitly, every node in the tree has a list of diagonal entries associated with it. It is important to keep in mind that, as shown in Theorem 2.6 above, each of those lists is split in two every time two new nodes are created, but no individual entry in the list is actually modified throughout the whole process.

Notice, finally, that for any vertex, at any stage in the tree's splitting process, every entry in the list corresponding to that vertex is an original element of the initial list Λ, except, possibly, the leftmost entry in the list.

As an immediate consequence of these two remarks we obtain the following result:

Lemma 2.7. *Let $\Lambda = \{\lambda_1, \dots, \lambda_n, -\mu_m, \dots, -\mu_1\}$ be a list of real numbers with $n > m$ and all λ_i and μ_j positive. Then Λ is EŠ-realizable if and only if there exists a subset $\Lambda^+ = \{\lambda_1, \dots\} \subset \{\lambda_1, \dots, \lambda_n\}$ of cardinal at most m such that $\Lambda^+ \cup \{-\mu_m, \dots, -\mu_1\}$ is EŠ-realizable.*

Proof. By Observation 6.3.7 and Lemma 6.3.9 in [3], the 'if' part is trivially true. Now, suppose the list Λ is EŠ-realizable. According to [3], this is equivalent to Λ belonging to the class $\mathcal{H}_{n+m}$, i.e., to the existence of an EŠ-tree associated with Λ. This tree has at least $n - m$ singleton leaves, each containing one single positive number. We claim that for each such singleton leaf there is a positive entry in Λ which can be removed from Λ without the list losing its EŠ-realizability. We distinguish two cases:

(1) The singleton leaf is obtained as $\lambda_2 + \epsilon$, on the right branch of the corresponding split. In that case λ_2 is an original entry in Λ, so we just backtrack from the singleton leaf $\lambda_2 + \epsilon$, level by level up the tree, until the original entry λ_2 is reached at the tree's root vertex. Removal of this whole sequence from the tree, one λ_2 entry for each level, starting from the singleton $\lambda_2 + \epsilon$ at the bottom, up to the corresponding original entry λ_2 at the root vertex Λ, leads to another EŠ-tree, which corresponds to EŠ-realizing a list which differs from Λ in one single (positive) entry λ_2 (in the EŠ-tree above, for instance, consider the rightmost singleton $\{3\}$, where $3 = \lambda_2 + \epsilon$ for $\lambda_2 = 2,\ \epsilon = 1$: we remove the singleton and then backtrack from there, removing a 2 entry at each level in the right half of the tree. We are left with a tree which represents a valid realizing procedure for the list obtained from Λ by removing one 2 entry). As for the diagonal entries of EŠ-realizing matrices, the rightmost singleton $\lambda_2 + \epsilon$ will have a certain diagonal entry attached to it, which can only be $\lambda_2 + \epsilon$ itself, since the realizing matrix is 1×1. This means that all lists of diagonal entries above in the tree will contain at least one $\lambda_2 + \epsilon$ entry. Our backtracking procedure

deletes every one of those entries. In the example above, one of the 3 entries in each of the 'parent' diagonal entry lists would be deleted.

Notice that this does not remove the full positivity $\lambda_2 + \epsilon$ of the singleton from the vertex list, but just λ_2. On the other hand, the full positivity $\lambda_2 + \epsilon$ has been removed from the list of diagonal entries. In order to reconcile these two quantities, we notice that removing the singleton makes the splitting unnecessary, so whenever $\epsilon > 0$ the remaining positivity ϵ is transferred to a new list whose sum is increased by ϵ (for the singleton $\{3\}$ above, for instance, removal of the 2 entry from $\{7, 2, -3, -3\}$ leaves the Suleǐmanova list $\{7, -3, -3\}$, whose sum is $\epsilon = 1$, instead of the neutral Suleǐmanova list $\{6, -3, -3\}$ which appeared in the original, unmodified EŠ-tree). As for the diagonal entries, we increase by ϵ the one corresponding to the Perron root, since increasing the dominant entry of an EŠ-realizable list preserves EŠ-realizability, and Theorem 2.6 expresses a necessary and sufficient condition. This balances again the sum of the diagonal entry list with the sum of the new list.

(2) The singleton leaf is obtained as $\lambda_1 - \epsilon$, on the left branch of the corresponding split. In that case we cannot in general remove λ_1 or any of its parent vertices, since their positivity may be needed to realize (take, for instance, the singleton $\{4\}$ at the bottom of the EŠ-tree above: the corresponding parent entry 5 cannot be removed, since this would leave $\{2, -3\}$ on its own, which is not realizable). However, we can actually remove λ_2 and keep λ_1: more precisely,
   (i) first, we backtrack from the list $\{\lambda_2 + \epsilon, \ldots\}$ on the right of the split and remove from the EŠ-tree one λ_2 entry at each level above, including the one at the root vertex Λ;
   (ii) next, we remove from the tree both vertices $\{\lambda_1 - \epsilon\}$ and $\{\lambda_2 + \epsilon, \ldots\}$ obtained from the split.

