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DIAGONALIZABLY REALIZABLE IMPLIES UNIVERSALLY REALIZABLE*

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Abstract. A spectrum $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ of complex numbers is said to be *realizable* if it is the spectrum of an entrywise nonnegative matrix A. The spectrum Λ is *diagonalizably realizable* ($\mathcal{DR}$) if the realizing matrix A is diagonalizable, and Λ is *universally realizable* ($\mathcal{UR}$) if it is realizable for each possible Jordan canonical form allowed by Λ. In 1981, Minc proved that if Λ is the spectrum of a diagonalizable positive matrix, then Λ is universally realizable. One of the main open questions about the problem of universal realizability of spectra is whether $\mathcal{DR}$ implies $\mathcal{UR}$. Here, we prove a surprisingly simple result, which shows how diagonalizably realizable implies universally realizable.

Key words. Spectra diagonalizably realizable, Spectra universally realizable, Nonnegative matrices, Jordan structure.

AMS subject classifications. 15A18, 15A29.

1. Introduction. The *Nonnegative Inverse Eigenvalue Problem* (NIEP) consists of: Given a list $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ of complex numbers, find necessary and sufficient conditions for the existence of an entrywise n-by-n nonnegative matrix A with spectrum Λ. In this case, we say that Λ is *realizable* and that the matrix A is a *realizing matrix* for Λ. If the matrix A is diagonalizable, we have the DNIEP, and we say that Λ is *diagonalizably realizable* ($\mathcal{DR}$). If the matrix A is symmetric, we have the SNIEP, and we say that Λ is *symmetrically realizable* ($\mathcal{SR}$). The NIEP, DNIEP and SNIEP are equivalent for n smaller than five. The list Λ is *universally realizable* ($\mathcal{UR}$) if it is realizable for every Jordan canonical form (JCF) allowed by Λ. The problem of the universal realizability of spectra is called the *universal realizability problem* (URP). The URP contains the NIEP, and both problems remain unsolved for $n \geq 5$. Both problems are equivalent if the prescribed eigenvalues $\lambda_1, \dots, \lambda_n$ are distinct. The URP seeks to determine the spectral properties allowed by a nonnegative matrix, not only regarding the eigenvalues themselves but also from the point of view of the corresponding JCF.

The first known results on the URP, formerly called *nonnegative inverse elementary divisors problem,* are due to Minc [17]. In terms of the URP, Minc proved that if $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ is the spectrum of a diagonalizable positive matrix, then Λ is $\mathcal{UR}$. There are spectra, not positively realizable, that are known to be $\mathcal{UR}$, as for instance, certain spectra in the left half-plane, that is, $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 > 0$, $\operatorname{Re} \lambda_i \leq 0$, $i = 2, \dots, n$. In particular, the following were progressively shown to be $\mathcal{UR}$: real Suleĭmanova spectra [19], that is $\lambda_1 > 0 > \lambda_2 \geq \cdots \geq \lambda_n$; complex Suleĭmanova spectra [20], that is

$$\lambda_1 > 0,\ \lambda_i \in \{z \in \mathbb{C} : \operatorname{Re} z \leq 0,\ \ |\operatorname{Re} z| \geq |\operatorname{Im} z|\},\ i = 2, \dots, n;$$

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and Šmigoc spectra [5],

$$\lambda_1 > 0,\ \lambda_i \in \left\{ z \in \mathbb{C} : \operatorname{Re} z \leq 0,\ \left|\sqrt{3}\operatorname{Re} z\right| \geq |\operatorname{Im} z| \right\},\ i = 2, \ldots, n.$$

These lists are realizable if and only if they are $\mathcal{UR}$, and both hold if and only if $\sum_{i=1}^{n} \lambda_i \geq 0$. The good behavior of this kind of lists led to the idea that any left half-plane list was $\mathcal{UR}$. Now, we know that this is not true (see Remark 3.1 in [12]).

Throughout this paper, if $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ is realizable, then λ_1 is the Perron eigenvalue of the realizing matrix. We denote by $\mathcal{CS}_\alpha$ the set of all n-by-n real matrices with constant row sums equal to α. It is clear that any matrix in $\mathcal{CS}_\alpha$ has the eigenvector $\mathbf{e}^{\mathrm{T}} = [1 \ \cdots \ 1]$ corresponding to the eigenvalue α. It is well known (see [8]) that the problem of finding a nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ is equivalent to the problem of finding a nonnegative matrix in $\mathcal{CS}_{\lambda_1}$ with spectrum Λ. We denote by $\mathbf{e}_k$ the vector with 1 in the k^{th} position and zeros elsewhere, by $E_{i,j}$ the matrix with 1 in position (i, j) and zeros elsewhere and we define the matrix

$$E_K = \sum_{i \in K} E_{i,i+1}, \quad K \subset \{1, 2, \ldots, n\}. \tag{1}$$

Diagonalizability is a necessary condition for a list of complex numbers to be $\mathcal{UR}$. It is also important because we know how to join Jordan blocks to obtain any coarser JCF in the case of positive realizations. The question of whether Minc's result holds for nonnegative realizations has been open for almost 40 years. Recently, two extensions have been obtained: the first, by Collao et al. [3], shows that if $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of a diagonalizable nonnegative matrix $A \in \mathcal{CS}_{\lambda_1}$ with a positive column, then Λ is $\mathcal{UR}$. Note that if A has a positive row and A^T has a positive eigenvector, then Λ is also $\mathcal{UR}$. The second extension, by Johnson et al. [9], shows that if Λ is realizable by a diagonalizable ODP matrix, that is, a diagonalizable nonnegative matrix having all its off-diagonal entries being positive (zeros on diagonal are permitted), then Λ is also $\mathcal{UR}$.

