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Universal realizability in low dimension ☆

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We say that a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of complex numbers is realizable, if it is the spectrum of a nonnegative matrix A (a realizing matrix). We say that Λ is universally realizable if it is realizable for each possible Jordan canonical form allowed by Λ. This work studies the universal realizability of spectra in low dimension, that is, realizable spectra of size $n \leq 5$. It is clear that for $n \leq 3$ the concepts of universally realizable and realizable are equivalent. The case $n = 4$ is easily deduced from previous results in [7]. We characterize the universal realizability of real spectra of size 5 and trace zero, and we describe a region for the universal realizability of nonreal 5-spectra with trace zero. As an important by-product of our study, we also show that realizable lists on the left half-plane, that is, lists $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, where λ_1 is the Perron eigenvalue and $\text{Re}\, \lambda_i \leq 0$, for $i = 2, \ldots, n$, are not necessarily universally realizable.

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1. Introduction

The *nonnegative inverse eigenvalue problem* (NIEP) is the problem of finding necessary and sufficient conditions for a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of complex numbers to be the spectrum of a nonnegative matrix. If there exists a nonnegative matrix A with spectrum Λ, we say that Λ is *realizable* and that A is a *realizing matrix*. In terms of n, the NIEP is completely solved only for $n \leq 4$. A number of sufficient conditions for the problem to have a solution are known for $n \geq 5$.

We say that a realizable list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, of complex numbers, is *universally realizable* ($\mathcal{UR}$) if, for every possible Jordan canonical form (JCF) allowed by Λ, there is a nonnegative matrix with spectrum Λ. The problem of finding necessary and sufficient conditions for a realizable list Λ, of complex numbers, to be universally realizable will be called the *universal realizability problem* (URP).

As far as we know, the first results concerning the URP are due to Minc [11]. He proved that if a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, of complex numbers, is the spectrum of an $n \times n$ positive diagonalizable matrix, then Λ is $\mathcal{UR}$. The URP has been completely solved, with different approaches, for the following types of lists:

i) In [14,15], for lists of complex numbers, of Suleĭmanova type, that is, $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, with

$$\text{Re } \lambda_i \leq 0, \quad |\text{Re } \lambda_i| \geq |\text{Im } \lambda_i|, \; \lambda_1 \geq |\lambda_i|, \; i = 2, \ldots, n.$$

ii) In [5], for lists of complex numbers, of Šmigoc type, that is, $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, with

$$\text{Re } \lambda_i \leq 0, \quad \sqrt{3}\, |\text{Re } \lambda_i| \geq |\text{Im } \lambda_i|, \; \lambda_1 \geq |\lambda_i|, \; i = 2, \ldots, n.$$

It is important to note that, in the above cases, the list is $\mathcal{UR}$ if and only if $\sum_{i=1}^{n} \lambda_i \geq 0$, that is, the list is realizable if and only if it is $\mathcal{UR}$. Sufficient conditions for more general lists have been obtained in [15] (and the references therein). In [2] the authors study the URP for spectra with two positive eigenvalues. In [7] the authors answer the question of whether certain properties that hold for the NIEP also hold for the URP. Recently, in [3], the authors proved that if a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, with $\lambda_1 > |\lambda_i|$ for $i = 2, \ldots, n$, of complex numbers, is diagonally realizable by a nonnegative matrix with constant row sums and a positive row or column, then Λ is $\mathcal{UR}$; while in [6], it is proved that if Λ is diagonally realizable by a matrix A, with all off-diagonal entries being positive (zeros are allowed on the main diagonal), then Λ is $\mathcal{UR}$.

In this work, we consider the URP for realizable lists, of complex numbers, of size $n \leq 5$. It is clear that for $n \leq 3$, both problems, the NIEP and the URP, are equivalent. For $n = 4$ and $n = 5$ there are lists which are realizable, but not universally realizable. In fact, from [7, Lemma 3.2] we have:

Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a list of complex numbers. If Λ is nonreal, then Λ is realizable if and only if Λ is $\mathcal{UR}$. If Λ is real with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, then Λ is $\mathcal{UR}$ if and only if Λ is realizable and it does not satisfy $\lambda_1 = \lambda_2 > 0 > \lambda_3 = \lambda_4$ and $\lambda_1 + \lambda_3 + \lambda_4 < 0$.

We also solve the URP for real spectra of size 5 and trace zero, and we describe a region for the universal realizability of nonreal 5-spectra with trace zero. To prove the universal realizability for these lists, we apply several criteria and results from the NIEP. In particular, we apply the criteria of Laffey and Meehan [8], Šmigoc [13], Soto and Rojo [16], Spector [17], and results due to Cantoni and Butler [1], Torre-Mayo et al. [19] and Johnson, Julio and Soto [6].

To prove the nonexistence of certain JCFs we use new methods based on Graph Theory. See Theorem 3.1 and Theorem 3.7.

As an important by-product of our results, we answer the question of whether a realizable list in the left-half plane, that is, a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of complex numbers where λ_1 is the Perron eigenvalue and Re $\lambda_i \leq 0$, for $i = 2, \ldots, n$, is $\mathcal{UR}$. The answer is negative.

The paper is organized as follows: In Section 2, we present a number of previous results, which will be used in this work. In Section 3, we study the universal realizability for lists, of complex numbers, of size 5 and trace zero. Finally, we include an Appendix A explaining a hypothesis required in Theorem 2.1 given in [7].

2. Background results

We start this section by listing some relevant results on the NIEP. The following result, due to Laffey and Meehan [8], solves the NIEP for $n = 5$ with trace zero.

Theorem 2.1. [8] *Let $\Lambda = \{\lambda_1, \ldots, \lambda_5\}$ be a list of complex numbers, with $s_k = \sum_{i=1}^{5} \lambda_i^k$, for $k \in \mathbb{N}$. Assume $s_1 = 0$. Then Λ is the spectrum of a 5×5 nonnegative matrix if and only if the following conditions hold:*

i) $s_k \geq 0$, *for* $k = 2, 3, 4, 5$;
ii) $4s_4 \geq s_2^2$ *and*
iii) $12s_5 - 5s_2s_3 + 5s_3\sqrt{4s_4 - s_2^2} \geq 0$.