   Thus, the list $\{\lambda_1, \lambda_2, \ldots\}$ before the split in the original EŠ-tree is replaced by the list $\{\lambda_1, \ldots\}$, with the dots denoting the exact same entries. Notice that the latter list is realizable, since we know $\{\lambda_2 + \epsilon, \ldots\}$ was, and $\lambda_1 \geq \lambda_1 - \epsilon \geq \lambda_2 + \epsilon$. Again, we arrive at a new EŠ-tree which describes a valid realizing procedure for the list obtained from Λ by removing its λ_2 entry (consider, for instance, the singleton $\{4\}$ at the bottom of the EŠ-tree above. The construction we just describe backtracks from the $\lambda_2 = 2$ entry at the vertex $\{5, 2, -3\}$ above it, removes one 2 entry at each level, and also removes the two leaves $\{4\}$ and $\{3, -3\}$. This amounts to deleting the last split and replacing the list $\{5, 2, -3\}$ in the original tree by $\{5, -3\}$ in the new one). As for the diagonal entries, the one associated with the singleton $\lambda_1 - \epsilon$ must be $\lambda_1 - \epsilon$ itself again. We just remove that diagonal entry from every list of diagonal entries in our backtracking route.

   Again, this does not remove the full positivity $\lambda_1 - \epsilon$ of the singleton, but just λ_2: the remaining positivity $\lambda_1 - \lambda_2 - \epsilon$ is the sum of the entries of a new Suleǐmanova list which replaces a neutral one from the original, unmodified EŠ-tree for Λ (in our example above with the singleton $\{4\}$, removing the 2 entry makes the last splitting unnecessary, which leads to the Suleǐmanova list $\{\mathbf{5}, \mathbf{-3}\}$ instead of the neutral one

$\{3, -3\}$ which was originally in the EŠ-tree). As for the list of diagonal entries, we increase by $\lambda_1 - \lambda_2 - \epsilon$ the diagonal entry corresponding to the dominant entry of the old Suleĭmanova list, and we can do this due to the same argument employed in case (1) above. □

Remark 2.8. As a consequence of the previous Lemma, the list $\{4, 1^p, -3, -3\}$, where the superindex $p \geq 2$ means multiplicity, cannot be C-realizable because the suppression of at least $p - 1$ positive elements leaves a list with negative trace. However, the list $\{4, 2, 1^{p-1}, -3, -3\}$ is C-realizable because the list $\{4, 2, -3, -3\}$ is.

Note that the original list $\{4, 1^p, -3, -3\}$ is realizable, since the sublist $\{4, 1, 1, -3, -3\}$ has trace zero and satisfies Spector's characterization (see [20]).

Observe that in Theorem 2.6 and in the above Remark, EŠ-realizable can be changed to C-realizable, and reciprocally, since we know from [3, Theorem 4.1] that they are equivalent conditions.

2.4. C-realization with zero trace

C-realizable sets were combinatorially characterized in [11] under the restriction of having zero sum. The two main concepts to describe such a characterization are those of *partition* and *nested bracket structure*. These two ingredients completely characterize each of the possible realizing procedures which allow us to reach a C-realizable list Λ starting from zero singletons and performing a sequence of moves of types 2 and 3 as described in §2.1 above (moves of type 1 are not needed when the trace is zero). The only lists we have to consider in the zero-trace case (see §2 in [11]) are the so-called T_0-admissible ones:

Definition 2.9. We say that a list

$$\Lambda = \{\lambda_1, \ldots, \lambda_n, -\mu_m, \ldots, -\mu_1\},$$

of real numbers is **$\mathbf{T_0}$-admissible** if $n \leq m$, $\lambda_1 \geq \ldots \geq \lambda_n > 0$, $\lambda_1 \geq \mu_1 \geq \ldots \geq \mu_m > 0$, and

$$\sum_{i=1}^{n} \lambda_i = \sum_{j=1}^{n} \mu_j. \tag{2}$$

Given a T_0-admissible list $\Lambda = \{\lambda_1, \ldots, \lambda_n, -\mu_m, \ldots, -\mu_1\}$, a **partition** of Λ is any splitting

$$\Lambda = \Lambda_1^{(0)} \cup \ldots \cup \Lambda_n^{(0)} \tag{3}$$

of Λ into a disjoint union of n lists, where each $\Lambda_j^{(0)}$ contains exactly one positive entry λ_j.

The sublists $\Lambda_j^{(0)}$ in (3) can be considered as the ultimate targets to be reached by the C-realizing procedure represented using that specific partition. The approximation to Λ is made gradually, as the procedure advances, producing new and longer sublists with zero sum, which get closer and closer to the target lists. We call *quasi C-realizations (QCRs)* those intermediate lists obtained in the C-realizing procedure after each move of type 2, since they are transient steps in the process of reaching the target lists.

The whole process can be interpreted as one of positivity transferences: each $\Lambda_j^{(0)}$ in the partition (3) is said to be positive or negative, according to the sum of its entries (sublists with zero sum can be disregarded, since they can be C-realized on their own), and every C-realizing procedure can be thought of as a process where the positive sublists in (3) gradually lend their positivity, step by step, to the negative ones, until everything balances out in the final zero-sum list Λ.

The process starts from the so-called *QCRs at level* 0, each of which is a zero-sum approximation of one of the sublists $\Lambda_j^{(0)}$, $j = 1, 2, \dots, n$, in the partition (3). By extension, each of the QCRs at level 0 is said to have the same sign, positive or negative, as the sublist $\Lambda_j^{(0)}$ it comes from. Each QCR at level 0 is constructed in one of two different ways, depending on the sign of the corresponding sublist $\Lambda_j^{(0)}$:

(i) if $\Lambda_j^{(0)}$ is positive, the corresponding QCR contains all negative entries of $\Lambda_j^{(0)}$, and its only positive entry equals minus the sum of its negative entries. Take, for instance, $\Lambda_1^{(0)} = \{7, -2, -3\}$, which is positive. The corresponding QCR at level 0 would be $\{5, -2, -3\}$.

