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Square matrices with the inverse diagonal property

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ABSTRACT

We identify the class of real square invertible matrices A for which the signs of the diagonal entries of A^{-1} match those of A, and begin their study. We say such matrices have the inverse diagonal property (IDP). This class includes many important classes: the positive definite matrices, the M-matrices, the totally positive matrices and some variants, the P-matrices, the diagonally dominant and H-matrices and their inverse classes, as well as triangular matrices. This class is closed under any real invertible diagonal multiplication on either the right or the left. So questions about this class can be reduced to the case of positive diagonal entries. Other basic properties are given. One theme is what conditions need be added to the IDP to insure membership in a familiar class. For example, the positive definite matrices are characterized as certain IDP matrices with special conditions on certain particular principal minors. The tridiagonal case is highlighted. Certain specially simple conditions on such matrices are mentioned that ensure them to be P-matrices, positive definite matrices or M-matrices. We also note that recent results about the invertibility of weakly diagonally dominant matrices are used. Examples are given throughout the paper.

Introduction

Suppose that $A \in M_n(\mathbb{R})$, $A = (a_{ij})$, is invertible. We are interested in the situation in which the i-th diagonal entry of A^{-1} has the same sign as a_{ii}, $i = 1, \ldots, n$, i.e. $diag(A) \circ diag(A^{-1}) > 0$ (no 0 diagonal entries). Here $\circ$ denotes the Hadamard product. For brevity, we say that A has the *inverse diagonal property* (IDP) or A is IDP. For example, if

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & -2 \end{bmatrix},$$

then

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ -1 & -3 & -1 \\ -1 & -1 & -1 \end{bmatrix},$$

and, as the diagonal entries match in sign, A has the IDP.

Several special types of IDP matrices will be of interest. If the diagonal entries of A are all positive, we say that A has the positive IDP, or P-IDP. If A has positive (negative) determinant, we indicate this as IDP$_+$ (IDP$_-$), and when A is symmetric, we say “symmetric IDP”. Of course, combinations are possible, such as symmetric P-IDP$_+$, which means that A is symmetric with positive diagonal entries and determinant.

As we will see, (1) for general matrices, the general IDP may be reduced to the P-IDP, and (2) the IDP unites many familiar and important classes of matrices.

Here, our purpose is two-fold: (1) to lay the groundwork for a problem we have found to be worthy of study; and (2) to give some specific results and limiting examples about the topic. In the next section, we give some examples (not necessarily all) of important classes that enjoy the IDP, mostly P-IDP$_+$. In Sections Closure Properties and Observations we give some closure properties for the IDP class(es) and some basic observations that underlay analysis of the IDP. We consider in Section The Positive Definite Case the relationship between positive definite (PD) matrices and the symmetric P-IDP$_+$. In Section Tridiagonal Matrices, we make some initial observations about the tridiagonal case. This suggests further study of related ideas about tridiagonal matrices, which we anticipate. Finally, in Section Diagonally Dominant Matrices, we consider strictly (and invertible weakly) diagonally dominant (DD) matrices. They are (both) IDP, and we extend some ideas of (Johnson et al., 2023; Johnson et al.). In (Johnson et al.), inequalities were obtained for the inverse diagonal entries of row stochastic diagonally dominant matrices. This

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raises the question of when a weakly diagonal dominant matrix is invertible, which was addressed in (Johnson et al., 2023). However, in (Johnson et al., 2023) it was not noticed that a weakly diagonal dominant invertible matrix is IDP, which we show in the last section. The example above is weakly DD.

There are other perspectives on matrix sign patterns including those of the inverse entries (Eschenbach et al., 1999; Ma and Zhan, 2014; Roy and Xue, 2021).

We use the standard principal submatrix notation throughout: $A[i_1, \ldots, i_k]$ means the principal submatrix of A lying in the rows and columns $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ and $A(i)$ means the principal submatrix of A obtained deleting the row and column $i \in \{1, \ldots, n\}$.

Examples of familiar IDP classes

We were motivated in part by a question posed by Jordan Norris (economist, NYU) to Charles Johnson, stemming from economic equilibrium analysis, about the case of strictly (row or column) diagonally dominant (DD) matrices. They do have the IDP property, as one might guess from (Johnson et al., 2023). We verify this directly in Section Diagonally Dominant Matrices and extend it to the weakly DD, but invertible case, as studied in (Johnson et al.).