The next result, due independently to Milić, Sachs and Spialter, [10,12,18], establishes a useful connection between Spectral Graph Theory and the NIEP. In [19], the authors apply this result, independently of the result of Meehan [9], to solve the NIEP for $n = 4$.

Theorem 2.2. *Let D be a weighted digraph, A its adjacency matrix and $P_D(x) = P_A(x) = |xI - A| = x^n + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_n$. Then, for each integer $1 \leq i \leq n$,*

$$k_i = \sum_{L \in \mathcal{L}_i} (-1)^{p(L)} \pi(L),$$

where $\mathcal{L}_i$ is the set of all collections of disjoint cycles L of D with exactly i vertices; $p(L)$ denotes the number of cycles of L; and $\pi(L)$ denotes the product of the weights of all arcs belonging to L.

In [19], the authors extend Theorem 2.1 and obtain, in particular, an independent solution to the NIEP, for $n = 5$ with trace zero.

Theorem 2.3. [19, Theorem 39 for $n = 5$ and $p = 2$] *Let $P(x) = x^5 + k_2x^3 + k_3x^2 + k_4x + k_5$. Then the following statements are equivalent:*

i) $P(x)$ is the characteristic polynomial of a nonnegative matrix;
ii) the coefficients of $P(x)$ satisfy:
   a) $k_2, k_3 \leq 0$;
   b) $k_4 \leq \frac{k_2^2}{4}$, and
   c) $k_5 \leq \begin{cases} k_2k_3 & \text{if } k_4 \leq 0, \\ k_3\left(\frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4}\right) & \text{if } k_4 > 0. \end{cases}$

Remark 2.1. The proofs of Theorem 2.1 and Theorem 2.3 are constructive. Below, we give realizing matrices used in the proofs of those results.

- Theorem 2.1:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{12s_5 - 5s_2s_3}{60} & \frac{4s_4 - s_2^2}{16} & \frac{s_3}{3} & \frac{s_2}{4} & 0 \end{bmatrix},$$

$$(\text{if } 12s_5 - 5s_2s_3 \geq 0)$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2 + \sqrt{4s_4 - s_2^2}}{4} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{12s_5 - 5s_2s_3 + 5s_3\sqrt{4s_4 - s_2^2}}{60} & 0 & \frac{s_3}{3} & \frac{s_2 - \sqrt{4s_4 - s_2^2}}{4} & 0 \end{bmatrix},$$

$$(\text{if } s_2^2 - 2s_4 \geq 0)$$

$$\text{and} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{s_5}{5} & \frac{2s_4-s_2^2}{8} & \frac{s_3}{3} & 0 & 0 \end{bmatrix}.$$

$$(\text{if } 2s_4 - s_2^2 > 0)$$

- Theorem 2.3:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -k_3 & 0 & 0 & 1 & 0 \\ -k_4 & 0 & 0 & 0 & 1 \\ k_2k_3 - k_5 & 0 & 0 & -k_2 & 0 \end{bmatrix} \text{and}$$

$$(\text{if } k_4 \le 0)$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4} & 0 & 1 & 0 & 0 \\ -k_3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -k_3\left(-\frac{k_2}{2} + \sqrt{\frac{k_2^2}{4} - k_4}\right) - k_5 & 0 & 0 & -\frac{k_2}{2} + \sqrt{\frac{k_2^2}{4} - k_4} & 0 \end{bmatrix}.$$

$$(\text{if } k_4 > 0)$$

Observe that, although the realizations obtained are different, all of them are Hessenberg matrices with JCF containing maximal Jordan blocks.

The next result, due to Spector [17, Theorem 3], characterizes the spectra of symmetric nonnegative matrices of size 5 and trace 0.

Theorem 2.4. [17] *Let* $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\} \subset \mathbb{R}$ *and* $s_k = \sum_{i=1}^{5} \lambda_i^k$ *for* $k \in \mathbb{N}$. *Suppose* $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \lambda_4 \ge \lambda_5 \ge -\lambda_1$ *and* $s_1 = 0$. *Then,* Λ *is symmetrically realizable if and only if the following conditions hold:*

i) $s_3 \ge 0$ *and*
ii) $\lambda_2 + \lambda_5 \le 0$.

To establish the universal realizability of realizable lists in dimension 5 and trace 0, we need the following results:

In [1] Cantoni and Butler study the eigenvalues of symmetric centrosymmetric matrices. The next result is the centrosymmetric, not necessarily symmetric, version of their results. This version has been an important and strong tool to universally realize certain realizable lists by centrosymmetric matrices. Here we only consider the odd case.

Lemma 2.1. [1] *If n is odd, the $n \times n$ matrices*

$$\begin{bmatrix} A & \mathbf{y} & JBJ \\ \mathbf{x}^T & q & \mathbf{x}^T J \\ B & J\mathbf{y} & JAJ \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A - JB & 0 & 0 \\ 0 & q & \sqrt{2}\mathbf{x}^T \\ 0 & \sqrt{2}\mathbf{y} & A + JB \end{bmatrix},$$

where $J = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$, *are orthogonally similar.*

We have been able to universally realize a number of realizable lists by applying the following result, due to Šmigoc [13].

Lemma 2.2. [13] *Suppose B is an $m \times m$ matrix with Jordan canonical form that contains at least one 1×1 Jordan block corresponding to the eigenvalue α:*

$$J(B) = \begin{bmatrix} \alpha & 0 \\ 0 & I(B) \end{bmatrix}.$$

Let $\mathbf{t}$ and $\mathbf{s}$, respectively, be the left and the right eigenvectors of B associated with the 1×1 Jordan block in the above canonical form. Furthermore, we normalize vectors $\mathbf{t}$ and $\mathbf{s}$ so that $\mathbf{t}^T \mathbf{s} = 1$. Let $J(A)$ be a Jordan canonical form for an $n \times n$ matrix

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{21}^T & \alpha \end{bmatrix},$$

where A_1 is an $(n-1) \times (n-1)$ matrix and A_{12} and A_{21} are vectors in $\mathbb{C}^{n-1}$. Then the matrix

$$C = \begin{bmatrix} A_1 & A_{12}\mathbf{t}^T \\ \mathbf{s}A_{21}^T & B \end{bmatrix}$$

has Jordan canonical form

$$J(C) = \begin{bmatrix} J(A) & 0 \\ 0 & I(B) \end{bmatrix}.$$

The following result, due to Soto and Rojo [16, Lemma 1], has been employed to construct symmetric realizations for lists of 5 real numbers.