(ii) when $\Lambda_j^{(0)}$ is negative, the QCR contains the same positive entry as $\Lambda_j^{(0)}$, with some of its negative entries modified so that the sublist has zero sum. Now, consider $\Lambda_2^{(0)} = \{4, -2, -3\}$, which is negative. Then the QCR at level 0 is $\{4, -2, -2\}$ (it might equally be $\{4, -1, -3\}$).

Once the QCRs at level 0 are known, the C-realizing procedure leading to Λ consists of a sequence of consecutive stages, producing longer and longer QCRs, ever closer to their corresponding target lists. Each of these stages can be divided in two steps:

(1) perform a move of type 3, merging one positive QCR with one or more negative ones, which leads to new, longer sublists; then

(2) perform a move of type 2 on the merged sublist obtained in step (1) above. This amounts to transferring a certain amount of positivity from the positive QCR chosen in step (1) into the negative ones. This produces a new, longer QCR, whose sign is defined as the sign of the sum of entries of the result of merging the target lists associated with the QCRs in step (1). Again, depending on the sign of the new QCRs, the move of type 2 is done either as in (i) or in (ii) above.

These two steps are repeated over and over again until a final, single QCR is obtained which coincides with Λ.

The **nested bracket structure** corresponding to this C-realizing procedure encodes all the moves of type 3 performed by the procedure. Instead of a formal definition, it may be better to illustrate the concept of nested bracket structure with a specific example: consider the C-realizable list

$$\Lambda = \{16,\ 13,\ 10,\ 10,\ 4,\ -2,\ -6,\ -6,\ -6,\ -9,\ -12,\ -12\},$$

which we choose to partition as

$$\begin{gathered}\Lambda_1^{(0)} = \{16,\ -12\}, \quad \Lambda_2^{(0)} = \{13,\ -12\}, \quad \Lambda_3^{(0)} = \{10,\ -2,\ -9\}, \\ \Lambda_4^{(0)} = \{10,\ -6,\ -6\}, \quad \Lambda_5^{(0)} = \{4,\ -6\},\end{gathered} \tag{4}$$

i.e., $\Lambda_1^{(0)}$ and $\Lambda_2^{(0)}$ are positive, while the remaining sublists in the partition are negative. To represent the C-realizing procedure we use the rooted tree in Fig. 1 below: each vertex in the tree is one of the intermediate sublists in the realizing procedure. A vertex u is adjacent to a vertex v whenever v is obtained from u via a move of type either 2 or 3 (recall that, since our lists have zero sum, no move of type 1 can be performed). Moves of type 2 are represented with vertical edges, while moves of type 3 are with oblique ones. Numbers in red indicate that the desired final value has not yet been reached, while numbers in black indicate that it has (see [11, §3.1] for more details on this tree representation).

In order to describe the nested bracket structure associated with the C-realizing procedure above, we begin by observing that the first round of merging, once the QCRs at level 0 are obtained, involves, on the one hand, the first, third and fifth sublists in the partition (4). This means that

$$\Lambda_1^{(1)} = \Lambda_1^{(0)} \cup \Lambda_3^{(0)} \cup \Lambda_5^{(0)},$$

where we write $\Lambda_1^{(1)}$ in blue because the positivity $+4$ of $\Lambda_1^{(0)}$ outweighs the negativity $-2-1=-3$ brought by $\Lambda_3^{(0)}$ and $\Lambda_5^{(0)}$ to the union. Thus, the leftmost bracket in the nested bracket structure associated with this C-realizing procedure is

$$[\,1,\ 3, 5\,]$$

(notice the blue outer brackets indicating that the entries in $\Lambda_1^{(1)}$ have positive sum). Similarly, the second merging in the fourth line joins the QCRs corresponding to the second and fourth sublists in the partition, i.e.,