In fact the IDP notion unites many familiar and important classes of matrices, and our purpose here is to mention several of them.

The PD matrices are symmetric P-IDP$_+$, as they are closed under inversion and necessarily have positive diagonal entries (Horn and Johnson, 2013). Not all symmetric P-IDP$_+$ matrices are PD, but in Section The Positive Definite Case we investigate the relationship more deeply and determine what additional hypotheses are needed to give a converse. This is especially interesting in the tridiagonal case, Section Tridiagonal Matrices.

A nonsingular M-matrix is a Z-matrix:

$$\begin{bmatrix} + & & & & \\ & + & & -/0 & \\ & & \ddots & & \\ & -/0 & & + & \\ & & & & + \end{bmatrix}$$

(positive diagonal entries and nonpositive off-diagonal entries), with entry-wise nonnegative inverse (Horn and Johnson, 2013). The properties of M-matrices often parallel those of PD matrices, and that is so here. The M-matrices are P-IDP$_+$ and, in pursuit of a converse, there are parallel statements.

Matrix A is *totally positive* (TP) if all its minors are positive (Fallat and Johnson, 2011). The TP matrices are also P-IDP$_+$.

For the totally nonnegative (TN) matrices, the minor conditions are relaxed to "nonnegative". Such matrices may, or may not, be nonsingular. However, when they are, the diagonal entries and, in fact, all principal minors must be positive, because the inequalities of Hadamard and Fischer hold (Fallat and Johnson, 2011). Thus, invertible TN-matrices are P-IDP$_+$. The "sign regular" matrices may or may not be IDP, but when they are not, the inverse diagonal signs will be the opposite of the diagonal ones. It depends on the sequence of signs defining the sign regular class.

Clearly, an invertible diagonal matrix is IDP. But, also invertible (upper or lower) triangular matrices are IDP. These matrices have an even stronger property. The diagonal entries of the inverse are the inverses of the diagonal entries of the original matrix.

The P-matrices (all principal minors positive) are also P-IDP$_+$ as they are inverse closed. They include several of the above classes, but are much less structured.

Closure properties

The IDP matrices are naturally closed under inversion, as are each IDP variant. So, for example, the inverse M-matrices are P-IDP$_+$, as the M-matrices are.

If $A \in M_n(\mathbb{R})$ is invertible and P is any permutation matrix in $M_n(\mathbb{R})$, since $(P^TAP)^{-1} = P^TA^{-1}P$, the IDP matrices, as well as each variant, are closed under permutation similarity. The same applies to any diagonal similarity.

If A is IDP and D is any invertible diagonal matrix then $(DA)^{-1} = A^{-1}D^{-1}$ ($(AD)^{-1} = D^{-1}A^{-1}$). Since the signs of the diagonal entries are changed in the same way in each, DA is IDP if and only if A is IDP. So, in particular, the IDP is unchanged by left or right multiplication by either a positive diagonal matrix or a signature matrix. Since a signature multiply can change the diagonal signs, the general IDP problem may always be changed to the P-IDP problem, the natural special case to study. Then P-IDP$_+ \cup$ P-IDP$_-$ is the general problem. In either case, using a further positive diagonal multiply (or positive diagonal congruence in the symmetric positive diagonal case), we may assume 1's on the diagonal of A if A is P-IDP.

Observations

In a sense, IDP may be checked in a simple way; however we want more interesting observations. If $\det(A) > 0$, since $(A^{-1})_{ii} = \frac{\det A(i)}{\det(A)}$, A is IDP$_+$ if and only if $\det A(i)$ and a_{ii} have the same (nonzero) sign, $i = 1, \ldots, n$, i.e. $a_{ii} \det A(i) > 0$. Similarly, if $\det A < 0$, then IDP$_-$ means $a_{ii} \det A(i) < 0$. In either event, for A to be IDP, $a_{ii} \det A(i)$ must have the same sign as $\det A$, $i = 1, \ldots, n$.

