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Diagonal entries of inverses of diagonally dominant matrices ☆

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We consider invertible, row diagonally dominant real matrices and give inequalities on their minors and diagonal entries of their inverses. A very special case is that all diagonal entries of an inverse, of a row stochastic, row diagonally dominant and invertible matrix, are at least 1, with strict inequality at least when the dominance is strict. This was conjectured in international trade theory in economics and motivated the present work (though much more is proven). Some of the results generalize previously known facts for M-matrices.

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The matrix A is considered to be n-by-n with entries a_{ij} throughout. We also suppose that A is *row diagonally dominant* (DD), *i.e.* $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$, and invertible. (There are corresponding statements for column DD.) Invertibility is often a consequence of DD, but not always. For example, strict diagonal dominance or irreducible diagonal dominance suffices [2], but is not necessary as indicated by

$$\begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Diagonal dominance means that the diagonal entry in each row is rather big, occupying at least half the weight. One of our purposes is to show that the diagonal entries of the inverse are also "big" in at least two senses, an absolute size sense and relative to column entries. At first this seems no surprise, as $AA^{-1} = I$, but it is, by no means, trivial from this. Some of what we say generalizes known facts about M-matrices. But, we were also motivated by questions coming from international trade theory in economics, whence M-matrices also arise [4,3]. In any event, these observations make a nice addition to core matrix analysis and will likely be useful elsewhere.

By $|A|$ we mean the entry-wise absolute value of A: $|A| = (|a_{ij}|)$. Of course, A is row (column) DD if and only if $|A|$ is. We take our matrices to be real-entried. If e, as usual, is the n-vector of all ones, let $r(A) = |A|e$, the vector of absolute row sums of A and let D_r be the diagonal matrix such that $D_r e = r(A)$. Then, assuming A has no zero rows, $|D_r^{-1}A|$ is row stochastic and row DD if A is row DD. It is the case of row stochastic, DD matrices that generated the original motivating interest in international trade theory and these will be special cases of work here.

In case A is real and DD, then $\operatorname{sgn}(\det A) = \operatorname{sgn}(\prod_{i=1}^n a_{ii})$, weakly. If A is invertible, the equality is precise.

Let $A(i;j)$ denote the $(n-1)$-by-$(n-1)$ submatrix of A resulting from deletion of row i and column j. Of course, $A(i;i)$, or for short $A(i)$, is a principal submatrix. Our first major fact is

Theorem 1. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be such that for some $R \geq 1$:*

$$R \sum_{j\neq i} |a_{ij}| \leq |a_{ii}|, \quad i = 1, \dots, n. \tag{1}$$

Then also

$$R\,|\det A(i;j)| \leq |\det A(i)| \qquad \textit{for } j \neq i. \tag{2}$$

For $R = 1$, Theorem 1 yields

Corollary 2. *If $A \in \mathcal{M}_n(\mathbb{R})$ is row DD, then*

$$|\det A(i;j)| \leq |\det A(i)|, \quad j = 1, 2, \dots, n.$$

(In case the DD is strict and $j \neq i$, the above inequality is strict.)

Proof of Theorem 1. We can assume that the diagonal of A is nonnegative because all the minors of

$$\operatorname{diag}\left(\operatorname{sgn}(a_{11}), \dots, \operatorname{sgn}(a_{nn})\right) A$$

in absolute value are equal to the ones of A and the previous matrix has nonnegative diagonal. Note that in this case $\det A(i)$ is nonnegative.

It suffices to show that

$$R|\det(A(1;j)| \leq \det A(1),$$

or, equivalently,

$$\det A(1) \pm R \det A(1;j) \geq 0; \tag{3}$$

for others values of i the proof will be analogous.

Let C_r denote the r-th column of A without the first entry:

$$C_r = (a_{2r}, \dots, a_{nr})^T,$$

so that

$$A(1;j) = (C_1, \dots, C_{j-1}, C_{j+1}, \dots, C_n) \quad \text{and} \quad A(1) = (C_2, \dots, C_n).$$

Then (3) can be rewritten as

$$\det(C_2, \dots, C_n) \pm R \det(C_1, \dots, C_{j-1}, C_{j+1}, \dots, C_n) \geq 0,$$

or, equivalently, $\det A_{\pm} \geq 0$, where

$$A_{\pm} =: (C_2, \dots, C_{j-1}, C_j \pm (-1)^j R C_1, C_{j+1}, \dots, C_n)$$
$$= \begin{pmatrix} a_{22} & \dots & a_{2j-1} & a_{2j} \pm (-1)^j R a_{21} & a_{2j+1} & \dots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & a_{jj} \pm (-1)^j R a_{j1} & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n2} & \dots & a_{nj-1} & a_{nj} \pm (-1)^j R a_{n1} & a_{nj+1} & \dots & a_{nn} \end{pmatrix}.$$

Observe that the matrices $A_{\pm}$ have nonnegative diagonal entries and are row DD. Indeed, $a_{rr} \geq 0$ by assumption, while $a_{jj} \pm R a_{j1} \geq 0$ due to (1). Furthermore, also due to (1):

$$a_{ii} \geq R \sum_{k \neq i} |a_{ik}| = R \sum_{k \neq i,j,1} |a_{ik}| + R\,|a_{i1}| + R\,|a_{ij}| \geq \sum_{k \neq i,j,1} |a_{ik}| + |a_{i1} \pm R a_{ij}|$$

and

$$|a_{jj} \pm R a_{j1}| \geq |a_{jj}| - R\,|a_{j1}| \geq R \sum_{k \neq j,1} |a_{kj}| \geq \sum_{k \neq j,1} |a_{kj}|\,.$$

The non-negativity of $\det A_{\pm}$ follows. $\square$

We say that a matrix A is *diagonally dominant of its (off-diagonal) column entries* if, for $i = 1, \ldots, n$,

$$|a_{ii}| \geq |a_{ji}|, \quad j = 1, \ldots, n.$$

This, of course, is strictly weaker than the traditional diagonal dominance. If the inequality is strict, for $j \neq i$, we refer to this dominance as *strict*. A restatement of Theorem 1 generalizes the known case of diagonally dominant M-matrices [2].