$$\Lambda_2^{(1)} = \Lambda_2^{(0)} \cup \Lambda_4^{(0)},$$

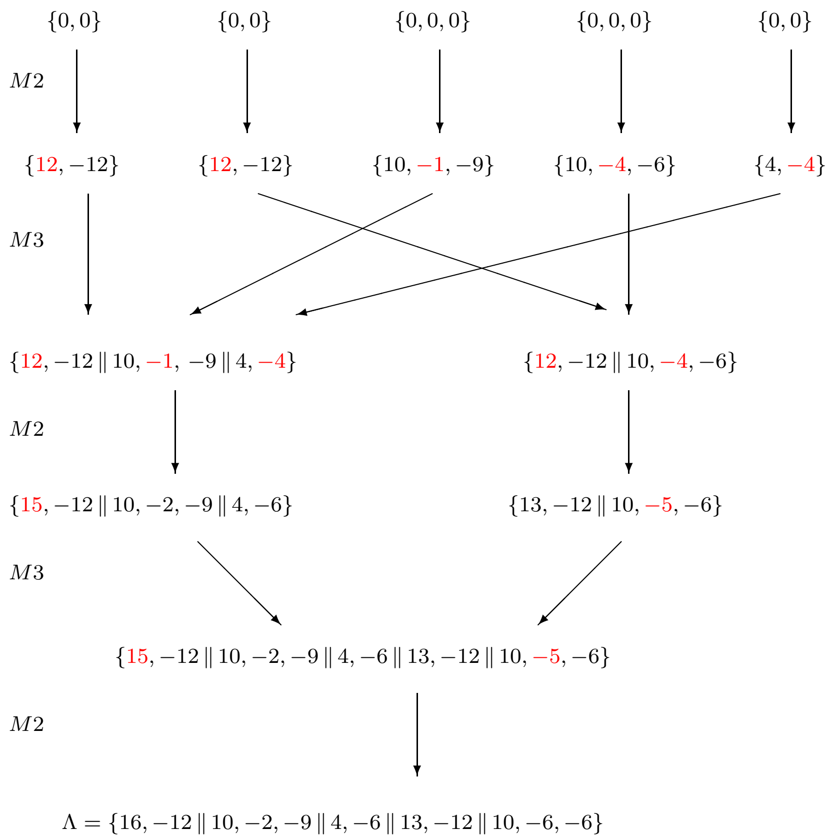


Fig. 1. Rooted tree associated with the C-realization procedure.

where we write $\Lambda_2^{(1)}$ in red because the positivity $+1$ of $\Lambda_2^{(0)}$ is smaller than the negativity -2 of $\Lambda_4^{(0)}$. Therefore, the first level of brackets in the nested bracket structure is given by the two brackets

$$[1, 3, 5], [2, 4]$$

(notice the red outer brackets in the rightmost one).

Once the first round of mergings is done, and appropriate moves of type 2 are performed, we proceed again to merge QCRs via moves of type 3. In this case there are only two remaining QCRs to be joined, leading to the desired zero-sum list Λ. Therefore, the second level of brackets gives us the complete nested bracket structure corresponding to the C-realizing procedure above, which is

$$[[1, 3, 5], [2, 4]].$$

The two outermost brackets in black indicate that the full list Λ is neutral (i.e., has zero sum). In general, any nested bracket structure will have a pair of black outer brackets, with several colored brackets inside. Each of these colored brackets includes one blue and, possibly, one or more red brackets. For more details on nested bracket structures, see [11, §3.3].

One can prove (see [11, Theorem 3.1]) that any procedure C-realizing a T_0-admissible list Λ can be uniquely represented via a rooted tree as above, and that every such tree is in turn completely described by two objects, namely a partition of Λ, together with the nested bracket structure associated with the C-realizing procedure. Furthermore, each nested bracket structure imposes a set of explicit conditions on the entries of Λ which are necessary (and sufficient) for the C-realizing procedure to be fully executed according to the moves described in the tree. More specifically, each vertex in the tree imposes one so-called *sign condition*, while each new positive bracket with more than one entry imposes a so-called *merging condition* (see [11, §3.4] for more details on these conditions).

Using these ingredients, we may state the main result in [11, Theorem 3.1], which combinatorially characterizes the zero-sum C-realizable lists of real numbers:

Theorem 2.10 *(Theorem 5.1 in [11]). Let Λ be a T_0-admissible list. Then Λ is C-realizable if and only if there exists a partition (3) of Λ and a nested bracket structure such that the entries in Λ satisfy all sign and merging conditions associated with that nested bracket structure.*

With regard to our example above, for instance, let

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 > 0 > -\mu_7 \geq -\mu_6 \geq -\mu_5 \geq -\mu_4 \geq -\mu_3 \geq -\mu_2 \geq -\mu_1 \tag{5}$$

be the entries of any list of real numbers with five positive and seven negative entries, partitioned in the same way as Λ, i.e.,

$$\Lambda_1^{(0)} = \{\lambda_1, -\mu_1\}, \quad \Lambda_2^{(0)} = \{\lambda_2, -\mu_2\}, \quad \Lambda_3^{(0)} = \{\lambda_3, -\mu_7, -\mu_3\},$$

$$\Lambda_4^{(0)} = \{\lambda_4, -\mu_4, -\mu_5\}, \quad \Lambda_5^{(0)} = \{\lambda_5, -\mu_6\}.$$

Then the C-realizing procedure shown in Fig. 1 above realizes the list (5) provided that the *sign conditions*

$$\lambda_1 - \mu_1 \geq 0, \quad \lambda_2 - \mu_2 \geq 0, \qquad \lambda_3 - \mu_7 - \mu_3 \leq 0$$

$$\lambda_4 - \mu_4 - \mu_5 \leq 0, \quad \lambda_5 - \mu_6 \leq 0,$$

corresponding to this partition, the merging condition

$$\mu_2 \geq \lambda_3$$

associated with the positive bracket $[1, 3, 5]$, and the two sign conditions

$$\lambda_1 + \lambda_3 + \lambda_5 - \mu_1 - \mu_3 - \mu_6 - \mu_7 \geq 0,$$

associated with [1, 3, 5], and

$$\lambda_2 + \lambda_4 - \mu_2 - \mu_4 - \mu_5 \leq 0,$$

associated with [2, 4], are satisfied.

Notice that both the sign and merging conditions above have a very particular structure: both are inequalities involving linear combinations of the absolute values of the entries in Λ with *all* coefficients equal either to 1 or to -1. This is not a coincidence, and is equally true for any other set of conditions associated with any nested bracket structure. Hence, for every particular C-realizing procedure, the set of T_0-admissible lists satisfying the sign and merging conditions imposed by that procedure (equivalently, by the corresponding nested bracket structure) is a polyhedral cone of quite a special kind. To be more specific, it is a set of the form $\{x : Ax \leq 0\}$, where the matrix A has dimension $p \times (n+m)$, with p being the number of both sign and merging conditions, and the point $x = (\lambda_1, \cdots, \lambda_n, -\mu_1, \cdots, -\mu_m)$ in $\mathbb{R}_+^{m+n}$ corresponds to the T_0-admissible list $\Lambda = \{\lambda_1, \ldots, \lambda_n, -\mu_m, \ldots, -\mu_1\}$. A very special property of A is that the nonzero entries on each of its rows are constant, equal either to 1 or to -1.