Lemma 2.3. [16] *Let* $(\lambda_1, \lambda_2, \lambda_3, \lambda_3, \lambda_2)$ *be a vector of real numbers (even-conjugate) such that*

$$\lambda_1 \geq |\lambda_j|, \quad j = 2, 3; \quad \lambda_1 \geq \lambda_2 \geq \lambda_3, \quad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_3 + \lambda_2 \geq 0.$$

A necessary and sufficient condition for $\{\lambda_1, \lambda_2, \lambda_3, \lambda_3, \lambda_2\}$ *to be the spectrum of a* 5×5 *nonnegative symmetric circulant matrix is*

$$\lambda_1 + (\lambda_3 - \lambda_2)\frac{\sqrt{5} - 1}{2} - \lambda_2 \geq 0.$$

In [6], the authors define a matrix $A = (a_{ij})$ to be *off-diagonally positive* (ODP) if $a_{ij} > 0$ for $i \neq j$, and zero entries on the diagonal are allowed. A is said to be *quasi*-ODP, if all off-diagonal entries are positive, except for one that is allowed to be zero. Then the authors in [6] prove the following result:

Corollary 2.1. [6] *If a spectrum* Λ *is diagonalizably ODP or quasi-ODP realizable, then* Λ *is universally realizable.*

We note that Corollary 2.1 contains the well known result of Minc [11] that diagonalizable positive realization implies universal realization.

3. Universal realizability for lists of size 5 with trace zero

If a spectrum has all its elements different, there is nothing to study because there is only one JCF and it is similar to any of the known realizations given in Theorem 2.1 and Theorem 2.3. Then, only spectra with repeated eigenvalues are of interest. In this light, we distinguish between the following cases:

3.1. Nonreal spectra with repeated complex eigenvalues

In order to study the universal realizability of

$$\Lambda = \left\{a, b + ci, b - ci, -\frac{a}{2} - b + di, -\frac{a}{2} - b - di\right\} \quad \text{with} \quad a, c, d > 0,$$

we start by assuming that Λ is realizable, *i.e.*, Λ satisfies Theorem 2.1, or equivalently Theorem 2.3.

Repeated elements in Λ implies that $b = -a/4$ and $d = c$. Then we study the universal realizability of lists

$$\Lambda = \left\{a, -\frac{a}{4} + ci, -\frac{a}{4} - ci, -\frac{a}{4} + ci, -\frac{a}{4} - ci\right\}.$$

These spectra have two JCF: the diagonal one and the nondiagonal one. The realizations given by Theorem 2.1 or Theorem 2.3 are all Hessenberg matrices, see Remark 2.1, and they have a nondiagonal JCF. So we need to obtain the diagonal JCF, if possible, for these spectra. Let us see first when Λ is realizable. By Theorem 2.3, Λ is realizable if and only if

$$k_2 = 2c^2 - \frac{5a^2}{8} = \sqrt{2}\left(\sqrt{2}c + \frac{\sqrt{5}a}{2\sqrt{2}}\right)\left(c - \frac{\sqrt{5}a}{4}\right) \leq 0 \Longleftrightarrow c \leq \frac{\sqrt{5}a}{4}$$

$$k_3 = -ac^2 - \frac{5a^3}{16} \leq 0$$

$$k_4 - \frac{k_2^2}{4} = -\frac{a^2}{4}\left(c^2 + \frac{5a^2}{8}\right) \leq 0$$

$$k_5 - k_2k_3 = a\left(\frac{c^2}{2}\left(2c^2 - \frac{a^2}{4}\right) - \frac{51a^4}{256}\right) \leq 0 \text{ if } k_2 \leq 0$$

$$(\text{Note that } k_4 = \frac{8\left(2c^2 - \frac{15a^2}{8}\right)(a^2 + 16a^2)}{256} < 0 \text{ if } k_2 \leq 0).$$

As a conclusion, Λ is realizable if and only if $c \leq \frac{\sqrt{5}a}{4}$. Now we look for realizations of Λ with diagonal JCF. Several procedures are known in the literature and they work in different regions. Here we give the realization found with the desired JCF that covers the biggest region.

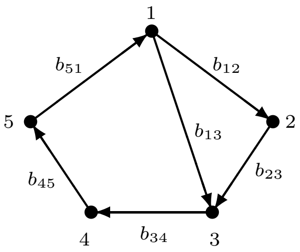
The spectrum Λ can be realized by a centrosymmetric matrix if, in the notation of Lemma 2.1, there exist matrices A and B, and vectors $\mathbf{x}$ and $\mathbf{y}$ so that $\left\{-\frac{a}{4} + ci, -\frac{a}{4} + ci\right\}$ is the spectrum of $A - JB$ real and $\left\{a, -\frac{a}{4} + ci, -\frac{a}{4} - ci\right\}$ is the spectrum of $\begin{bmatrix} 0 & \sqrt{2}\,\mathbf{x}^T \\ \sqrt{2}\,\mathbf{y} & A + JB \end{bmatrix}$. Let us take $A - JB = \begin{bmatrix} -\frac{a}{4} & 1 \\ -c^2 & -\frac{a}{4} \end{bmatrix}$. Then we need to realize the other spectrum with diagonal $0, a/4, a/4$ in order to obtain a nonnegative matrix. For

$$\begin{bmatrix} 0 & \sqrt{2}\,\mathbf{x}^T \\ \sqrt{2}\,\mathbf{y} & A + JB \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{a}{4} & 1 \\ \frac{a^3}{16} + ac^2 & \frac{a^2}{2} - c^2 & \frac{a}{4} \end{bmatrix}$$

we have the matrix, with diagonal JCF,

$$\begin{bmatrix} A & \mathbf{y} & JBJ \\ \mathbf{x}^T & q & \mathbf{x}^T J \\ B & J\mathbf{y} & JAJ \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \frac{a}{4} \\ \frac{a^2}{4} - c^2 & 0 & \frac{(a^3+16\,ac^2)\sqrt{2}}{32} & \frac{a}{4} & \frac{a^2}{4} \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ \frac{a^2}{4} & \frac{a}{4} & \frac{(a^3+16\,ac^2)\sqrt{2}}{32} & 0 & \frac{a^2}{4} - c^2 \\ \frac{a}{4} & 0 & 0 & 1 & 0 \end{bmatrix} \geq 0 \Longleftrightarrow c \leq \frac{a}{2}.$$

Note that the realization above does not cover the complete region of realizability of Λ. We do not know if the spectra corresponding to $\frac{a}{2} < c < \frac{\sqrt{5}}{4}a$ are universally realizable.