Corollary 3. *If $A \in \mathcal{M}_n(\mathbb{R})$ is row DD and invertible, then A^{-1} is diagonally dominant of its column entries. If the row DD is strict, then the invertibility of A is ensured and the diagonal dominance of the column entries in A^{-1} is also strict.*

Proof. This is a restatement of Corollary 2, using the co-factor form of the inverse. $\square$

Recall that C is the *comparison matrix* for A if $c_{ii} = |a_{ii}|, c_{ij} = -|a_{ij}|$, $i, j = 1, \ldots, n; i \neq j$. An H-matrix is a matrix with its comparison matrix an M-matrix. Of course, the comparison matrix of an M-matrix is this matrix itself, and so M-matrices are H-matrices by default. According to [1, Theorem 5.7.5], if $A, B \in \mathcal{M}_n(\mathbb{R})$ are M-matrices, then so is the Hadamard product $A \circ B^{-1}$. With the use of Corollary 3, we have the following generalization.

Theorem 4. *Let $A, B \in \mathcal{M}_n(\mathbb{R})$ be H-matrices. Then $A \circ B^{-1}$ also is an H-matrix.*

Proof. The characteristic property of H-matrices is that they become strictly row DD upon multiplying on the right by a suitable diagonal matrix with positive diagonal entries. So, let D be the respective diagonal matrix for A, and E for B^T. Applying Corollary 3 to $B^T E$ we find that $B^{-1}E^{-1}$ is diagonally dominant of its row entries. A direct computation shows that then $(AD) \circ (B^{-1}E^{-1})$ is strictly row DD along with AD. It remains to observe that $(AD) \circ (B^{-1}E^{-1}) = (A \circ B^{-1})(DE^{-1})$ and invoke the invariance of H-matrices under right multiplication by positive diagonal matrices. $\square$

Corollary 5. *If $A \in \mathcal{M}_n(\mathbb{R})$ is row DD, then, for $i = 1, \dots, n$,*

$$|\det A| \leq \left(\sum_{j=1}^{n} |a_{ij}| \right) |\det A(i)|,$$

with strict inequality when the dominance is strict.

Proof. We have

$$\begin{aligned} |\det A| &= \left| \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A(i;j) \right| \leq \sum_{j=1}^{n} |a_{ij} \det A(i;j)| \\ &= |a_{ii} \det A(i)| + \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij} \det A(i;j)| \\ &\leq |a_{ii} \det A(i)| + \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij} \det A(i)| = \left(\sum_{j=1}^{n} |a_{ij}| \right) |\det A(i)|. \quad \square \end{aligned}$$

Corollary 6. *If $A \in \mathcal{M}_n(\mathbb{R})$ is row stochastic, and row DD and invertible, then each diagonal entry of A^{-1} is at least 1. If the dominance is strict, the diagonal entries of A^{-1} are all strictly greater than* 1.

Proof. Note that because A is invertible, nonnegative, and row DD, then $\operatorname{sgn}(\det A) = \operatorname{sgn}(\prod_{i=1}^{n} a_{ii}) > 0$ and $\det A(i) \geq 0$ for $i = 1, \dots, n$. If we apply Corollary 5 to the matrix A we have $\det A \leq \det A(i)$ for $i = 1, \dots, n$. So the element (i, i) of A^{-1} is

$$\frac{\det A(i)}{\det A} \geq 1. \quad \square$$

The row stochastic case may be generalized as follows.

Theorem 7. *If $A \in \mathcal{M}_n(\mathbb{R})$ is row DD and invertible, then*

$$\left|(A^{-1})_{ii}\right| \geq \frac{1}{(|A|e)_i}.$$

This inequality is strict when the dominance is strict.

Proof. The theorem follows from Corollary 6 via left multiplication by D_r^{-1} and calculation. $\square$

We note that when the dominance is weak (equality in each row), the matrix may be invertible, without further assumption, and if it is invertible the inequalities, given for the diagonal entries of the inverse may or may not be strict. We illustrate this in the row stochastic case. We note that in the 2-by-2 row stochastic case, invertibility cannot occur, though it can in the non-row-stochastic, as illustrated by the example, earlier.

Example 8. The row stochastic matrix

$$A = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

has all dominance inequalities weak, but

$$A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

exists. But the diagonal entries are all 1, so that equality can occur in Corollary 6.

Example 9. On the other hand

$$A = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

is also invertible,

$$A^{-1} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix},$$

but its diagonal entries are all > 1.

It appears that for $n \geq 3$, row stochastic, row DD matrices are generically invertible and usually have inverse diagonal entries > 1. Real (weakly) row DD matrices seem also to be invertible and usually satisfy the inequalities of Theorem 7 strictly.

Declaration of competing interest

None declared.

Data availability

No data was used for the research described in the article.

References

[1] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1994, Corrected reprint of the 1991 original.
[2] R.A. Horn, C.R. Johnson, Matrix Analysis, second ed., Cambridge University Press, Cambridge, 2013.
[3] J. Norris, The home market effect in a home-biased geography, Working Papers 20210072, New York University Abu Dhabi, Department of Social Science, revised Sep 2021, https://ideas.repec.org/p/nad/wpaper/20210072.html.
[4] J. Norris, Private communication.