With regard to our example above, for instance, A is the 8×12 matrix

$$A = \left(\begin{array}{ccccc|ccccccc}
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{array}\right).$$

Consequently, the whole set of zero-sum C-realizable real lists is a union of such polyhedral cones. As will be seen below, this geometric property can be extended (in an affine sense) to the set of all real C-realizable lists (see Theorem 3.4 and Remark 3.6 below).

3. Main result

We first define the appropriate type of lists we shall characterize as C-realizable. By analogy with Definition 2.9,

Definition 3.1. We say that a list

$$\Lambda = \{\lambda_1, \ldots, \lambda_n, -\mu_m, \ldots, -\mu_1\},$$

of real numbers is **admissible** if $\lambda_1 \geq \ldots \geq \lambda_n > 0$, $\lambda_1 \geq \mu_1 \geq \ldots \geq \mu_m > 0$, and

$$\sum_{i=1}^{n} \lambda_i \geq \sum_{j=1}^{n} \mu_j. \tag{6}$$

We already know from Lemma 2.7 that, even if the number n of positive entries in such a list Λ exceeds the number m of its negative entries, we may replace Λ by a shorter list, with at most m of Λ's original positive entries and all its n original negative entries. Furthermore, this latter list is C-realizable if and only if Λ is. What we are going to see is that an admissible list is C-realizable if and only if this modified, shorter list can be shifted to a T_0-admissible list satisfying the conditions of Theorem 2.10.

Before we state the main result of this paper we need some definitions:

Definition 3.2. Let $\Lambda = \{\lambda_1, \ldots, \lambda_n, -\mu_m, \ldots, -\mu_1\}$ be an admissible list. A **pruning** of Λ is any list

$$\Lambda' = \Lambda^+ \cup \{-\mu_m, \ldots, -\mu_1\},$$

where $\Lambda^+ = \{\lambda_1, \ldots\}$ is a sublist of $\{\lambda_1, \ldots, \lambda_n\}$. The pruning is said to be **feasible** if the cardinal of Λ^+ is less than or equal to m.

For instance, given $\Lambda = \{12, 6, 5, 3, 2, 1, -3, -5, -8\}$, the sublist $\Lambda_1 = \{12, 6, 5, 2, -3, -5, -8\}$ is a non-feasible pruning of Λ, while $\Lambda_2 = \{12, 6, 2, -3, -5, -8\}$ is a feasible one.

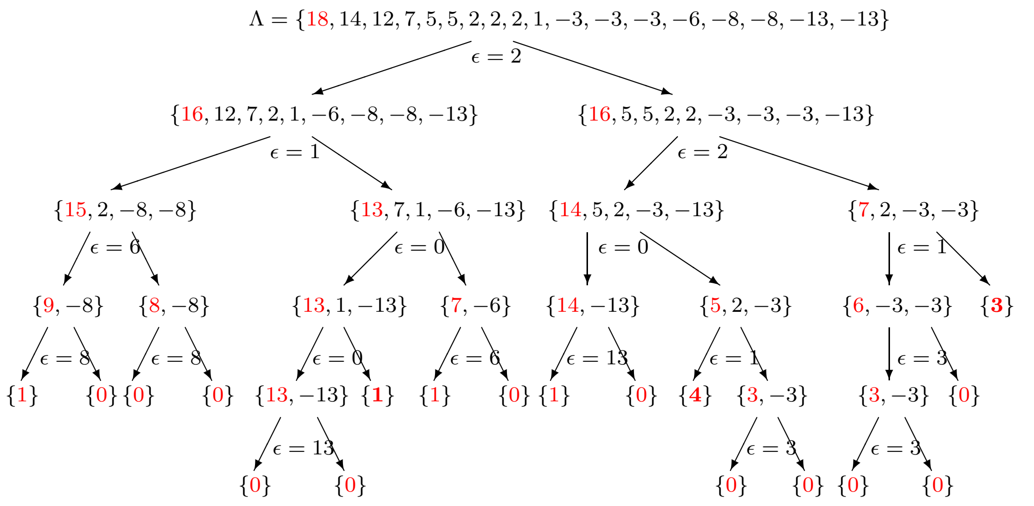
With this new concept, Lemma 2.7 can be summarized by saying that an admissible list is EŠ-realizable if and only if it has an EŠ-realizable pruning which is feasible. If that feasible pruning happens to have zero trace, since EŠ-Realizable and C-realizable are equivalent, Theorem 2.10 readily applies. We will show that even if that is not the case, the positive entries in the feasible pruning can be shifted in such a way that a zero-trace list is obtained while preserving ES-realizability.

Definition 3.3. Let $\Lambda = \{\lambda_1, \ldots, \lambda_n, -\mu_m, \ldots, -\mu_1\}$ and $\widetilde{\Lambda} = \{\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_n, -\mu_m, \ldots, -\mu_1\}$, with $\lambda_i \geq \widetilde{\lambda}_i$ for $i = 1, \ldots, n$, be admissible lists. Then, we say that $\widetilde{\Lambda}$ is a **downward shift** of Λ, and that Λ is an **upward shift** of $\widetilde{\Lambda}$.

