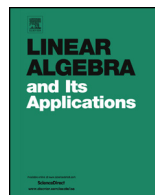




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## A note for the SNIEP in size 5 <sup>☆</sup>



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### ABSTRACT

The purpose of this note is to establish the current state of the knowledge about the SNIEP (symmetric nonnegative inverse eigenvalue problem) in size 5 with just one repeated eigenvalue.

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The SNIEP (symmetric nonnegative inverse eigenvalue problem) is the problem of characterizing all possible real spectra of entrywise symmetric nonnegative matrices. A complete solution of this problem is known only for spectra of size  $n \leq 4$ . For these  $n$ 's the most basic necessary conditions are also sufficient. That is, the Perron and the trace conditions characterize the SNIEP for  $n \leq 4$ . Spectra of size 5 are not characterized and this problem has proven to be a very challenging one.

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The open case for size 5,  $\sigma = \{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5\}$ , is when there are 3 positive eigenvalues, the trace is positive,  $\lambda_1 \geq |\lambda_5|$  and  $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 < 0$  (see [1]). Loewy in [4] studies this case. In fact, as he shows, when Loewy’s result, see Theorem 1 below, is applied to the case of two repeated eigenvalues we have a wider area than the one excluded in [1, Theorem 1].

**Theorem 1.** ([4, Theorem 2.1]) *Let  $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$  be a list of real numbers, where the elements of  $\sigma$  are arranged in monotonically decreasing order and with  $\lambda_3 - s_1(\sigma) \geq 0$ . If  $\sigma$  is the spectrum of a nonnegative symmetric matrix, then  $s_3(\sigma) \geq s_1(\sigma)^3 + 6\lambda_3 s_1(\sigma)(\lambda_3 - s_1(\sigma))$ .*

$$(s_k(\sigma) \text{ is the } k\text{th moment of } \sigma, \text{ i.e. } s_k(\sigma) = \sum_{i=1}^5 \lambda_i^k)$$

Before this result was published, another result appeared about symmetric realization of spectra with 5 eigenvalues.

**Theorem 2.** ([3, Theorem 4]) *Let  $\sigma = \{\lambda_1, \dots, \lambda_5\}$  be a list of monotonically decreasing real numbers such that  $\sum_{i=1}^5 \lambda_i \geq \frac{\lambda_1}{2}$ . Necessary and sufficient conditions for  $\sigma$  to be the spectrum of a nonnegative symmetric matrix are:*

$$(1) \lambda_1 = \max_{\lambda \in \sigma} |\lambda|, \quad (2) \lambda_2 + \lambda_5 \leq \sum_{i=1}^5 \lambda_i \quad \text{and} \quad (3) \lambda_3 \leq \sum_{i=1}^5 \lambda_i.$$

Previously, unresolved spectra with just one repeated eigenvalue are shown not to occur in [2]. The repetition could be either positive or negative, but the two situations are different.

**Theorem 3.** ([2, Theorem 3]) *Let  $a, d_1 > 0, d_2 > d_1$  satisfy  $a + d_2, d_1 + d_2 < 1 < a + d_1 + d_2$ . If  $(a + d_1)^3 + (a + d_2)^3 > 1 + a^3 + (a + d_1 + d_2 - 1)^3$ , then  $1, a, a, -(a + d_1), -(a + d_2)$  are not the eigenvalues of a 5-by-5 symmetric nonnegative matrix.*

**Theorem 4.** ([2, Theorem 4]) *The spectrum  $1, a, a - r, -(a + d), -(a + d)$  with  $d, r > 0, a > r$  and  $a + d, r + 2d < 1 < a + 2d$  is not realizable by a symmetric nonnegative 5-by-5 matrix if*

$$2(a + d)^3 > 1 + a^3 + (a + 2d - 1)^3.$$

The purpose of this note is to establish the current state of the knowledge about the SNIEP in size 5 with just one repeated eigenvalue. The next theorems show that Loewy’s result is strictly stronger than the results in [2] when it is particularized to one repeated eigenvalue.