D

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} & 0 \\ 0 & 0 & b_{23} & b_{24} & b_{25} \\ 0 & 0 & 0 & b_{34} & b_{35} \\ b_{41} & b_{42} & 0 & 0 & b_{45} \\ b_{51} & b_{52} & b_{53} & 0 & 0 \end{bmatrix}$$

Fig. 1. Realizing digraph D and general adjacency matrix B.

Using Graph Theory techniques, we prove that the spectra corresponding to $c = \frac{\sqrt{5}}{4}a$ are not diagonalizably realizable. As a conclusion we have the following result:

Theorem 3.1. *Let* $\Lambda = \left\{a, b+ci, b-ci, -\frac{a}{2}-b+di, -\frac{a}{2}-b-di\right\}$, *with* $a, c, d > 0$.

i) If $b \neq -\frac{a}{4}$ *or* $d \neq c$, *then* Λ *is realizable if and only if* Λ *is* $\mathcal{UR}$.
ii) If $b = -\frac{a}{4}$ *and* $d = c$, *then* $\Lambda = \left\{a, -\frac{a}{4}+ci, -\frac{a}{4}-ci, -\frac{a}{4}+ci, -\frac{a}{4}-ci\right\}$ *is realizable if and only if* $c \leq \frac{\sqrt{5}}{4}a$, *and* Λ *is* $\mathcal{UR}$ *if* $c \leq \frac{a}{2}$.
iii) If $\Lambda = \left\{a, -\frac{a}{4}+\frac{\sqrt{5}a}{4}i, -\frac{a}{4}-\frac{\sqrt{5}a}{4}i, -\frac{a}{4}+\frac{\sqrt{5}a}{4}i, -\frac{a}{4}-\frac{\sqrt{5}a}{4}i\right\}$, *then* Λ *is not diagonalizably realizable, and therefore is not* $\mathcal{UR}$.

Proof. *iii)* It is enough to prove the result for the "normalized" spectrum

$$\sigma = \left\{4, -1+\sqrt{5}i, -1-\sqrt{5}i, -1+\sqrt{5}i, -1-\sqrt{5}i\right\}.$$

Its characteristic polynomial is $P(x) = x^5 - 40x^2 - 60x - 144$. Then, from Theorem 2.2, any digraph D realizing σ has no loops nor 2-cycles, but has cycles of lengths three, four and five. Without loss of generality, we can choose a 5-cycle with any diagonal giving a 4-cycle, as in Fig. 1. Note that the realizing digraph D is an antisymmetric digraph with the property: if uv is an arc, then vu is not an arc. Then the digraphs D, as in Fig. 1, must also include some of the arcs $14, 24, 25$ and 35, or their opposites, in order to have a 3-cycle. Thus, the more general adjacency matrix of these digraphs is the matrix B, as in Fig. 1, where the entries $b_{12}, b_{23}, b_{34}, b_{45}, b_{51}$ and b_{13} are positive (given the existence of cycles of lengths 5 and 4), and $b_{14}b_{41} = b_{24}b_{42} = b_{25}b_{52} = b_{35}b_{53} = 0$ (by the not existence of 2-cycles).

It is a simple exercise to verify that a matrix for the form B realizes the spectrum σ for some choice of b_{ij}, since $P(B) = 0$, except if $b_{41} = b_{25} = b_{42} = b_{53} = 0$, in which case B is the adjacency matrix of the directed Petersen digraph, the only one of these antisymmetric digraphs that does not realize the spectrum σ.

The minimal polynomial of a matrix realizing diagonally the spectrum σ is

$$Q(x) = (x-4)(x+1-\sqrt{5}i)(x+1+\sqrt{5}i) = x^3 - 2x^2 - 2x - 24.$$

To prove that the spectrum σ is not diagonalizably realizable, it is sufficient to verify that the evaluation of the minimal polynomial $Q(x)$ in the matrix B is not null. Let $C = (c_{ij}) = Q(B)$. Our argument only involves the following entries which must be null:

$$\begin{aligned} c_{51} &= b_{24}b_{41}b_{52} + b_{34}b_{41}b_{53} - 2b_{51} \\ c_{34} &= b_{14}b_{35}b_{51} + b_{24}b_{35}b_{52} - 2b_{34} \\ c_{52} &= b_{14}b_{42}b_{51} + b_{34}b_{42}b_{53} - 2b_{12}b_{51} - 2b_{52} \\ c_{33} &= b_{13}(b_{34}b_{41} + b_{35}b_{51}) + b_{23}(b_{34}b_{42} + b_{35}b_{52}) + b_{34}b_{35}b_{52} - 24, \end{aligned}$$

where the 2-cycles $b_{ij}b_{ji}$ have been eliminated. Substituting $2b_{51} = b_{24}b_{41}b_{52} + b_{34}b_{41}b_{53}$, in the others, we have:

$$\begin{aligned} c_{34} &= b_{24}b_{35}b_{52} - 2b_{34} \\ c_{52} &= -b_{12}b_{41}(b_{24}b_{52} + b_{34}b_{53}) + b_{34}b_{42}b_{53} - 2b_{52} \\ c_{33} &= \frac{1}{2}b_{13}b_{41}(b_{24}b_{35}b_{52} + 2b_{34}) + b_{23}(b_{34}b_{42} + b_{35}b_{52}) + b_{34}b_{45}b_{53} - 24, \end{aligned}$$

and taking $2b_{34} = b_{24}b_{35}b_{52}$, we have

$$c_{52} = -b_{12}b_{24}b_{41}b_{52} - 2b_{52} \quad \text{and} \quad c_{33} = b_{13}b_{24}b_{35}b_{41}b_{52} + b_{23}b_{35}b_{52} - 24,$$

which imply $b_{52} = 0$, and then $c_{33} = -24$. So finally, $Q(B) \neq 0$, the matrix B is not diagonalizable, and the spectrum σ is not diagonalizably realizable. □