Before we state our main result, let us illustrate the two concepts defined above with a couple of auxiliary EŠ-trees, all of them related with the example shown in §2.3: we start with the full EŠ-tree associated with the list

$$\Lambda = \{18, 14, 12, 7, 5, 5, 2, 2, 2, 1, -3, -3, -3, -6, -8, -8, -13, -13\},$$

extended to include the last, bottom level, with all terminal nodes of the tree being nonnegative singletons. This is just a trivial extension of the tree already shown in §2.3, but it will be useful to explain certain aspects in the proof of our main result:

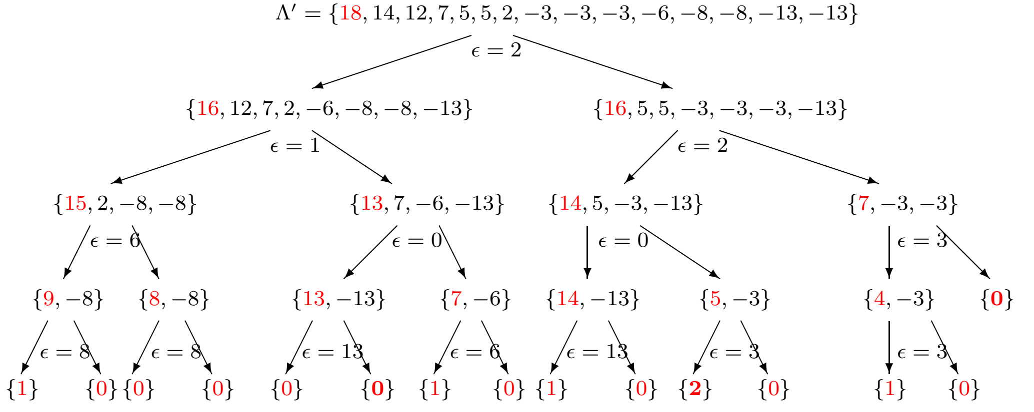


Tree #1

According to Lemma 2.7, this tree can be transformed into another EŠ-realizing tree, associated with the pruning

$$\Lambda' = \{18, 14, 12, 7, 5, 5, 2, -3, -3, -3, -6, -8, -8, -13, -13\}$$

of Λ. The corresponding ES-tree is as follows:

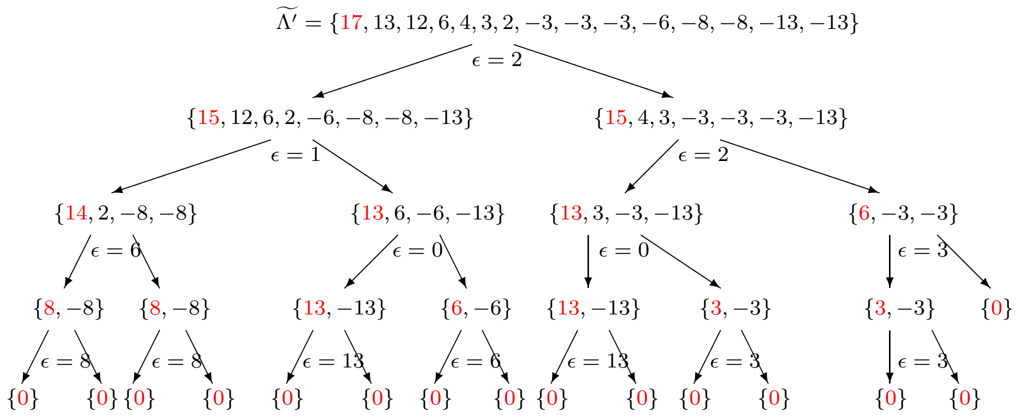


Tree #2

Our main result (Theorem 3.4 below) will show that one can further transform the EŠ-tree above into a third tree, associated with a zero-trace downward shift

$$\widetilde{\Lambda}' = \{17, 13, 12, 6, 4, 3, 2, -3, -3, -3, -6, -8, -8, -13, -13\}$$

of Λ' above. The corresponding EŠ-tree is as follows:



Tree #3

We are now in the position to state and prove our main result:

Theorem 3.4. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n, -\mu_m, \ldots, -\mu_1\}$ be an admissible list. Then Λ is C-realizable if and only if there exists a feasible pruning Λ' which admits a C-realizable downward shift $\widetilde{\Lambda'}$ with zero sum.*

Proof: First, if $n > m$ we know from Lemma 2.7 that Λ is C-realizable if and only if it has a C-realizable feasible pruning. Thus, from now on we may assume that the pruning has already been performed and that $n \leq m$ in the statement of the theorem. In the example above, this would amount to replacing Λ by Λ' and, consequently, Tree #1 by Tree #2.

Now, we know that Λ is C-realizable if and only if it is EŠ-realizable. Consequently, there exists an EŠ-tree associated to Λ. Since we are assuming that the pruning has already been performed, the EŠ-tree would resemble Tree #2 above.