Since the inverse of a direct sum of matrices is the direct sum of the inverses, the direct sum of two (or more) matrices is IDP if and only if each is IDP. Similarly, a block triangular matrix is IDP if and only if each diagonal block is IDP. So, it suffices to consider irreducible IDP matrices.

Another interesting variant of IDP is "inheritted IDP" by which we mean that A and each leading principal submatrix of A has the IDP. Again, the variants mentioned before are possible. Of course, a strictly DD matrix is naturally inheritted IDP. As in the strictly DD case, the sign of the determinant of an inheritted IDP matrix is the same as that of the product of its diagonal entries (by induction).

We also note that the set of IDP matrices (and of each variant) is open in $M_n(\mathbb{R})$.

The positive definite case

As mentioned, all PD matrices are symmetric P-IDP$_+$. In the presence of symmetry, to what extent is there a converse? To see that the converse is not generally so, consider the following.

Example 1. The matrix

$$A = \begin{bmatrix} 1 & -5 & 10 & -2 \\ -5 & 1 & -5 & 10 \\ 10 & -5 & 1 & -5 \\ -2 & 10 & -5 & 1 \end{bmatrix}$$

is symmetric P-IDP$_+$ and not PD. Observe that

$$A^{-1} = \frac{1}{483} \begin{bmatrix} 39 & 40 & 75 & 53 \\ 40 & 8 & 15 & 75 \\ 75 & 15 & 8 & 40 \\ 53 & 75 & 40 & 39 \end{bmatrix}.$$

This raises the question of what additional (minimal?) hypotheses permit a converse, i.e., the implication that P-IDP$_+$ implies PD. First, we note.

Theorem 2. If $n < 4$, then a symmetric P-IDP$_+$ matrix is PD.

Proof. The cases $n = 1$ and $n = 2$ are straightforward. Suppose that $n = 3$. As is well-known (Horn and Johnson, 2013), we only need show that the leading principal minors of such a matrix are positive. (We will use this fact again.) As the diagonal entries are positive, the first leading principal minor is positive. The determinant is positive, as we are in

IDP_+. The second leading principal minor is positive as it is the numerator of the third diagonal entry of the inverse (with positive denominator, as the determinant is positive). See Section Observations.

Theorem 3. For a symmetric $A \in M_n(\mathbb{R})$, A is PD if and only if A is an inheritted P-IDP_+ matrix.

Proof. Since PD is an inheritted property and we have mentioned that PD matrices are symmetric P-IDP_+, the forward implication is clear. Conversely, if A is inheritted P-IDP_+, then each leading principal minor is positive and A is PD.

In fact, the proof of the backward implication only needs to use inheritance by the leading principal minors 3 to $(n-2)$. We also note that, for $n = 4$, when just P-IDP_+ is not sufficient, the only counterexamples may be shown to be essentially positive (entry-wise) (signature similar to a positive matrix) as in Example 1. In general, we have.

Theorem 4. For symmetric $A \in M_n(\mathbb{R})$, the following are equivalent:

1. *A is PD;*
2. *A is P-IDP_+ and* $\det A[1, 2, \ldots, k] > 0$, $k = 2, \ldots, n-2$ (equivalently, $\det A[k, \ldots, n-1, n] > 0$, $k = 3, \ldots, n-1$); and
3. *A is P-IDP_+ and* $A[1, \ldots, n-2]$ *is PD* (equivalently, $A[3, \ldots, n]$ *is PD).*

Proof. That 1. implies 2. and 3. is clear from prior comments and the fact that all principal minors are positive in a PD matrix. Conversely, we show that either 2. or 3. imply that the leading principal minors are all positive. We proof the result assuming the hypotheses on the leading principal submatrices. For the trailing ones, the proof is analogous. In the case of 2., the only ones not explicitly mentioned are the first diagonal entry and the last 2 leading principal minors. The diagonal entry and the determinant are part of the definition of P-IDP_+, and the penultimate leading principal minor is the numerator of the last diagonal entry of A^{-1}. The argument for 3. implies 1. is similar. Notice that in the presence of P-IDP_+, the additional hypotheses of 2. and 3. are equivalent.