Remark 3.1. The fact that the lists $\Lambda = \left\{a, -\frac{a}{4} + \frac{\sqrt{5}a}{4}i, -\frac{a}{4} - \frac{\sqrt{5}a}{4}i, -\frac{a}{4} + \frac{\sqrt{5}a}{4}i, -\frac{a}{4} - \frac{\sqrt{5}a}{4}i\right\}$ are not $\mathcal{UR}$ answer negatively the question: Is a realizable list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, of complex numbers, in the left-half plane, that is, λ_1 being the Perron eigenvalue and $\operatorname{Re}\lambda_i \leq 0$, for $i = 2, \ldots, n$, universally realizable? Taking into account the good behavior of Suleĭmanova and Šmigoc type spectra, it is rather surprising that spectra in the left half plane need not be universally realizable.

3.2. Nonreal spectra with repeated real eigenvalues

In order to study the universal realizability of

$$\Lambda = \left\{a, b, c, -\frac{a+b+c}{2} + di, -\frac{a+b+c}{2} - di\right\} \quad \text{with} \quad a \geq b \geq c \quad \text{and} \quad d > 0,$$

we start by assuming that Λ is realizable, *i.e.*, Λ satisfies Theorem 2.1, or equivalently Theorem 2.3.

The cases $a = b > c$ or $a = b = c$ are clearly impossible. So we just have to study the case $a > b = c$, that is, $\Lambda = \left\{a, b, b, -\frac{a}{2} - b + di, -\frac{a}{2} - b - di\right\}$. These spectra have two

JCF: the diagonal one and the nondiagonal one. The realizations given by Theorem 2.1 or Theorem 2.3 are all Hessenberg matrices, see Remark 2.1, and they have nondiagonal JCFs. So we need to separate the real eigenvalue b into two 1×1 blocks. Before going on, let us see how the realizability region is in the bd-space for a fixed a. By Theorem 2.3, Λ is realizable if and only if

$$k_2 = \frac{5a^2}{8}\left(\frac{d^2}{\left(\frac{\sqrt{5}\,a}{2\sqrt{2}}\right)^2} - \frac{\left(b+\frac{a}{4}\right)^2}{\left(\frac{\sqrt{5}\,a}{4}\right)^2} - 1\right) \le 0,$$

$$k_3 = -2bd^2 + ab^2 - ad^2 + \frac{a^2 b}{2} - \frac{a^3}{4} \le 0,$$

$$k_4 - \frac{k_2^2}{4} = 2b^2d^2 - \frac{d^4}{4} + \frac{5abd^2}{2} + \frac{a^2b^2}{4} + \frac{3a^2d^2}{8} + \frac{a^3 b}{8} - \frac{9a^4}{64} \le 0,$$

$$k_5 - k_2k_3 = -\frac{ab^2\left((a+2b)^2 + 4d^2\right)}{4} - k_2k_3 \le 0 \text{ if } k_2, k_3 \le 0,$$

$$k_5 - k_3\left(\frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4}\right) = -\frac{ab^2\left((a+2b)^2 + 4d^2\right)}{4} - k_3\left(\frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4}\right) \le 0 \text{ if } \begin{array}{l} k_2, k_3 \le 0 \\ k_4 - \frac{k_2^2}{4} \le 0 \end{array}.$$

It can be checked that if $k_2, k_3, k_4 - \frac{k_2^2}{4} \le 0$, k_4 can have either sign nonpositive and positive. As a conclusion, the realizability region in the bd-space is the region on the first and second quadrants bounded by $k_2, k_3 = 0$ and $k_4 = \frac{k_2^2}{4}$, see the gray region in Fig. 2. We look for realizations of spectra corresponding to the gray region with diagonal JCF. Several procedures are known to do this and they work in different regions. Here we give some of the realizations with the desired JCF:

- The spectrum Λ is realized by the circulant matrix, with diagonal JCF,

$$\begin{bmatrix} 0 & c_1 & c_2 & c_3 & c_4 \\ c_4 & 0 & c_1 & c_2 & c_3 \\ c_3 & c_4 & 0 & c_1 & c_2 \\ c_2 & c_3 & c_4 & 0 & c_1 \\ c_1 & c_2 & c_3 & c_4 & 0 \end{bmatrix} \ge 0 \Longleftrightarrow c_2, c_4 \ge 0,$$

  where

$$c_1 = \left(\frac{1}{4} + \frac{\sqrt{5}}{20}\right)a + \frac{\sqrt{5}\,b}{5} + \frac{\sqrt{10-2\sqrt{5}}\,d}{10}$$

$$c_2 = \left(\frac{1}{4} - \frac{\sqrt{5}}{20}\right)a - \frac{\sqrt{5}\,b}{5} - \frac{\sqrt{10+2\sqrt{5}}\,d}{10}$$