**Theorem 5.** *Let  $\sigma = \{1, a, a, -(a + d_1), -(a + d_2)\}$  with  $a, d_1 > 0, d_2 > d_1$  and  $a + d_2, d_1 + d_2 < 1 < a + d_1 + d_2$ . If  $(a + d_1)^3 + (a + d_2)^3 - 1 - a^3 - (a + d_1 + d_2 - 1)^3 > 0$ , then  $s_1(\sigma)^3 + 6as_1(\sigma)(a - s_1(\sigma)) - s_3(\sigma) > 0$ . The reverse is not true.*

**Proof.** First of all, note that the hypothesis  $\lambda_3 > s_1(\sigma)$  of Theorem 1 applied to our list is the hypothesis  $1 < a + d_1 + d_2$ . It is straightforward to show that

$$\begin{aligned} & s_1(\sigma)^3 + 6as_1(\sigma)(a - s_1(\sigma)) - s_3(\sigma) \\ &= (1 - d_1 - d_2)^3 + 6a(1 - d_1 - d_2)(a + d_1 + d_2 - 1) - 1 - 2a^3 + (a + d_1)^3 + (a + d_2)^3 \\ &= (a + d_1)^3 + (a + d_2)^3 - 1 - a^3 - a^3 + (1 - d_1 - d_2)^3 + 6a(1 - d_1 - d_2)(a + d_1 + d_2 - 1) \\ &\quad > (a + d_1 + d_2 - 1)^3 - a^3 + (1 - d_1 - d_2)^3 + 6a(1 - d_1 - d_2)(a + d_1 + d_2 - 1) \\ &= 3a(1 - d_1 - d_2)(a + d_1 + d_2 - 1) > 0, \end{aligned}$$

where both inequalities are from the hypothesis of the theorem.

For  $a = \frac{1}{2}$ ,  $d_1 = \frac{1}{4}$  and  $d_2 = \frac{3}{8}$  we are under the hypothesis of the theorem and for these values we have

$$s_1(\sigma)^3 + 6as_1(\sigma)(a - s_1(\sigma)) - s_3(\sigma) = \frac{9}{256} > 0$$

and

$$(a + d_1)^3 + (a + d_2)^3 - 1 - a^3 - (a + d_1 + d_2 - 1)^3 = -\frac{9}{256} < 0. \quad \square$$

Let  $a, d_1$  and  $d_2$  be under the hypothesis of Theorem 5 and let define the functions

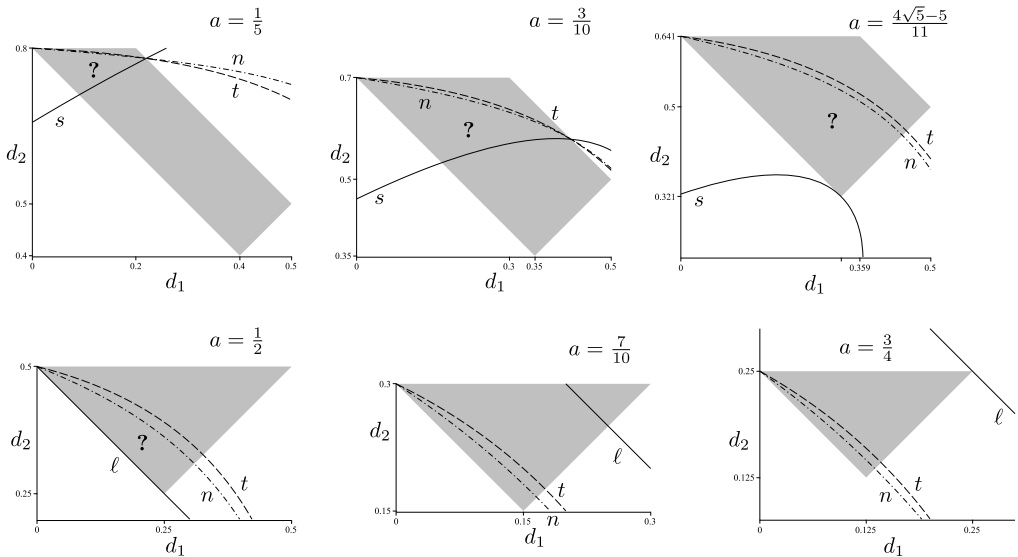
$$\begin{aligned} F(a, d_1, d_2) &= \left(\frac{4 + d_1 + d_2}{5}\right)^3 + 2\left(\frac{5a - 1 + d_1 + d_2}{5}\right)^3 \\ &\quad + \left(\frac{d_2 - 5a - 1 - 4d_1}{5}\right)^3 + \left(\frac{d_1 - 5a - 1 - 4d_2}{5}\right)^3 \\ G(a, d_1, d_2) &= (a + d_1)^3 + (a + d_2)^3 - 1 - a^3 - (a + d_1 + d_2 - 1)^3 \\ L(a, d_1, d_2) &= s_1(\sigma)^3 + 6as_1(\sigma)(a - s_1(\sigma)) - s_3(\sigma) \end{aligned}$$

and, for a fixed  $a$ , the curves  $s \equiv F(a, d_1, d_2) = 0$ ,  $t \equiv G(a, d_1, d_2) = 0$ ,  $n \equiv L(a, d_1, d_2) = 0$  and  $\ell = \{d_1 + d_2 = \frac{1}{2}\} \cap \{X = a\}$ . See [2] for a more detailed explanation. For the spectra  $\sigma$  under the hypothesis of Theorem 5 we have:

- If  $a < \frac{4\sqrt{5}-5}{11}$ , for some  $(d_1, d_2)$  the spectrum  $\sigma$  is symmetrically realizable with constant diagonal, those under or on curve  $s$  in Fig. 1. And for others,  $\sigma$  is not symmetrically realizable those above curve  $t$  in Fig. 1 (Theorem 3) and those above curve  $n$  in Fig. 1 (Theorem 1). The question mark in Fig. 1 means that the region between  $s$  and  $n$  (including  $n$ ) is unresolved.

- If  $\frac{4\sqrt{5}-5}{11} \leq a \leq \frac{1}{2}$ , for some  $(d_1, d_2)$  the spectrum  $\sigma$  is not symmetrically realizable, those above curve  $t$  in Fig. 1 (Theorem 3) and those above curve  $n$  in Fig. 1 (Theorem 1). The question mark in Fig. 1 means that the region under or on  $n$  is unresolved.

- If  $\frac{1}{2} < a < \frac{3}{4}$ , for some  $(d_1, d_2)$  the spectrum  $\sigma$  is not symmetrically realizable, those above curve  $t$  (Theorem 3), or those above curve  $n$  (Theorem 1) in Fig. 1, and for others neither, those under or on line  $\ell$  in Fig. 1 (Theorem 2). Since all the section is covered between both,  $\sigma$  is not symmetrically realizable.



**Fig. 1.**  $d_1 d_2$ -sections of the domain of Theorem 5 with curve  $s \equiv F(a, d_1, d_2) = 0$ , curve  $t \equiv G(a, d_1, d_2) = 0$ , curve  $n \equiv L(a, d_1, d_2) = 0$  and curve  $\ell = \{d_1 + d_2 = \frac{1}{2}\} \cap \{X = a\}$  for  $a \in \{\frac{1}{5}, \frac{3}{10}, \frac{4\sqrt{5}-5}{11}, \frac{1}{2}, \frac{7}{10}, \frac{3}{4}\}$ .

- If  $a \geq \frac{3}{4}$ , the spectrum  $\sigma$  is not symmetrically realizable by Theorem 2, see Fig. 1. The new area that was unresolved is the one between curves  $n$  and  $t$  for  $a \leq \frac{1}{2}$ .

**Theorem 6.** Let  $\sigma = \{1, a, a-r, -(a+d), -(a+d)\}$  with  $d, r > 0, a > r$  and  $a+d, r+2d < 1 < a+2d$ . If  $2(a+d)^3 - 1 - a^3 - (a+2d-1)^3 > 0$ , then  $s_1(\sigma)^3 + 6(a-r)s_1(\sigma)(a-r-s_1(\sigma)) - s_3(\sigma) > 0$ . The reverse is not true.

**Proof.** The hypothesis  $\lambda_3 > s_1(\sigma)$  of Theorem 1 applied to our list is the hypothesis  $1 < a+2d$ . It is straightforward to show that

$$\begin{aligned}
 & s_1(\sigma)^3 + 6(a-r)s_1(\sigma)(a-r-s_1(\sigma)) - s_3(\sigma) \\
 &= (1-r-2d)^3 + 6(a-r)(1-r-2d)(a+2d-1) - 1 - a^3 - (a-r)^3 + 2(a+d)^3 \\
 &= 2(a+d)^3 - 1 - a^3 + (1-r-2d)^3 + 6(a-r)(1-r-2d)(a+2d-1) - (a-r)^3 \\
 &> (a+2d-1)^3 + (1-r-2d)^3 + 6(a-r)(1-r-2d)(a+2d-1) - (a-r)^3 \\
 &= 3(a-r)(1-r-2d)(a+2d-1) > 0,
 \end{aligned}$$

where both inequalities are from the hypothesis of the theorem.