First, let us show that if a given pruned list is EŠ-realizable, then we can construct a downward shift with zero trace from it which is also EŠ-realizable. Notice that once pruning is done, whatever positivity is left in the trace must be stored in the Suleǐmanova leaves, either the ones originally in the EŠ-tree for Λ or the ones created in the process of pruning (see the proof of Lemma 2.7). To achieve a global zero sum we just (i) replace the unique positive entry at each Suleǐmanova leaf by minus the sum of the negative entries in the leaf, and (ii) backtrack up the EŠ-tree, and accordingly decrease the positive entry obtained from backtracking at every previous level of the tree. In Tree #2 above, for instance, the Suleǐmanova leaf $\{9, -8\}$ is transformed into $\{8, -8\}$, which in turn lowers 15 to 14 in $\{14, 2, -8 - 8\}$ at the level above, then lowers 16 to 15 at the level immediately above that, and finally lowers 18 to 17 at the root of the tree. Notice that, since Theorem 2.6 is a necessary and sufficient condition, and EŠ-realizability is guaranteed after every change in the lists; every change at every level induces a corresponding change in the associated list of diagonal entries.

The same procedure can be repeated on every Suleĭmanova list with positive sum until the sum of the entries of the list at the root of the tree is zero. Therefore, we arrive at a downward shift $\widetilde{\Lambda}'$ of Λ' with zero trace.

Conversely, let us show that the process is reversible, i.e., that we may start at the EŠ-tree of a zero-trace downward shift $\widetilde{\Lambda}'$ and construct from it the EŠ-tree of its upward shift Λ': first, notice that whatever positivity we may pump into the system will be done through one of the zero *leftmost* singleton leaves at the bottom level of the EŠ-tree for $\widetilde{\Lambda}'$, since rightmost positive singleton leaves have already been exhausted at the pruning stage.

In order to choose which leftmost singleton to increase, one just has to follow the entry one wants to change down the tree and increase the appropriate zero singleton (in Tree #3, for instance, we need to increase the entry 13 to 14: the 13 entry at the top of Tree #3 is transformed into 15 using $\epsilon = 2$ in the first application of Theorem 2.6. The subdivision process goes down to the Suleĭmanova list $\{13, -13\}$, on the fifth branch (out of eight) from the left at the second-lowest level of the tree. Thus, to increase 13 to 14, all we have to do is to increase the leftmost singleton below $\{13, -13\}$ from 0 to 1). Once all appropriate leftmost zero singletons have been conveniently increased, a new tree is constructed, which is a valid EŠ-tree for Λ', which is an upward shift of $\widetilde{\Lambda}'$. As for the diagonal entries, there is no difference between backtracking and forward tracking, since Theorem 2.6 is a necessary and sufficient condition. □

Remark 3.5. Although the three Trees #1 − #3 were useful to illustrate the proof of our main result, the simplified tree in §2.3 is probably better to illustrate the process described in Theorem 3.4: we start with the admissible list

$$\Lambda = \{18, 14, 12, 7, 5, 5, 2, 2, 2, 1, -3, -3, -3, -6, -8, -8, -13, -13\},$$

which has trace 11. Removal of the three singletons, $\{1\}$, $\{4\}$ and $\{3\}$ in the EŠ-tree leads to the feasible pruning

$$\Lambda' = \{18, 14, 12, 7, 5, 5, 2, -3, -3, -3, -6, -8, -8, -13, -13\}$$

of Λ, which has trace 6. Recall that, as explained in the proof of Lemma 2.7, the removal of the singletons may produce additional Suleĭmanova lists with positive sum. Removal of the $\{1\}$ singleton produces no such list, since $\epsilon = 0$, but removing the $\{3\}$ singleton gives rise to $\{7, -3, -3\}$, with sum one, instead of the neutral $\{6, -3, -3\}$, while removing the $\{4\}$ singleton produces a new list $\{5, -3\}$, with sum two, replacing the neutral list $\{5, 2, -3\}$.

Next, one removes the positivity corresponding to the five Suleĭmanova leaves with positive sum (the three ones in the original EŠ-tree plus the two ones created while pruning): we have already seen in the proof of Theorem 3.4 that removing the positivity from the Suleǐmanova leaf $\{5, 2, -3\}$ lowers the 18 entry to 17 in Λ'. Similarly, the backtracking associated with replacing the positive Suleǐmanova leaf $\{7, -6\}$ by $\{6, -6\}$ transforms the 7 entry in Λ' into a 6. Lowering $\{14, -13\}$ to $\{13, -13\}$ and the subsequent backtracking decreases the 14 entry in Λ' to 13. Replacement of the Suleǐmanova leaf $\{5, -3\}$ (which was created in the pruning phase) by the neutral one $\{3, -3\}$ decreases one of the 5 entries in Λ' to 3. Finally, removing the +1 positivity from the Suleǐmanova leaf $\{7, -3, -3\}$, also created in the process of pruning, leads to replacing the second 5 entry of Λ' by 4. All these operations lead therefore, ultimately, to the downward shift

$$\widetilde{\Lambda}' = \{17, 13, 12, 6, 4, 3, 2, -3, -3, -3, -6, -8, -8, -13, -13\}$$

of Λ', which has zero sum. Each of the three lists, Λ, Λ' and $\widetilde{\Lambda}$, is both EŠ- and C-realizable, since each one has a corresponding valid EŠ-tree.

Remark 3.6. Since C-realizable lists with zero sum have been combinatorially characterized in Theorem 2.10, arbitrary C-realizable lists may be characterized similarly, as those admissible lists of the form of Definition 3.1 allowing for a feasible pruning, which in turn admits a downward shift with zero sum satisfying the conditions of Theorem 2.10. Since downward-shifting means to decrease positive elements of the admissible list and pruning amounts just to disregard some of the entries in the list, we may conclude that the set of C-realizable lists is the set of positive expansions[3] of translations, by means of nonnegative vectors, of unions of polyhedral cones, with each cone given by an inequality involving only linear combinations, with coefficients either 1 or -1, of the absolute values of the entries in the list (as described at the end of subsection 2.4).