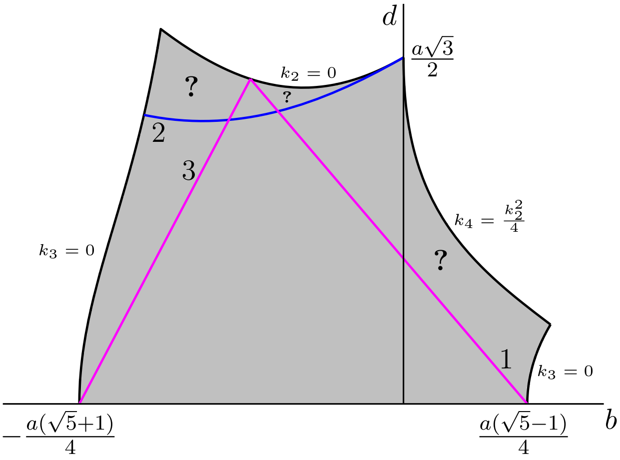


Fig. 2. ■ region of realizability for $\{a, b, b, -\frac{a}{2} - b + d, -\frac{a}{2} - b - di\}$ in the bd-space for a fixed a, line 1 $\equiv \frac{\sqrt{5}}{5}b + \frac{\sqrt{10+2\sqrt{5}}}{10}d = \left(\frac{1}{4} - \frac{\sqrt{5}}{20}\right)a$, curve 2 $\equiv -\frac{\left(b+\frac{a}{2}\right)^2}{\left(\frac{a}{\sqrt{2}}\right)^2} + \frac{d^2}{\left(\frac{a}{\sqrt{2}}\right)^2} = 1$, line 3 $\equiv \frac{\sqrt{5}}{5}b - \frac{\sqrt{10-2\sqrt{5}}}{10}d = -\left(\frac{1}{4} + \frac{\sqrt{5}}{20}\right)a$, and regions with a question mark are the unknown region for universal realizability.

$$c_3 = \left(\frac{1}{4} - \frac{\sqrt{5}}{20}\right)a - \frac{\sqrt{5}\,b}{5} + \frac{\sqrt{10+2\sqrt{5}}\,d}{10}$$
$$c_4 = \left(\frac{1}{4} + \frac{\sqrt{5}}{20}\right)a + \frac{\sqrt{5}\,b}{5} - \frac{\sqrt{10-2\sqrt{5}}\,d}{10}.$$

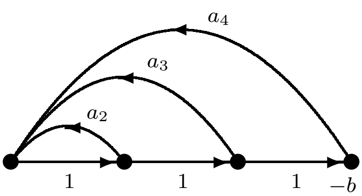
These are the spectra corresponding to the pairs (b, d) under or on lines 1 and 3 in Fig. 2.

- Certain spectra Λ, for $b \leq 0$, can be realized with Lemma 2.2. If $\{-b, b\}$ is the spectrum of

$$B = \begin{bmatrix} 0 & -b \\ -b & 0 \end{bmatrix} \sim \begin{bmatrix} \alpha = -b & 0 \\ 0 & b \end{bmatrix},$$

then, to realize Λ, we need to realize $\{a, b, -\frac{a}{2} - b + di, -\frac{a}{2} - b - di\}$ by a matrix A with diagonal entries $0, 0, 0, -b$. Following the technique in [19], we look for

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \\ \hline a_4 & 0 & 0 & -b \end{array}\right]$$



Since $|A - xI| = x^4 + bx^3 - a_2x^2 - (a_3 + ba_2)x - a_4 - ba_3 = (x - a)(x - b)(x + \frac{a}{2} + b - di)(x + \frac{a}{2} + b + di)$, by identifying the coefficients, we have

$$a_2 = b^2 + ab - d^2 + \frac{3a^2}{4} = \frac{a^2}{2}\left(\frac{\left(b+\frac{a}{2}\right)^2}{\left(\frac{a}{\sqrt{2}}\right)^2} - \frac{d^2}{\left(\frac{a}{\sqrt{2}}\right)^2} + 1\right)$$

$$a_3 = \frac{a^3}{4} + \frac{a^2 b}{4} + ad^2 + b^3 + bd^2 - ba_2 = -k_3 \geq 0$$

$$a_4 = -abd^2 - \frac{a^3 b}{4} - a^2b^2 - ab^3 - ba_3 = \frac{-b\left(a^2+4d^2\right)(a+b)}{2} \geq 0$$

where the second inequality is due to the fact that Λ is realizable. Now, the application of Lemma 2.2 with $\mathbf{t}^T = (1/2, 1/2)$ and $\mathbf{s}^T = (1,1)$ provides the matrix C, with diagonal JCF,

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ a_2 & 0 & 1 & 0 & 0 \\ a_3 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ a_4 & 0 & 0 & 0 & -b \\ a_4 & 0 & 0 & -b & 0 \end{bmatrix} \geq 0 \Longleftrightarrow \begin{cases} b \leq 0 \\ -\frac{\left(b+\frac{a}{2}\right)^2}{\left(\frac{a}{\sqrt{2}}\right)^2} + \frac{d^2}{\left(\frac{a}{\sqrt{2}}\right)^2} \leq 1. \end{cases}$$

These are the spectra corresponding to pairs (b,d) under or on curve 2 in Fig. 2. Other partitions of Λ also work, but they cover a smaller area than the one shown.

The previous realizations do not cover the complete region of realizability of Λ. We do not know if the spectra corresponding to the three portions marked with a question mark on Fig. 2 are universally realizable. As a conclusion, we have the following result.

Theorem 3.2. *Let* $\Lambda = \left\{a, b, c, -\frac{a+b+c}{2} + di, -\frac{a+b+c}{2} - di\right\}$, *with* $a \geq b \geq c$ *and* $d > 0$, *and let* $x^5 + k_2x^3 + k_3x^2 + k_4x + k_5$ *be its characteristic polynomial.*

i) If $a > b > c$, *then* Λ *is realizable if and only if* Λ *is* $\mathcal{UR}$.
ii) If $a = b > c$ *or* $a = b = c$, *then* Λ *is not realizable.*
iii) If $a > b = c$, *then* $\Lambda = \left\{a, b, b, -\frac{a}{2} - b + di, -\frac{a}{2} - b - di\right\}$ *(see Fig. 2), and*
  1) Λ *is realizable if and only if* $k_2, k_3, k_4 - \frac{k_2^2}{4} \leq 0$;
  2) if Λ *is realizable and satisfies one of the following conditions*

$$b^2 + ab - d^2 + \frac{3a^2}{4} \geq 0, \text{ or}$$

$$\left(\frac{1}{4} - \frac{\sqrt{5}}{20}\right)a - \frac{\sqrt{5}\,b}{5} - \frac{\sqrt{10+2\sqrt{5}}\,d}{10}, \left(\frac{1}{4} + \frac{\sqrt{5}}{20}\right)a + \frac{\sqrt{5}\,b}{5} - \frac{\sqrt{10-2\sqrt{5}}\,d}{10} \geq 0,$$

then Λ *is* $\mathcal{UR}$.