There are still numerous open questions about the URP. One of them, which motivates our interest in this paper, is under what conditions a $\mathcal{DR}$ list of complex numbers is $\mathcal{UR}$. In [11], the authors showed that the list

$$\Lambda = \{\lambda_1, \lambda_1, \lambda_2, \lambda_2\},\ \text{with } \lambda_1 > 0 > \lambda_2 \geq -\lambda_1,\ \lambda_1 + 2\lambda_2 < 0, \tag{2}$$

is not $\mathcal{UR}$, although it is $\mathcal{DR}$. Since Λ in (2) has a reducible realization, it is worth asking whether $\mathcal{DR}$ implies $\mathcal{UR}$ is valid for irreducible realizations. However, in [12] it was also shown that there exist lists, as

$$\Lambda = \left\{ a, \frac{\sqrt{5}-1}{4}a, \frac{\sqrt{5}-1}{4}a, -\frac{\sqrt{5}+1}{4}a, -\frac{\sqrt{5}+1}{4}a \right\},\ a > 0 \tag{3}$$

that are irreducibly diagonalizably realizable but not $\mathcal{UR}$

In this paper, we prove a surprisingly simple result, which shows how $\mathcal{DR}$ implies $\mathcal{UR}$ for general lists of complex numbers. The paper is organized as follows: In Section 2, we introduce some theorems which are used to obtain our results. In Section 3, we prove that there is a nonnegative real number λ_0 such that $\Lambda_\mu = \{\mu, \lambda_2, \ldots, \lambda_n\}$ is $\mathcal{DR}$ ($\mathcal{UR}$) if $\mu \geq \lambda_0$, and we define, for a list Λ of complex numbers, the indices of

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diagonalizable realizability and of *universal realizability*. We also define the concepts *diagonalizably realizable extreme* and *diagonalizably realizable non-extreme*. This is a key definition to establish that $\mathcal{DR}$ implies $\mathcal{UR}$ under certain conditions. We also introduce in this section the main results, Theorems 3.5 and 3.6, which show when $\mathcal{DR}$ implies $\mathcal{UR}$. Other results, which establish a connection between $\mathcal{DR}$ and $\mathcal{UR}$, are also introduced. In Section 4, we consider the case $\mathcal{DR}$ implies $\mathcal{UR}$ for lists $\mathcal{DR}$ extreme, and we propose realizations that are $\mathcal{DR}$ and consequently $\mathcal{UR}$. Examples are given to illustrate the results.

2. Preliminaries. In this paper, we use the following results: Theorems 2.1 to 2.6 below. Theorem 2.1, due to Brauer [2], is a perturbation result that shows how to change one single eigenvalue of an n-by-n matrix without changing any of the remaining $(n-1)$ eigenvalues. Theorem 2.2, by Soto and Ccapa [19], establishes the JCF of the Brauer perturbation $A + \mathbf{e}\mathbf{q}^T$. Theorem 2.3, by Laffey and Šmigoc [14], gives a necessary and sufficient condition for the realizability of a left half-plane list of complex numbers. Theorem 2.4, by Šmigoc [18], gives a procedure to obtain, from two matrices A and B, a new matrix C, preserving in certain way, the corresponding JCFs of A and B. This procedure is called Šmigoc's glue technique. Theorem 2.5, by Torre et al. [22], solves the NIEP on size 5 with trace zero from the coefficients of the characteristic polynomial.

Theorem 2.1. *[2] Brauer. Let A be an n-by-n matrix with spectrum $\{\lambda_1, \ldots, \lambda_n\}$. Let $\mathbf{v}^T = [v_1 \ \cdots \ v_n]$ be an eigenvector of A associated with the eigenvalue λ_k and let $\mathbf{q}$ be any n-dimensional vector. Then, the matrix $A + \mathbf{v}\mathbf{q}^T$ has eigenvalues $\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + \mathbf{v}^T\mathbf{q}, \lambda_{k+1}, \ldots, \lambda_n$.*

Theorem 2.2. *[19] Soto and Ccapa. Let $\mathbf{q}^T = [q_1 \ \cdots \ q_n]$ be an arbitrary n-dimensional vector. Let $A \in \mathcal{CS}_{\lambda_1}$ with Jordan canonical form*

$$J(A) = S^{-1}AS = diag\,\{J_1(\lambda_1), J_{n_2}(\lambda_2), \ldots, J_{n_k}(\lambda_k)\}\,.$$

If $\lambda_1 + \sum_{i=1}^n q_i \neq \lambda_i$, $i = 2, \ldots, n$, then the matrix $A + \mathbf{e}\mathbf{q}^T$ has Jordan canonical form $J(A) + \left(\sum_{i=1}^n q_i\right) E_{11}$. In particular, if $\sum_{i=1}^n q_i = 0$ then A and $A + \mathbf{e}\mathbf{q}^T$ are similar.

Theorem 2.3. *[14] Laffey and Šmigoc. Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a left half-plane list of complex numbers. Then, Λ is realizable if and only if*

$$s_1 = \sum_{i=1}^n \lambda_i \geq 0; \quad s_2 = \sum_{i=1}^n \lambda_i^2 \geq 0; \quad s_1^2 \leq n s_2.$$

Theorem 2.4. *[18] Šmigoc. Suppose B is an m-by-m matrix with a Jordan canonical form that contains at least one 1-by-1 Jordan block corresponding to the eigenvalue c:*

$$J(B) = \begin{bmatrix} c & 0 \\ 0 & I(B) \end{bmatrix}.$$

Let $\mathbf{t}$ and $\mathbf{s}$, respectively, be the left and the right eigenvectors of B associated with the 1-by-1 Jordan block in the above canonical form. Furthermore, we normalize vectors $\mathbf{t}$ and $\mathbf{s}$ so that $\mathbf{t}^T\mathbf{s} = 1$. Let $J(A)$ be a Jordan canonical form for the n-by-n matrix

$$A = \begin{bmatrix} A_1 & \mathbf{a} \\ \mathbf{b}^T & c \end{bmatrix},$$

where A_1 is an $(n-1)$-by-$(n-1)$ matrix and $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{C}^{n-1}$. Then, the matrix

$$C = \begin{bmatrix} A_1 & \mathbf{a}\mathbf{t}^T \\ \mathbf{s}\mathbf{b}^T & B \end{bmatrix}$$

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has Jordan canonical form

$$J(C) = \begin{bmatrix} J(A) & 0 \\ 0 & I(B) \end{bmatrix}.$$

THEOREM 2.5. *[22, Theorem 39 for $n = 5$ and $p = 2$] Torre et al. The polynomial $P_5(x) = x^5 + k_2x^3 + k_3x^2 + k_4x + k_5$ is realizable if and only if the coefficients of $P_5(x)$ satisfy*

$$k_2,\, k_3,\, k_4 - \tfrac{k_2^2}{4} \leq 0, \quad \textit{and} \quad k_5 \leq \begin{cases} k_2k_3 & \textit{if} \ \ k_4 \leq 0, \\ k_3\left(\frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4}\right) & \textit{if} \ \ k_4 > 0. \end{cases}$$

In the case $k_4 \leq 0$, the polynomial $P_5(x)$ is realizable by the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -k_3 & 0 & 0 & 1 & 0 \\ -k_4 & 0 & 0 & 0 & 1 \\ k_2k_3 - k_5 & 0 & 0 & -k_2 & 0 \end{bmatrix}.$$

Finally, although mentioned in the Introduction (cases *ii.* to *iv.*) and proven in [1, 15] (case *i.*) we recall here, for the sake of completeness, the following results:

THEOREM 2.6. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a diagonalizably realizable list of complex numbers with a diagonalizable realizing matrix A, where:*
i. A is irreducible with a positive row or column, or
ii. $A \in \mathrm{CS}_{\lambda_1}$ has a positive column, or
iii. A has a positive row and A^T has a positive eigenvector, or
iv. A is an ODP matrix.
Then, Λ is universally realizable.