If we assume that $A \in M_n(\mathbb{R})$ has the Z-sign pattern, there are parallel theorems in which “PD” is replaced by “M-matrix”.

Tridiagonal matrices

When A is tridiagonal, the $(n-1)$-by-$(n-1)$ principal submatrices missing an interior row and column break into direct sums. This is helpful. Again, positive definite and M-matrices may be studied in parallel.

First, we ask how large n may be so that all symmetric tridiagonal P-IDP_+ matrices are PD, the analog of Theorem 2.

Lemma 5. A 3-by-3 symmetric tridiagonal matrix with positive diagonal entries is PD if and only if its determinant is positive.

Proof. A simple calculation shows that the positivity of the determinant (and that of either the 1,1 or 3,3 diagonal entry) implies that of either the leading or trailing 2-by-2 principal minors. Thus, both the leading and trailing principal minors are positive and the matrix is PD.

We note that Lemma 5 has generalizations that we do not need here.

Theorem 6. *Suppose* $A \in M_n(\mathbb{R})$ is symmetric tridiagonal and $n < 6$. Then, *A is P-IDP_+* if and only if *A is PD.*

Proof. Again, we have already noted the reverse implication (without the tridiagonal hypothesis). For the forward implication, the case $n < 4$ follows from Theorem 2. When $n = 4, 5$, the hypothesis implies that each of the $(n-1)$-*by*-$(n-1)$ principal minors is positive. Noting that the interior ones reduce in the tridiagonal case, we have

$$\det A[1,2,3] > 0$$
$$\det A[1,2] > 0 \ \ (\text{as} \ \ a_{44} > 0),$$

when $n = 4$, and

$$\det A[1,2,3,4] > 0$$
$$\det A[1,2,3] > 0 \, (\text{as} \ \ a_{55} > 0)$$
$$\det A[1,2] > 0 \, (\text{by} \ \ \text{Lemma} \ \ 5),$$

when $n = 5$. Since a_{11}, $\det A > 0$ by the definition of P-IDP_+, this means that the leading principal minors of A are all positive, and because of symmetry, A is PD.

Example 7. A 6-by-6 symmetric tridiagonal P-IDP_+ matrix that is not PD is

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 \\ 3 & 1 & 5 & 0 & 0 & 0 \\ 0 & 5 & 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 & 5 & 0 \\ 0 & 0 & 0 & 5 & 1 & 3 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{bmatrix}.$$

Now, we may say exactly what need be added to P-IDP_+ in the symmetric tridiagonal case to ensure PD. Note that a forward/backward symmetry occurs in the next result (as happened in Theorem 4).

Theorem 8. Suppose $A \in M_n(\mathbb{R})$ is symmetric and tridiagonal.

1. *If $n = 6$, A is PD if and only if A is P-IDP_+ and* $\det A[1,2] > 0$ *(*equivalently, $\det A[5,6] > 0$*).*
2. *If $n > 6$,* the following are equivalent:
   (a) *A is PD;*
   (b) *A is P-IDP_+ and* $\det A[1, 2, \ldots, k] > 0$, $k = 3, \ldots, n-4$ (equivalently, $\det A[k, \ldots, n-1, n] > 0$, $k = 5, \ldots, n-2$); and
   (c) *A is P-IDP_+ and* $A[1, \ldots, n-4]$ *is PD* (equivalently, $A[5, \ldots, n]$ *is PD).*

Proof. In 1. and 2., PD is sufficient and we wish to show the reverse implication by showing that the hypothesis implies that the leading principal minors of A are positive. Again P-IDP_+ implies that $\det A(i) > 0$, $i = 1, \ldots, n$, and some of the $A(i)$ are reducible. In case 1., taking into account Theorem 4, *we just need to see that* $\det A[1,2,3] > 0$ and $\det A[1,2,3,4] > 0$. The latter inequality follows because $\det A(5) > 0$ and $a_{66} > 0$. As for the former inequality, we have