3.3. Real spectra

We study

$$\Lambda = \{a, b, c, d, -(a+b+c+d)\} \quad \text{with} \quad a \geq b \geq c \geq d,\ a > 0 \quad \text{and} \quad b+c+d \leq 0$$

in terms of the number of positive elements. Note that trace 0 implies no more than three positive elements. If there is only one positive eigenvalue, then the list is of Suleĭmanova type, which is $\mathcal{UR}$ (see [14]). We study the cases of two and three positive elements separately.

Case of two positive eigenvalues
Let $\Lambda = \{a, b, -r_1, -r_2, -(a+b-r_1-r_2)\}$, with $a \geq b > 0$ and $0 \leq r_1 \leq r_2 \leq a+b-r_1-r_2 \leq a$. Let us study the case $b = a$.

Theorem 3.3. *Let $\Lambda = \{a, a, -r_1, -r_2, -(2a-r_1-r_2)\}$, with $a > 0$ and $0 < r_1 \leq r_2 \leq (2a-r_1-r_2) \leq a$. Then Λ is realizable if and only if Λ is $\mathcal{UR}$.*

Proof. The necessary reducible realization of Λ implies $a \geq r_1 + r_2$, and the Perron condition implies $a \geq 2a - r_1 - r_2$. Therefore $a = r_1 + r_2$ and we can write $\Lambda = \{a, a, -r, -(a-r), -a\}$ with $a > 0$ and $0 < r \leq a - r \leq a$. On the one hand, the realizations given by Theorem 2.1 or Theorem 2.3 are all Hessenberg matrices, see Remark 2.1, and they have a JCF with maximal Jordan blocks. On the other hand, Λ clearly satisfies Theorem 2.4, so it is symmetrically realizable, which implies it is diagonally realizable. Therefore, if $a - r \neq r$, then Λ is realizable if and only if Λ is $\mathcal{UR}$. If $a - r = r$, we have $\Lambda = \left\{a, a, -\frac{a}{2}, -\frac{a}{2}, -a\right\}$, so we need to realize Λ, if possible, with the forms $J_2(a) \oplus J_1\left(-\frac{a}{2}\right) \oplus J_1\left(-\frac{a}{2}\right) \oplus J_1(-a)$ and $J_1(a) \oplus J_1(a) \oplus J_2\left(-\frac{a}{2}\right) \oplus J_1(-a)$.

Since the Suleĭmanova list $\left\{a, -\frac{a}{2}, -\frac{a}{2}\right\}$ is $\mathcal{UR}$, the separated realizations of $\{a, -a\}$ and $\left\{a, -\frac{a}{2}, -\frac{a}{2}\right\}$ give the form $J_1(a) \oplus J_1(a) \oplus J_2\left(-\frac{a}{2}\right) \oplus J_1(-a)$, with realizing matrices, for example,

$$\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \frac{a}{2} & \frac{a}{2} \\ \frac{a}{2} & 0 & \frac{a}{2} \\ 0 & a & 0 \end{bmatrix}.$$

Finally, perturbing a diagonalizable realization of Λ, for example,

$$\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \frac{a}{2} & \frac{a}{2} \\ \frac{a}{2} & 0 & \frac{a}{2} \\ \frac{a}{2} & \frac{a}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & 1 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{a}{2} & \frac{a}{2} \\ 0 & 0 & \frac{a}{2} & 0 & \frac{a}{2} \\ 0 & 0 & \frac{a}{2} & \frac{a}{2} & 0 \end{bmatrix}$$

we obtain the other form $J_2(a) \oplus J_1\left(-\frac{a}{2}\right) \oplus J_1\left(-\frac{a}{2}\right) \oplus J_1(-a)$. □

Note that the spectra $\Lambda = \{a, a, -r_1, -r_2, -(2a - r_1 - r_2)\}$ for $r_1 = 0$ are reduced to $\Lambda = \{a, a, 0, -a, -a\}$, which we consider in the following result.

Theorem 3.4. *The spectra* $\Lambda = \{a, a, 0, -a, -a\}$*, with* $a > 0$*, have all possible JCFs allowed except* $J_1(a) \oplus J_1(a) \oplus J_1(0) \oplus J_2(-a)$*, and therefore are not* $\mathcal{UR}$*.*

Proof. On the one hand, the realizations given by Theorem 2.1 or Theorem 2.3 are all Hessenberg matrices, see Remark 2.1, and they have a JCF with maximal Jordan blocks. On the other hand Λ clearly satisfies Theorem 2.4, so it is symmetrically realizable, which implies it is diagonally realizable.

Perturbing a diagonalizable realization of Λ, for example,

$$\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & 1 & 1 & 0 \\ a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we obtain the form $J_2(a) \oplus J_1\left(-\frac{a}{2}\right) \oplus J_1\left(-\frac{a}{2}\right) \oplus J_1(-a)$.

Suppose there exists a nonnegative realization M of Λ similar to $J_1(a) \oplus J_1(a) \oplus J_1(0) \oplus J_2(-a)$. This means that the minimal polynomial of M is $(x-a)x(x+a)^2 = x^4 + ax^3 - a^2x^2 - a^3x$, so $M^4 + aM^3 - a^2M^2 - a^3M = 0$, and that $\text{rank}(M + aI) = 4$. It is clear that Λ only admits reducible realizations and must be partitioned as $\{a, -a\} \cup \{a, -a\} \cup \{0\}$ or $\{a, -a\} \cup \{a, 0, -a\}$. Let us study each case separately.