3. $\mathcal{DR}$ non-extreme implies $\mathcal{UR}$. It is well known (see [8]) that an n-by-n irreducible nonnegative matrix A is similar, via a positive eigenvector, to an irreducible nonnegative matrix B with constant row sums. In fact, since A has a positive eigenvector $[x_1 \ \cdots \ x_n]$ associated with its spectral radius $\rho(A)$, then $B = D^{-1}AD$, where $D = diag\{x_1 \ \cdots \ x_n\}$, is a nonnegative matrix similar to A and $B \in \mathcal{CS}_{\rho(A)}$. If A is a reducible nonnegative matrix, then only co-spectrality with a nonnegative matrix $B \in \mathcal{CS}_{\rho(A)}$ can be assured. In this case, we consider a permutationally similar Frobenius normal form $\widetilde{A} = P^{-1}AP$. Thus, if a list Λ is $\mathcal{DR}$ by a diagonalizable nonnegative matrix A, then the matrix $\widetilde{A}$ is also $\mathcal{DR}$. The matter is that the transformation of a diagonalizable matrix in a matrix with constant row sums, in general, does not preserve the diagonalizability. From the Perron Frobenius Theory, we know that, to preserve diagonalizability, it is necessary and sufficient the existence of a positive eigenvector, as is said in the following result.

THEOREM 3.1. *(Theorem 6, [6]) Let A be a reducible nonnegative matrix with spectral radius $\rho(A)$. Then, A has a positive eigenvector $[x_1 \ \cdots \ x_n]$ associated with $\rho(A)$ if and only if, in the Frobenius normal form $\widetilde{A}$, all the final components have a spectral radius $\rho(A)$ and the non-final components have a smaller spectral radius.*

In this case, as in the irreducible case, $B = D^{-1}AD$, where $D = diag\{x_1 \ \cdots \ x_n\}$, is a nonnegative matrix similar to A and $B \in \mathcal{CS}_{\rho(A)}$.

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Let $\Lambda = \{\lambda_2, \ldots, \lambda_n\}$ be a self-conjugate list of complex numbers. Guo proved (Theorem 2.1 in [7]) that there is a minimum nonnegative number $g_r(\Lambda)$ such that $\Lambda_\mu = \{\mu, \lambda_2, \ldots, \lambda_n\}$ is realizable for all $\mu \geq g_r(\Lambda)$. We call $g_r(\Lambda)$ the *realizability index* of Λ. Analogously, for a list $\Lambda = \{\lambda_2, \ldots, \lambda_n\}$ of real numbers (Theorem 4.1 in [7]), there is a minimum nonnegative number $g_s(\Lambda)$ such that $\Lambda_\mu = \{\mu, \lambda_2, \ldots, \lambda_n\}$ is symmetrically realizable for all $\mu \geq g_s(\Lambda)$. We call $g_s(\Lambda)$ the *symmetric realizability index* of Λ. The minimality of these two indexes is a consequence of the property: If $\Lambda_\epsilon = \{\lambda_1 + \epsilon, \lambda_2, \cdots, \lambda_n\}$ is realizable ($\mathcal{SR}$) for all $\epsilon > 0$, then $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is realizable ($\mathcal{SR}$).

In a similar way, we obtain an index for diagonalizable realizability.

Theorem 3.2. *Let $\Lambda = \{\lambda_2, \ldots, \lambda_n\}$ be a self-conjugate list of complex numbers. Then, there is a nonnegative number $\lambda_0 \geq g_r(\Lambda)$ such that $\Lambda_\mu = \{\mu, \lambda_2, \ldots, \lambda_n\}$ is diagonalizably realizable for every $\mu \geq \lambda_0$.*

Proof. First, we exhibit a value $\lambda_0 \geq g_r(\Lambda)$ such that $\Lambda_{\lambda_0} = \{\lambda_0, \lambda_2, \ldots, \lambda_n\}$ is $\mathcal{DR}$. Let A be a realizing matrix for $\Lambda \cup \{g_r(\Lambda)\}$. Without loss of generality, we assume that $A \in \mathcal{CS}_{g_r(\Lambda)}$, that is, $A\mathbf{e} = g_r(\Lambda)\mathbf{e}$. If A is diagonalizable, we are done (λ_0 is $g_r(\Lambda)$). If not, we take $A = SJS^{-1}$, where J is the JCF of A and $S\mathbf{e}_1 = \mathbf{e}$. Now let $\widetilde{J}$ be the same as J, except that any nonzero superdiagonal number is replaced with $0s$. So, $\widetilde{J}$ is diagonal or block diagonal with spectrum Λ. Define $\widetilde{A} = S\widetilde{J}S^{-1}$. If $\widetilde{A}$ is nonnegative, we are done. If not, since

$$\widetilde{A}\mathbf{e} = S\widetilde{J}S^{-1}\mathbf{e} = g_r(\Lambda)\mathbf{e},$$

that is, $\widetilde{A} \in CS_{g_r(\Lambda)}$, we apply Brauer's Theorem 2.1 to produce a nonnegative matrix $A' = \widetilde{A} + eq^{\mathrm{T}}$, where $q^{\mathrm{T}} = [q_1 \ \cdots \ q_n]$ is an appropriate nonnegative vector. From Theorem 2.2, since $\widetilde{A}$ is diagonalizable, then A' is also diagonalizable with spectrum $\left\{ g_r(\Lambda) + \sum_{i=1}^{n} q_i, \lambda_2, \ldots, \lambda_n \right\}$. Let $\lambda_0 = g_r(\Lambda) + \sum_{i=1}^{n} q_i$. Thus, we have established the existence of a value $\lambda_0 \geq g_r(\Lambda)$ such that $\Lambda_{\lambda_0} = \{\lambda_0, \lambda_2, \ldots, \lambda_n\}$ is $\mathcal{DR}$. Now, from [11, Theorem 3.1], we have that if Λ_{λ_0} is $\mathcal{DR}$, then $\Lambda_{\lambda_0+\epsilon} = \{\lambda_0 + \epsilon, \lambda_2, \ldots, \lambda_n\}$, $\epsilon > 0$, is also $\mathcal{DR}$. Thus, Λ_μ is $\mathcal{DR}$ for all $\mu \geq \lambda_0$. ∎