$$\det A(3) = \det A[1,2] \det A[4,5,6] > 0, \quad \text{and} \tag{1}$$

$$\det A(4) = \det A[1,2,3] \det A[5,6] > 0. \tag{2}$$

From (1) and the hypothesis, $\det A[4,5,6] > 0$, implying, by Lemma 5, $\det A[5,6] > 0$. Thus, from (2), $\det A[1,2,3] > 0$. For 2., because of Theorem 4, we just need to show that (b) implies $\det A[1,2] > 0$, $\det A[1, 2, \ldots, n-3] > 0$ and $\det A[1, 2, \ldots, n-2] > 0$. The latter determinant is positive since $\det A(n-1) > 0$ and $a_{nn} > 0$. The former determinant is positive by Lemma 5, as, by hypothesis, $\det A[1,2,3] > 0$. As for the second determinant, we have

$$\det A(n-3) = \det A[1, \ldots, n-4] \det A[n-2, n-1, n] > 0,$$

implying $\det A[n-2, n-1, n] > 0$ and, by Lemma 5, $\det A[n-1, n] > 0$. Then, since

$$\det A(n-2) = \det A[1, \ldots, n-3] \det A[n-1, n] > 0,$$

we have $\det A[1, \ldots, n-3] > 0$.

There are again parallel results identifying the M-matrices among the Z-matrices in the tridiagonal case.

Diagonally dominant matrices

Matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ is (row) DD if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, n.$$

If all inequalities are strict, we say *strictly DD*; in this event, A is invertible, and a simple Gersgorin argument [3, Ch 6] shows that the sign of $\det A$ is the same as $\prod_{i=1}^{n} a_{ii}$. If some, or all, inequalities are equalities, A may be singular. When the dominance is weak (as weak as possible, *i.e.* all equalities) the occurrence of invertibility was recently studied in (Johnson et al., 2023). If some (and not all) inequalities are strict, it follows from Gersgorin Theory ([3, Ch 6]) that A is singular if and only if A has an irreducible component that is weakly (with all inequalities equalities) DD and singular.

We show that strictly DD matrices are IDP. As mentioned in Section 3, it suffices to consider the case of $a_{ii} > 0$, $i = 1, \ldots, n$; then $\det A > 0$. So, we only need to show that A is P-IDP$_+$ in this case. For this, according to Section 4, we need show that $\det A(i) > 0$, $i = 1, \ldots, n$. But, as each of the submatrices $A(i)$, $i = 1, \ldots, n$, inherits strict diagonal dominance, as well as positive diagonal entries, from A, then each has positive determinant. This verifies that a strictly diagonally dominant matrix with positive diagonal entries is P-IDP$_+$. We may conclude

Theorem 9. *A strictly DD matrix is IDP.*

This leaves the question of what happens if the diagonal dominance is not strict, including the case in which all inequalities are equalities. The answer is largely combinatorial. Again, we may assume the diagonal entries are positive. Suppose that A is invertible. Consider $A_\epsilon = A + \epsilon I$, for small $\epsilon > 0$; then $\det A_\epsilon > 0$, as A_ϵ is strictly DD with positive diagonal entries. Taking a limit as $\epsilon \to 0$ implies that $\det A (= \det A_0) \geq 0$. Since $A = A_0$ is invertible, then $\det A > 0$ and the sign of $\det A$ is the same as that of its diagonal entries, an addition to the theory developed in (Johnson et al., 2023). Now, $A(i)$ is either strictly DD or is DD with some equalities. The former case, or the latter when all entries in column i off the diagonal are 0, are straightforward; $\det A(i) > 0$. In the other cases, if $\det A(i)$ were 0, $A(i)$ would have to have its own principal submatrix, corresponding to a connected component of $A(i)$ in which all inequalities were equalities and the determinant is 0. The entries of $A(i)$ in the rows corresponding to this submatrix and outside it would then be 0. But the corresponding entries in column i of A would also have to be 0. Then $\det A$ would also be 0, a contradiction. Thus, $\det A(i) > 0$. We conclude again that A is P-IDP$_+$. Then, we have

Theorem 10. *If $A \in M_n(\mathbb{R})$ is DD and invertible (even if all inequalities are equalities), then A is IDP.*

Again, if all diagonal entries were positive, A would be a P-matrix. It is also interesting that in the case of all equalities, the analysis of invertibility, and thus the taking of limits, is primarily combinatorial (sign patterns and cycle structure). See (Johnson et al., 2023; Johnson et al.).

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