In our example above, for instance, the list

$$\Lambda = \{18, 14, 12, 7, 5, 5, 2, 2, 2, 1, -3, -3, -3, -6, -8, -8, -13, -13\}$$

is a positive expansion of the list $\Lambda' = \{18, 14, 12, 7, 5, 5, 2, -3, -3, -3, -6, -8, -8, -13, -13\}$ which is in turn a translation, by means of the nonnegative vector $(1, 1, 0, 1, 1, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0)$, of the zero-trace list $\widetilde{\Lambda}' = \{17, 13, 12, 6, 4, 3, 2, -3, -3, -3, -6, -8, -8, -13, -13\}$.

As for the zero-trace example in subsection 2.4, if we rename the T_0-admissible list as

$$\widetilde{\Sigma'} = \{16,\ 13,\ 10,\ 10,\ 4,\ -2,\ -6,\ -6,\ -6,\ -9,\ -12,\ -12\} = \{\lambda_1, \ldots, \lambda_5, -\mu_7, \ldots, -\mu_1\},$$

then any admissible list

$$\Sigma = \{\widetilde{\lambda_1}, \ldots, \widetilde{\lambda_q}, -\mu_7, \ldots, \mu_1\}$$

[3] By positive expansion we just mean a new list with some positive entries added.

with $q \geq 5$, $\widetilde{\lambda}_j > 0$ for $j = 1, \ldots, q$, and $\widetilde{\lambda}_i = \lambda_i + \delta_i$, $\delta_i \geq 0$, $i = 1, \ldots, 5$, is C-realizable (obviously, Σ is a positive expansion of $\Sigma' = \{\widetilde{\lambda_1}, \ldots, \widetilde{\lambda_5}, -\mu_7, \ldots, \mu_1\}$, who in turn is an upward shift of $\widetilde{\Sigma'}$).

The arguments outline in Remark 3.6 have one remarkable consequence, which to the authors' knowledge is not present in any other context of the NIEP. Arguments so far make it clear that if a list is C-realizable, then any of its positive entries can be slightly increased without ever losing C-realizability. In particular, this operation does not change the EŠ-tree. In fact, one can show that even if the increase of the positive entries is large enough to disturb their relative order, C-realizability is still preserved. This is detailed in the following result:

Theorem 3.7. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n, -\mu_m, \ldots, -\mu_1\}$ be an admissible list and let $\Delta = (\delta_1, \ldots, \delta_n)$ be an entrywise nonnegative vector. If Λ is C-realizable (EŠ-realizable), then $\Lambda + \Delta$, the non-increasing reordering of $\{\lambda_1 + \delta_1, \ldots, \lambda_n + \delta_n, -\mu_m, \ldots, -\mu_1\}$, is also C-realizable (EŠ-realizable).*

Proof. For the sake of simplicity, suppose only one positive entry λ_i of Λ is increased up to $\widetilde{\lambda}_i = \lambda_i + \delta_i$ (the general case follows from repeating the argument as many times as needed). We re-arrange the new λs in non-increasing order to make the new list Λ' admissible. The key to the proof is in showing that any such re-ordering is equivalent to an appropriate upward shift. Two possibilities have to be explored:

(i) the new $\widetilde{\lambda}_i$ does not exceed the Perron root λ_1. Suppose, for instance, that $\lambda_k \geq \widetilde{\lambda}_i > \lambda_{k+1} \geq \lambda_i$ for a certain index $k < i$. Then we go from the list

$$\Lambda = \{\lambda_1, \cdots, \lambda_k, \lambda_{k+1}, \lambda_{k+2}, \cdots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \cdots, \lambda_n\}$$

to the list

$$\Lambda' = \{\lambda_1, \cdots, \lambda_k, \widetilde{\lambda}_i, \lambda_{k+1}, \cdots, \lambda_{i-2}, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n\}$$

which is just an upward shift of Λ by means of the entrywise nonnegative vector

$$(0, \cdots, 0, \widetilde{\lambda}_i - \lambda_{k+1}, \lambda_{k+1} - \lambda_{k+2}, \cdots, \lambda_{i-2} - \lambda_{i-1}, \lambda_{i-1} - \lambda_i, 0, \cdots, 0);$$

(ii) the new $\widetilde{\lambda}_i$ exceeds the Perron root λ_1. Then we go from the list

$$\Lambda = \{\lambda_1, \cdots, \lambda_{i-2}, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \cdots, \lambda_n\}$$

to the list

$$\Lambda' = \{\widetilde{\lambda}_i, \lambda_1, \cdots, \lambda_{i-2}, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n\}$$

which is just an upward shift of Λ by means of the entrywise nonnegative vector

$$(\tilde{\lambda_i} - \lambda_1, \lambda_1 - \lambda_2, \cdots, \lambda_{i-2} - \lambda_{i-1}, \lambda_{i-1} - \lambda_i, 0, \cdots, 0).$$

In either of the two cases above, the new list Λ' is just an upward shift of Λ. According to Theorem 3.4, if Λ admits a feasible pruning, which in turn has a C-realizable downward shift with zero sum, then so does Λ': the pruning is the same, and the downward shift is just the sum of the shift from Λ' and the one from Λ. □

We end by noting that monotonicity is true not only when some of the positive entries are increased, but also when new positive entries are added to the list (these, of course, can always be realized separately on their own).

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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