Note that the set of μs such that Λ_μ is $\mathcal{DR}$ is infinite and bounded below by the realizability index $g_r(\Lambda)$. Then, there exists the infimum (the greatest lower bound) $g_d(\Lambda)$ that we call the *diagonalizable realizability index* of Λ. We have not been able to prove that if $\Lambda_\epsilon = \{\lambda_1 + \epsilon, \lambda_2, \cdots, \lambda_n\}$ is $\mathcal{DR}$ for all $\epsilon > 0$, then $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is $\mathcal{DR}$. Neither have we found a counterexample, so we do not know if the index $g_d(\Lambda)$ is or not a minimum.

Now we obtain an index for universal realizability.

Theorem 3.3. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a diagonalizably realizable list of complex numbers with a diagonalizable realizing matrix A having a positive eigenvector. Then, there is a nonnegative number $\lambda_0 \geq g_d(\Lambda/\lambda_1)$ such that $\Lambda_\mu = \{\mu, \lambda_2, \ldots, \lambda_n\}$ is $\mathcal{UR}$ for every $\mu \geq \lambda_0$.*

Proof. First we exhibit a value $\lambda_0 \geq \lambda_1$ such that $\Lambda_{\lambda_0} = \{\lambda_0, \lambda_2, \cdots, \lambda_n\}$ is $\mathcal{UR}$. Without loss of generality, we assume that $A \in \mathcal{CS}_{\lambda_1}$. Let S with $S\mathbf{e}_1 = \mathbf{e}$ be such that $A = SJ_AS^{-1}$, where $J_A = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. If Λ is $\mathcal{UR}$, there is nothing to do (λ_0 is λ_1). If not, let $J_A + E_K$, with E_K as defined in (1) be the desired JCF. Then,

$$J_A + E_K = S^{-1}AS + E_K = S^{-1}(A + SE_KS^{-1})S,$$

and $A + SE_KS^{-1}$ has the spectrum Λ and the desired JCF, although it is not necessarily nonnegative. Note that $SE_KS^{-1} \in \mathcal{CS}_0$ and, if $0 < \epsilon < 1$, then the absolute value of each entry of ϵSE_KS^{-1} is smaller

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than the absolute value of the corresponding entry of SE_KS^{-1}. Then, we may choose a nonnegative vector $\mathbf{q}^{\mathrm{T}} = [q_1 \ \cdots \ q_n]$ and $\epsilon > 0$ sufficiently small such that

$$B = (A + \epsilon S E_k S^{-1}) + \mathbf{e}\mathbf{q}^{\mathrm{T}} \tag{4}$$

is nonnegative, with spectrum $\Lambda_{\lambda_0} = \{\lambda_0, \lambda_2, \cdots, \lambda_n\}$ with $\lambda_0 = \lambda_1 + \sum_{i=1}^{n} q_i$ and, from Theorem 2.2, with the desired JCF. Now, from [11, Theorem 3.1], if Λ_{λ_0} is $\mathcal{UR}$, then $\Lambda_\mu = \{\mu, \lambda_2, \ldots, \lambda_n\}$ is $\mathcal{UR}$ for every $\mu \geq \lambda_0$. ∎

As above, there exists the infimum $g_u(\Lambda)$ that we call the *universal realizability index* of Λ. Later, we consider the minimality question about this index.

The following definition is key to understand the relation between $\mathcal{DR}$ and $\mathcal{UR}$.

Definition 3.4. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a diagonalizably realizable list of complex numbers. We say that Λ is diagonalizably realizable extreme if for all $\epsilon > 0$, $\Lambda_{-\epsilon} = \{\lambda_1 - \epsilon, \lambda_2, \ldots, \lambda_n\}$ is not diagonalizably realizable. We say that Λ is diagonalizably realizable non-extreme if there is a number $\epsilon > 0$, such that $\Lambda_{-\epsilon}$ is diagonalizably realizable.*

Remark 1. *From Theorem 3.2, if $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is $\mathcal{DR}$ extreme, then $g_d(\Lambda/\lambda_1) = \lambda_1$ and $g_d(\Lambda/\lambda_1)$ is a minimum. If Λ is $\mathcal{DR}$, extreme or not, we can only state that $g_r(\Lambda/\lambda_1) \leq g_d(\Lambda/\lambda_1)$.*

The number λ_0 in Theorem 3.3 is λ_1 if Λ is $\mathcal{UR}$, or strictly greater than λ_1 if Λ is not $\mathcal{UR}$. The following results establish the biggest possible refinement of the result in Theorem 3.3 and clarify the connection between $\mathcal{DR}$ and $\mathcal{UR}$. In particular, they show how $\mathcal{DR}$ non-extreme implies $\mathcal{UR}$.

Theorem 3.5. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a diagonalizably realizable list of complex numbers with a diagonalizable realizing matrix A having a positive eigenvector. Then , $\Lambda_\epsilon = \{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}$ is universally realizable for all $\epsilon > 0$.*

Proof. Note that in the proof of the Theorem 3.3, if Λ is not $\mathcal{UR}$, we can decrease ϵ in such a way that $\sum_{i=1}^{n} q_i$ is as small as we want to and, consequently, λ_0 converge to λ_1. ∎

Theorem 3.6. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a diagonalizably realizable list of complex numbers. Then*

(1) If Λ is non-extreme with $\Lambda_{-\epsilon} = \{\lambda_1 - \epsilon, \lambda_2, \ldots, \lambda_n\}$, $\epsilon > 0$, being diagonalizably realizable by a matrix A with a positive eigenvector, then Λ is universally realizable.

(2) If Λ is extreme, then $g_u(\Lambda) = g_d(\Lambda/\lambda_1)$.