For  $a = \frac{1}{2}, d = \frac{8}{25}$  and  $r = \frac{1}{10}$  we are under the hypothesis of the theorem and for these values we have

$$s_1(\sigma)^3 + 6(a-r)s_1(\sigma)(a-r-s_1(\sigma)) - s_3(\sigma) = \frac{1167}{62500} > 0$$

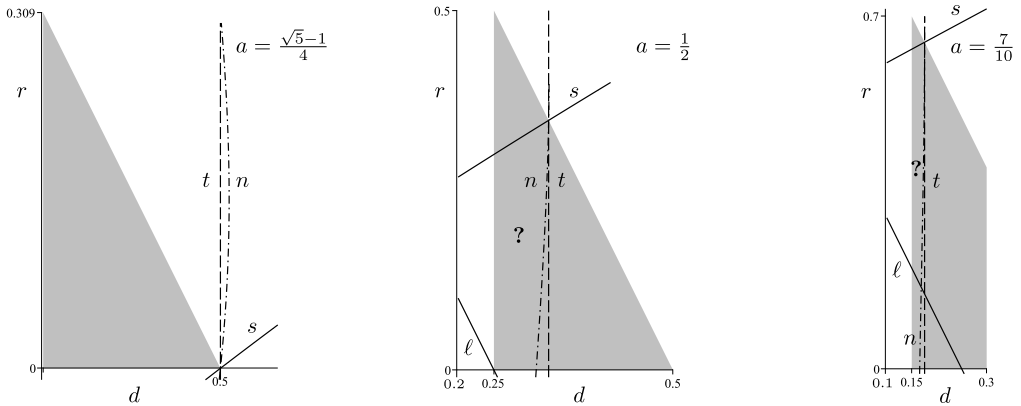


Fig. 2.  $dr$ -sections of the domain of Theorem 6 with curve  $s \equiv H(a, d, r) = 0$ , curve  $t \equiv J(a, d, r) = 0$ , curve  $n \equiv L(a, d, r) = 0$  and curve  $\ell = \{r + 2d = \frac{1}{2}\} \cap \{X = a\}$  for  $a \in \{\frac{\sqrt{5}-1}{4}, \frac{1}{2}, \frac{7}{10}\}$ .

and

$$2(a + d)^3 - 1 - a^3 - (a + 2d - 1)^3 = -\frac{1563}{62500} < 0. \quad \square$$

Let  $a, d$  and  $r$  be under the hypothesis of Theorem 6 and let define the functions

$$H(a, d, r) = \left(\frac{4 + r + 2d}{5}\right)^3 + \left(\frac{5a - 1 + r + 2d}{5}\right)^3 + \left(\frac{5a - 4r - 1 + 2d}{5}\right)^3 + 2\left(\frac{r - 5a - 3d - 1}{5}\right)^3$$

$$J(a, d, r) = 2(a + d)^3 - 1 - a^3 - (a + 2d - 1)^3$$

$$L(a, d, r) = s_1(\sigma)^3 + 6(a - r)s_1(\sigma)(a - r - s_1(\sigma)) - s_3(\sigma)$$

and, for a fixed  $a$ , the curves  $s \equiv H(a, d, r) = 0$ ,  $t \equiv J(a, d, r) = 0$ ,  $n \equiv L(a, d, r) = 0$  and  $\ell = \{r + 2d = \frac{1}{2}\} \cap \{X = a\}$ . See [2] for a more detailed explanation. For the spectra  $\sigma$  under the hypotheses of Theorem 6 we have:

- If  $a \leq \frac{\sqrt{5}-1}{4}$ , the spectrum  $\sigma$  is always symmetrically realizable with constant diagonal.

- If  $\frac{\sqrt{5}-1}{4} < a \leq \frac{1}{2}$ , for some  $(d, r)$  the spectrum  $\sigma$  is symmetrically realizable with constant diagonal, those above or on curve  $s$  in Fig. 2. And for others  $\sigma$  is not symmetrically realizable, those on the right hand side of curve  $t$  (Theorem 4) and those on the right hand side of curve  $n$  (Theorem 1) in Fig. 2. The question mark in Fig. 2 means that the region under  $s$  and on the left hand side of  $n$  (including  $n$ ) is unresolved.

- If  $a > \frac{1}{2}$ , for some  $(d, r)$  the spectrum  $\sigma$  is symmetrically realizable with constant diagonal, those above or on curve  $s$ , for others  $\sigma$  is not symmetrically realizable, those on the right hand side of curve  $t$  (Theorem 4), those on the right hand side of curve  $n$

(Theorem 1), and those under or on line  $\ell$  (Theorem 2) in Fig. 2. The question mark in Fig. 2 means that the region among  $\ell$ ,  $n$  and  $s$  (including only  $n$ ) is unresolved.

The new area that was unresolved is the one between curves  $n$  and  $t$  for  $a > \frac{\sqrt{5}-1}{4}$ .

### Declaration of competing interest

None declared.

### Data availability

No data was used for the research described in the article.

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