Proof. (1) Let $[x_1 \ \cdots \ x_n]$ be a positive eigenvector of A corresponding to $\lambda_1 - \epsilon$, and let $D = diag\{x_1 \ \cdots \ x_n\}$. Then, $B = D^{-1}AD \in \mathcal{CS}_{\lambda_1 - \epsilon}$ is $\mathcal{DR}$ with spectrum $\Lambda_{-\epsilon}$. Therefore, from Theorem 2.2, $B + \mathbf{e}\mathbf{q}^T$, where $\mathbf{q}^T = \frac{\epsilon}{n}\mathbf{e}^T$, is a diagonalizable positive matrix with spectrum Λ. Therefore, Λ is $\mathcal{UR}$. In this case, $\mathcal{DR}$ *and* $\mathcal{UR}$ are equivalent.

(2) If Λ is extreme, from Remark 1, $g_d(\Lambda/\lambda_1) = \lambda_1$ is a minimum. Since Λ_ϵ is universally realizable for all $\epsilon > 0$, then $g_u(\Lambda) = g_d(\Lambda/\lambda_1)$. ∎

Remark 2. *Lists Λ as in (3) are $\mathcal{DR}$ extreme, but not $\mathcal{UR}$. However, from Theorem 2.2, the corresponding lists Λ_ϵ are still $\mathcal{DR}$ but non-extreme, and since the realizations of Λ are irreducible, from Theorem 3.5, the lists Λ_ϵ are $\mathcal{UR}$ and $g_u(\Lambda) = g_d(\Lambda/\lambda_1)$ with $g_u(\Lambda)$ not minimum. Then, the statement "If Λ_ϵ is $\mathcal{UR}$*

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for all $\epsilon > 0$, then Λ is $\mathcal{UR}$" is false, and so, in general, the universal realizability index $g_u(\Lambda/\lambda_1)$ is not a minimum.

COROLLARY 3.7. *From Theorem 3.6, if Λ is $\mathcal{DR}$ extreme, then $g_u(\Lambda)$ can be or not a minimum. If Λ is $\mathcal{DR}$ non-extreme, we can only state that $g_d(\Lambda/\lambda_1) \leq g_u(\Lambda)$.*

The following example illustrates the Theorems 3.1 and 3.5:

EXAMPLE 1. *The list $\Lambda = \{2, 2, 1, -1, -1, -1, -1\}$ is realizable by the reducible $\mathcal{DR}$ matrix A and, by means of the permutation (7146352), we obtain the Frobenius normal form $\widetilde{A}$:*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \widetilde{A} = \left[\begin{array}{ccc|ccc|c} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

The basic components of $\widetilde{A}$ are just the final components, then by Theorem 3.1, the matrix A has a positive eigenvector, for instance $[3\ 1\ 3\ 3\ 1\ 3\ 1]$. Then, with $D = diag\{3\ 1\ 3\ 3\ 1\ 3\ 1\}$, we have that

$$D^{-1}AD = \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \in \mathcal{CS}_2$$

is diagonalizable with spectrum Λ and, for $\mathbf{q}^T = \frac{\epsilon}{7}\mathbf{e}^T$,

$$D^{-1}AD + \mathbf{e}\mathbf{q}^T,$$

is diagonalizable positive with spectrum $\Lambda_\epsilon = \{2 + \epsilon, 2, 1, -1, -1, -1, -1\}$. Therefore, Λ_ϵ is $\mathcal{UR}$ for all $\epsilon > 0$.

The following example illustrates the Theorem 3.6 in the non-extreme case:

EXAMPLE 2. *The list $\Lambda = \{7, -1, -1\pm 3i, -1\pm 3i\}$ is $\mathcal{DR}$ non-extreme since $\Lambda' = \{6, -1, -1\pm 3i, -1\pm 3i\}$ is the spectrum of the diagonalizable nonnegative matrix*

$$B = \begin{bmatrix} 0 & 0 & \frac{6}{17} & \frac{24}{17} & \frac{96}{17} & \frac{384}{17} \\ 1 & 0 & \frac{1}{85} & \frac{4}{85} & \frac{16}{85} & \frac{64}{85} \\ 0 & 10 & 0 & 0 & 0 & 40 \\ 0 & 12 & 1 & 0 & 0 & 38 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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having a positive eigenvector $\mathbf{x}^T = [10\ 2\ 10\ 12\ 3\ 1]$. *Then, for* $D = diag\{10\,2\,10\,12\,3\,1\}$,

$$D^{-1}BD = \begin{bmatrix} 0 & 0 & \frac{6}{17} & \frac{144}{85} & \frac{144}{85} & \frac{192}{85} \\ 5 & 0 & \frac{1}{17} & \frac{24}{85} & \frac{24}{85} & \frac{32}{85} \\ 0 & 2 & 0 & 0 & 0 & 4 \\ 0 & 2 & \frac{5}{6} & 0 & 0 & \frac{19}{6} \\ 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 3 & 1 \end{bmatrix} \in \mathcal{CS}_6$$

is diagonalizable with spectrum Λ' *and, for* $\mathbf{q}^T = \frac{1}{6}\mathbf{e}^T$,

$$A = D^{-1}BD + \mathbf{e}\mathbf{q}^T$$

is diagonalizable positive with spectrum Λ. *Therefore* Λ *is* $\mathcal{UR}$.

The following example illustrates the Theorem 3.6 in the extreme case:

Example 3. *It is clear that* $\Lambda = \{3, 3, -2, -2, -2\}$ *is not realizable. In [16], it was proved that the list* $\Lambda_t = \{3 + t, 3 - t, -2, -2, -2\}$ *is realizable if and only if* $t \geq \sqrt{16\sqrt{6} - 39} = 0.43799\cdots$, *that is, with* $g_r(\Lambda/3) = \sqrt{16\sqrt{6} - 39}$ *while in [4] it was proved that* Λ_t *is* $\mathcal{DR}$ *if and only if* $t \geq 1$. *Then,* $\Lambda_1 = \{4, 2, -2, -2, -2\}$ *is* $\mathcal{DR}$ *extreme, and* $g_d(\Lambda_1/4) = 4$ *is the minimum. From Theorem 3.5 in [12], we also know that* $g_u(\Lambda_1) = 4$ *is the minimum. Obviously, the list* Λ_1 *is* $\mathcal{SR}$ *and* $g_s(\Lambda_1/4) = g_d(\Lambda_1/4)$ *(see Remark 12 in [10]).*

Note that if Λ is a left half-plane list, then from Theorem 2.3, we can always compute the exact value of $g_r(\Lambda/\lambda_1)$.

Example 4. *It is easy to see, by applying Šmigoc's glue technique in Theorem 2.4, that the list*

$$\Lambda = \{11, -2, -1 \pm 3i, -1 \pm 3i, -1 \pm 3i, -1 \pm 3i\} \text{ is } \mathcal{DR}.$$

Is Λ $\mathcal{UR}$*? Our procedure, to answer this question, consists of identifying a real number* μ, *with* $g_r(\Lambda/\lambda_1) \leq \mu < 11$, *such that*

$$\Lambda_\mu = \{\mu, -2, -1 \pm 3i, -1 \pm 3i, -1 \pm 3i, -1 \pm 3i\},$$

is also $\mathcal{DR}$. *From Theorem 2.3, we compute* $g_r(\Lambda/\lambda_1) = 10$. *Then, we choose* $\mu = \frac{21}{2}$. *From Šmigoc's glue technique, we consider the decomposition of* Λ_μ

$$\Lambda_1 = \left\{\frac{21}{2}, -1 \pm 3i\right\},\ \Lambda_2 = \{-1 \pm 3i\},\ \Lambda_3 = \{-1 \pm 3i\},\ \Lambda_4 = \{-2, -1 \pm 3i\},$$

with the auxiliary lists

$$\Gamma_1 = \left\{\frac{21}{2}, -1 \pm 3i\right\},\ \Gamma_2 = \left\{\frac{17}{2}, -1 \pm 3i\right\},$$

$$\Gamma_3 = \left\{\frac{13}{2}, -1 \pm 3i\right\},\ \Gamma_4 = \left\{\frac{9}{2}, -2, -1 \pm 3i\right\},$$

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which are the spectrum of diagonalizable nonnegative companion matrices

$$A_1 = \begin{bmatrix} 0 & 0 & 105 \\ 1 & 0 & 11 \\ 0 & 1 & \frac{17}{2} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 85 \\ 1 & 0 & 7 \\ 0 & 1 & \frac{13}{2} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 65 \\ 1 & 0 & 3 \\ 0 & 1 & \frac{9}{2} \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 & 0 & 90 \\ 1 & 0 & 0 & 43 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix},$$

respectively. The right and left eigenvectors of A_2 *are*

$$s = \begin{bmatrix} 10 \\ 2 \\ 1 \end{bmatrix}, \quad t^T = \begin{bmatrix} \dfrac{4}{397} & \dfrac{34}{397} & \dfrac{289}{397} \end{bmatrix}.$$

Then, with $a = \begin{bmatrix} 105 \\ 11 \end{bmatrix}$ *and* $b^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$, *we have*

$$at^T = \begin{bmatrix} \dfrac{420}{397} & \dfrac{3570}{397} & \dfrac{30\,345}{397} \\[2ex] \dfrac{44}{397} & \dfrac{374}{397} & \dfrac{3179}{397} \end{bmatrix}, \quad sb^T = \begin{bmatrix} 0 & 10 \\ 0 & 2 \\ 0 & 1 \end{bmatrix},$$

and the Šmigoc's glue of A_1 *with* A_2 *is*

$$C_2 = \begin{bmatrix} 0 & 0 & \frac{420}{397} & \frac{3570}{397} & \frac{30\,345}{397} \\ 1 & 0 & \frac{44}{397} & \frac{374}{397} & \frac{3179}{397} \\ 0 & 10 & 0 & 0 & 85 \\ 0 & 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 1 & \frac{13}{2} \end{bmatrix}.$$

Next, the glue of C_2 *with* A_3 *produces*

$$C_3 = \begin{bmatrix} 0 & 0 & \frac{420}{397} & \frac{3570}{397} & \frac{40\,460}{34\,539} & \frac{262\,990}{34\,539} & \frac{1709\,435}{34\,539} \\ 1 & 0 & \frac{44}{397} & \frac{374}{397} & \frac{12\,716}{103\,617} & \frac{82\,654}{103\,617} & \frac{537\,251}{103\,617} \\ 0 & 10 & 0 & 0 & \frac{340}{261} & \frac{2210}{261} & \frac{14\,365}{261} \\ 0 & 2 & 1 & 0 & \frac{28}{261} & \frac{182}{261} & \frac{1183}{261} \\ 0 & 10 & 0 & 10 & 0 & 0 & 65 \\ 0 & 2 & 0 & 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & 1 & \frac{9}{2} \end{bmatrix},$$

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and the glue of C_3 with A_4 produces

$$C_4 = \begin{bmatrix} 0 & 0 & \frac{420}{397} & \frac{3570}{397} & \frac{40\,460}{34\,539} & \frac{262\,990}{34\,539} & \frac{1051\,960}{5422\,623} & \frac{1577\,940}{1807\,541} & \frac{7100\,730}{1807\,541} & \frac{31\,953\,285}{1807\,541} \\ 1 & 0 & \frac{44}{397} & \frac{374}{397} & \frac{12\,716}{103\,617} & \frac{82\,654}{103\,617} & \frac{330\,616}{16\,267\,869} & \frac{165\,308}{1807\,541} & \frac{743\,886}{1807\,541} & \frac{3347\,487}{1807\,541} \\ 0 & 10 & 0 & 0 & \frac{340}{261} & \frac{2210}{261} & \frac{8840}{40\,977} & \frac{4420}{4553} & \frac{19\,890}{4553} & \frac{89\,505}{4553} \\ 0 & 2 & 1 & 0 & \frac{28}{261} & \frac{182}{261} & \frac{728}{40\,977} & \frac{364}{4553} & \frac{1638}{4553} & \frac{7371}{4553} \\ 0 & 10 & 0 & 10 & 0 & 0 & \frac{40}{157} & \frac{180}{157} & \frac{810}{157} & \frac{3645}{157} \\ 0 & 2 & 0 & 2 & 1 & 0 & \frac{24}{2041} & \frac{108}{2041} & \frac{486}{2041} & \frac{2187}{2041} \\ 0 & 20 & 0 & 20 & 0 & 20 & 0 & 0 & 0 & 90 \\ 0 & 14 & 0 & 14 & 0 & 14 & 1 & 0 & 0 & 43 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix},$$

which is diagonalizable irreducible nonnegative with spectrum Λ and a positive column. Therefore, from Theorem 2.6, Λ is $\mathcal{UR}$.

Since diagonalizability is a necessary condition for universal realizability, it is an advantage to have diagonalizable realizing matrices. That is the reason why we recall here some basic properties of circulant matrices, which are diagonalizable. An n-by-n circulant matrix is a matrix of the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \ddots & \ddots & c_{n-2} \\ c_{n-2} & c_{n-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & c_2 \\ c_2 & \ddots & \ddots & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} & c_0 \end{bmatrix},$$

which we denote by $C = circ\,(c_0\ c_1\ \cdots\ c_{n-1})$. The matrix C has eigenvalues

$$\lambda_j = c_0 + c_1\omega^{j-1} + c_2\omega^{2(j-1)} + \cdots + c_{n-1}\omega^{(n-1)(j-1)},$$

$j = 1, 2, \ldots, n$, where $\omega = \exp(2\pi i/n)$, $i^2 = -1$. If C is a real matrix, then

$$\lambda_{n-j+2} = \overline{\lambda_j},\ j = 2, 3, \ldots, \left[\frac{n+1}{2}\right],$$

that is, the vector of eigenvalues of C is a conjugate-even vector.

Let $F = (f_{kj}) = [\mathbf{1} \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n]$, where

$$\begin{aligned} \mathbf{1}^T &= [1\ \cdots\ 1] \\ \mathbf{v}_j^T &= \left[1\ \omega^{j-1}\ \omega^{2(j-1)}\ \cdots\ \omega^{(n-1)(j-1)}\right],\ j = 2, \ldots, n. \end{aligned}$$

Then, $f_{kj} = \omega^{(k-1)(j-1)}$, $1 \le k,\ j \le n$, and $F\overline{F} = \overline{F}F = nI$. Let

$$\lambda^T = [\lambda_1\ \lambda_2\ \cdots\ \lambda_n] \text{ and } \mathbf{c} = [c_0\ c_1\ \cdots\ c_{n-1}]\,.$$

Then, $F\mathbf{c} = \lambda$ and $\mathbf{c} = \frac{1}{n}\overline{F}\lambda$.

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THEOREM 3.8. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the spectrum of a circulant nonnegative matrix and let $\lambda^T_{-\epsilon} = [\lambda_1 - \epsilon\ \lambda_2\ \cdots\ \lambda_n]$, $\epsilon > 0$, $\lambda_1 - \epsilon \geq |\lambda_j|$, $j = 2, \ldots, n$. If*

$$\mathbf{c} = \frac{1}{n}\overline{F}\lambda_{-\epsilon},$$

is a nonnegative vector, then Λ is universally realizable.

Proof. Since Λ is the spectrum of a circulant nonnegative matrix, then the vectors

$$\lambda^T = [\lambda_1\ \lambda_2\ \cdots\ \lambda_n] \text{ and } \lambda^T_{-\epsilon} = [\lambda_1 - \epsilon\ \lambda_2\ \ldots\ \lambda_n]$$

are conjugate-even vectors. If $\mathbf{c} = \frac{1}{n}\overline{F}\lambda_{-\epsilon}$ is a nonnegative vector, then there is a circulant nonnegative matrix $B \in \mathcal{CS}_{\lambda_1-\epsilon}$ with spectrum $\Lambda_{-\epsilon} = \{\lambda_1 - \epsilon, \lambda_2, \ldots, \lambda_n\}$. Thus, $A = B + \mathbf{eq}^T$, with $\mathbf{q} = \frac{\epsilon}{n}\mathbf{e}$, is a circulant positive matrix with spectrum Λ. Hence Λ is $\mathcal{UR}$. ∎

REMARK 3. *Nonnegative circulant matrices $circ\,(c_0\ c_1\ \cdots\ c_{n-1})$ with $c_0 < c_k$, $k = 1, 2, \ldots, n-1$ are diagonalizable ODP matrices. Then, from Theorem 2.6, their spectra are $\mathcal{UR}$.*

4. $\mathcal{DR}$ extreme implies $\mathcal{UR}$.

Now we want to discuss $\mathcal{DR}$ implies $\mathcal{UR}$ for the case of lists diagonalizably realizable extreme, which can be of two kinds: $\mathcal{DR}$ extreme lists with trace zero or $\mathcal{DR}$ extreme lists with positive trace. In this case, we have Theorem 3.6 and we can apply certain procedures that allow us to universally realize this type of lists. First, we consider the case trace zero. It is clear that the Minc realization and the extension in [3] do not work in this case.

In [12], the authors study the $\mathcal{UR}$ in low dimension for the cases where the NIEP is solved. In particular, in dimension five with trace zero they prove the following result.

COROLLARY 4.1. *Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ be a list of complex numbers with zero trace. If Λ is $\mathcal{DR}$, then it is $\mathcal{UR}$ except for the lists of real numbers: $\{a, a, 0, -a, -a\}$ and the lists in (3).*

Note that for nonreal lists (of the form $\left\{a, -\frac{a}{4} \pm ci, -\frac{a}{4} \pm ci\right\}$ with $a, c > 0$, or $\left\{a, b, b, -\frac{a}{2} - b \pm ci\right\}$ with $a, b, c > 0$), $\mathcal{DR}$ and $\mathcal{UR}$ are equivalent (see Theorems 3.1 and 3.2 in [12] for more details). For real lists, $\mathcal{DR}$ and $\mathcal{UR}$ are equivalent, except for the spectra $\{a, a, 0, -a, -a\}$, with $a > 0$, and the spectra in (3) that are $\mathcal{DR}$ but not $\mathcal{UR}$ (see Theorems 3.3 to 3.9 in [12] for more details).

In the particular case of left half-plane lists with trace zero, from Theorem 2.3, they are realizable if and only if $s_2 \geq 0$. In dimension five we have:

COROLLARY 4.2. *Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ be a left half-plane list of complex numbers with trace zero. Then $\mathcal{DR}$ implies $\mathcal{UR}$.*

Proof. If Λ is a real list, and thus of Suleĭmanova type, then realizable, $\mathcal{DR}$, $\mathcal{SR}$, and $\mathcal{UR}$ are all equivalent. If Λ is nonreal, then $\mathcal{DR}$ implies $\mathcal{UR}$ is a consequence of the previous corollary. ∎

In dimension higher than five, there are a number of procedures to realize a $\mathcal{DR}$ extreme list with zero trace.

EXAMPLE 5. *Consider the list*

$$\Lambda = \{26, -4, -10, -10, -1 + 11i, -1 - 11i\},$$

which can be ordered as the conjugate-even vector

$$\lambda = [26\ \ -10\ \ -1 + 11i\ \ -4\ \ -1 - 11i\ \ -10].$$

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In [21], it is shown that Λ is the spectrum of a circulant ODP matrix. Therefore, from Theorem 2.6, Λ is $\mathcal{UR}$.

EXAMPLE 6. *Consider the list $\Lambda = \{21, -13, -2 \pm 8i, -2 \pm 8i\}$. By applying the Šmigoc's glue, with*

$$\Lambda_1 = \{21, -2 \pm 8i\} \quad \Lambda_2 = \{-13, -2 \pm 8i\},$$

and

$$\Gamma_1 = \{21, -2 \pm 8i\}, \quad \Gamma_2 = \{17, -13, -2 \pm 8i\},$$

we can show that Λ is $\mathcal{DR}$. In fact, Γ_1 and Γ_2 are the spectrum of

$$A = \begin{bmatrix} 0 & 0 & 1428 \\ 1 & 0 & 16 \\ 0 & 1 & 17 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 15\,028 \\ 1 & 0 & 0 & 1156 \\ 0 & 1 & 0 & 169 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

respectively. Now, the glue of A with B produces the diagonalizable nonnegative matrix

$$C_1 = \begin{bmatrix} 0 & 0 & \frac{14}{125} & \frac{238}{125} & \frac{4046}{125} & \frac{68\,782}{125} \\ 1 & 0 & \frac{8}{6375} & \frac{8}{375} & \frac{136}{375} & \frac{2312}{375} \\ 0 & 884 & 0 & 0 & 0 & 15\,028 \\ 0 & 120 & 1 & 0 & 0 & 1156 \\ 0 & 17 & 0 & 1 & 0 & 169 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

with spectrum Λ. The other non-diagonal JCF is the companion matrix associated with the characteristic polynomial

$$p(x) = (x - 21)(x + 13)((x + 2)^2 + 64)^2,$$

that is

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1262\,352 \\ 1 & 0 & 0 & 0 & 0 & 185\,504 \\ 0 & 1 & 0 & 0 & 0 & 41\,224 \\ 0 & 0 & 1 & 0 & 0 & 2856 \\ 0 & 0 & 0 & 1 & 0 & 185 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore, Λ is $\mathcal{UR}$.

The $\mathcal{DR}$ extreme case with positive trace is more difficult to investigate. Here we introduce the following examples:

EXAMPLE 7. *Let us consider the nonreal list with positive trace*

$$\Lambda = \left\{4, -1 \pm \frac{5}{\sqrt{3}} i\right\}.$$

Note that Λ is not a Šmigoc spectrum since it does not satisfy the condition $|\sqrt{3} Re\, z| \geq |Im\, z|$. However, Λ is realizable, in fact it is on the border of the realizability region since $s_1(\Lambda)^2 = ns_2(\Lambda)$, where $s_n(\Lambda)$ is the moment of order n of Λ. As the eigenvalues are distinct, then Λ is also $\mathcal{DR}$ and $\mathcal{UR}$.

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It is easy to check that Λ *is* $\mathcal{DR}$ *extreme because the list*

$$\Lambda_{-\epsilon} = \left\{4-\epsilon, 1 \pm \frac{5}{\sqrt{3}}i\right\}$$

is not realizable since it does not satisfy the necessary moment condition $s_1(\Lambda_{-\epsilon})^2 \leq ns_2(\Lambda_{-\epsilon})$ *in Theorem 2.3.*

EXAMPLE 8. *From Theorem 3.1 in [12], we know that the 5-spectra with trace zero* $\left\{a, -\frac{a}{4} \pm ci, -\frac{a}{4} \pm ci\right\}$ *are realizable* $\Leftrightarrow c \leq \frac{\sqrt{5}}{4}a$ *and are* $\mathcal{UR}$ *if* $c \leq \frac{a}{2}$*. Then, we consider the realizable spectrum*

$$\Lambda = \{14, -1 \pm \sqrt{5}i, -1 \pm \sqrt{5}i\},$$

with $g_r(\Lambda/14) = 4$ *and* $g_d(\Lambda/14) > 4$*. To obtain a realizing matrix of* Λ*, we translate it to trace zero*

$$\Lambda_0 = \{12, -3 \pm \sqrt{5}i, -3 \pm \sqrt{5}i\}.$$

The characteristic polynomial of Λ_0 *is* $(x-12)((x+3)^2+5)^2 = x^5 - 80x^3 - 600x^2 - 1820x - 2352 = x^5 + k_2x^3 + k_3x^2 + k_4x + k_5$*, with* $k_2k_3 - k_5 = 50352$*. From Theorem 2.5, we have the following realizing matrices for* Λ_0 *and* Λ

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 600 & 0 & 0 & 1 & 0 \\ 1820 & 0 & 0 & 0 & 1 \\ 50352 & 0 & 0 & 80 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 600 & 0 & 2 & 1 & 0 \\ 1820 & 0 & 0 & 2 & 1 \\ 50352 & 0 & 0 & 80 & 2 \end{bmatrix}.$$

On the one hand, A is a Hessenberg matrix, and therefore, it has a Jordan canonical form with a maximal Jordan block. On the other hand, in a similar way to Theorem 3.3 in [13], we obtain that in general

$$g_d(\{\lambda_2, \cdots, \lambda_n\}) \leq (n-1) \max_{2 \leq j \leq n} |\lambda_j|.$$

In our case, there exists $g_d(\Lambda/14) \leq 4\sqrt{6} = 9.7979... < 14$*, then* Λ *is* $\mathcal{DR}$ *and therefore* $\mathcal{UR}$*.*

This kind of examples are also possible in the real case:

EXAMPLE 9. *The list* $\{9, 9, -4, -6, -8\}$ *is not realizable, since it does not admit a realizable partition, but* $\{10, 9, -4, -6, -8\}$ *is. Then, there exists a minimum* λ_1*, with* $9 < \lambda_1 \leq 10$*, such that the list* $\Lambda = \{\lambda_1, 9, -4, -6, -8\}$ *is realizable. Thus,* Λ *is* $\mathcal{DR}$ *(distinct eigenvalues) extreme and with positive trace. The minimum number* λ_1 *is unknown.*

Finally, note that the spectrum $\Lambda = \{4, \sqrt{5}-1, \sqrt{5}-1, -(\sqrt{5}+1), -(\sqrt{5}+1)\}$ is $\mathcal{SR}$, by the matrix

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 0 \end{bmatrix},$$

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but not $\mathcal{UR}$ (see (3)), and the spectrum $\Lambda = \{15, 11, -7, -9, -10\}$ is $\mathcal{UR}$, since it satisfies Theorem 2.5, but not $\mathcal{SR}$, because it does not satisfy the necessary Spector's condition $\lambda_2 + \lambda_5 \leq 0$. Then,

Theorem 4.3. *The SNIEP and the URP are independent.*

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