Case $\{a, -a\} \cup \{a, -a\} \cup \{0\}$: we can assume, without loss of generality, that M is of the form

$$M = \begin{bmatrix} A & 0 & 0 \\ B & C & 0 \\ D & E & F \end{bmatrix}.$$

We have three possibilities for the realization of $\{0\}$:

- If $A = 0$, then $\{a, -a\}$ is realized by C and F. Hence, by the Cayley-Hamilton Theorem, $C^2 = F^2 = a^2I$. Simple calculations give

$$M^2 = \begin{bmatrix} 0 & 0 & 0 \\ CB & a^2I & 0 \\ EB + FD & EC + FE & a^2I \end{bmatrix},$$

$$M^3 = \begin{bmatrix} 0 & 0 & 0 \\ a^2B & a^2C & 0 \\ ECB + FEB + a^2D & FEC + 2a^2E & a^2F \end{bmatrix}$$

and

$$M^4 = \begin{bmatrix} 0 & 0 & 0 \\ a^2CB & a^4I & 0 \\ FECB + 2a^2EB + a^2FD & 2a^2EC + 2a^2FE & a^4I \end{bmatrix}.$$

Now, by equating the block in position $(3, 2)$ of $M^4 + aM^3 - a^2M^2 - a^3M$ to zero, we have $aFEC + a^2EC + a^2FE + a^3E = 0$. Since the matrices involved are nonnegative and $a > 0$ we obtain $E = 0$ and, observing the columns,

$$\text{rank}(M + aI) = \text{rank} \begin{bmatrix} a & 0 & 0 \\ B & C + aI & 0 \\ D & 0 & F + aI \end{bmatrix} = 3$$

which gives a contradiction.

- If $C = 0$, then $\{a, -a\}$ is realized by A and F. Hence, by the Cayley-Hamilton Theorem, $A^2 = F^2 = a^2I$. Simple calculations give

$$M^2 = \begin{bmatrix} a^2I & 0 & 0 \\ BA & 0 & 0 \\ DA + EB + FD & FE & a^2I \end{bmatrix},$$

$$M^3 = \begin{bmatrix} a^2A & 0 & 0 \\ a^2B & 0 & 0 \\ EBA + FDA + FEB + 2a^2D & a^2E & a^2F \end{bmatrix}$$

and

$$M^4 = \begin{bmatrix} a^4I & 0 & 0 \\ a^2BA & 0 & 0 \\ FEBA + 2a^2DA + 2a^2EB + 2a^2FD & a^2FE & a^4I \end{bmatrix}.$$

By equating the block in position $(3, 1)$ of $M^4 + aM^3 - a^2M^2 - a^3M$ to zero, we have $FEBA + aEBA + aFDA + aFEB + a^2DA + a^2EB + a^2FD + a^3D = 0$. Since the matrices involved are nonnegative and $a > 0$, we obtain $D = EB = 0$. Note that $EB = 0$ where E is a nonnegative column matrix and B a nonnegative row matrix guarantees that $B = 0$ or $E = 0$. If $B = 0$, observing the columns, and if $E = 0$, observing the rows, we have

$$\text{rank}(M + aI) = \left\{ \begin{matrix} \text{rank} \begin{bmatrix} A + aI & 0 & 0 \\ 0 & a & 0 \\ 0 & E & F + aI \end{bmatrix} \\ \text{rank} \begin{bmatrix} A + aI & 0 & 0 \\ B & a & 0 \\ 0 & 0 & F + aI \end{bmatrix} \end{matrix} \right\} = 3$$

which gives a contradiction.

- If $F = 0$, then $\{a, -a\}$ is realized by A and C. Hence, by the Cayley-Hamilton Theorem, $A^2 = C^2 = a^2 I$. Simple calculations give

$$M^2 = \begin{bmatrix} a^2 I & 0 & 0 \\ BA + CB & a^2 I & 0 \\ DA + EB & EC & 0 \end{bmatrix}, \; M^3 = \begin{bmatrix} a^2 A & 0 & 0 \\ CBA + 2a^2 B & a^2 C & 0 \\ EBA + ECB + a^2 D & a^2 E & 0 \end{bmatrix}$$

and

$$M^4 = \begin{bmatrix} a^4 I & 0 & 0 \\ 2a^2 BA + 2a^2 CB & a^4 I & 0 \\ ECBA + a^2 DA + 2a^2 EB & a^2 EC & 0 \end{bmatrix}.$$

Now, by equating the block in position $(2, 1)$ of $M^4 + aM^3 - a^2 M^2 - a^3 M$ to zero, we have $aCBA + a^2 BA + a^2 CB + a^3 B = 0$, so $B = 0$ and, observing the rows,

$$\operatorname{rank}(M + aI) = \operatorname{rank} \begin{bmatrix} A + aI & 0 & 0 \\ 0 & C + aI & 0 \\ D & E & a \end{bmatrix} = 3$$

which gives a contradiction.

Case $\{a, -a\} \cup \{a, 0, -a\}$: we can assume, without loss of generality, that M is of the form

$$M = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix},$$

with A and C irreducible. This means, in particular, that none of them have a zero column or row. We have two possibilities for the realization of $\{a, -a\}$:

- If $\{a, -a\}$ is realized by A, then $\{a, 0, -a\}$ is realized by C. Hence, by the Cayley-Hamilton Theorem, $A^2 = a^2 I$ and $C^3 = a^2 C$. Simple calculations give

$$M^2 = \begin{bmatrix} a^2 I & 0 \\ BA + CB & C^2 \end{bmatrix}, \; M^3 = \begin{bmatrix} a^2 A & 0 \\ a^2 B + CBA + C^2 B & a^2 C \end{bmatrix}$$

and

$$M^4 = \begin{bmatrix} a^4 I & 0 \\ a^2 BA + 2a^2 CB + C^2 BA & a^2 C^2 \end{bmatrix}.$$

Now, by equating the block in position $(2,1)$ of $M^4 + aM^3 - a^2M^2 - a^3M$ to zero, we have $a^2CB + C^2BA + aCBA + aC^2B = 0$, so $CB = 0$. However, this implies that $B = 0$, because B and C are nonnegative and C has no zero column. Then,

$$\operatorname{rank}(M + aI) = \operatorname{rank}\begin{bmatrix} A + aI & 0 \\ 0 & C + aI \end{bmatrix} = 3$$

which gives a contradiction.

- If $\{a, 0, -a\}$ is realized by A, then $\{a, -a\}$ is realized by C. Hence, by the Cayley-Hamilton Theorem, $A^3 = a^2A$ and $C^2 = a^2I$. Simple calculations give

$$M^2 = \begin{bmatrix} A^2 & 0 \\ BA + CB & a^2I \end{bmatrix},\ M^3 = \begin{bmatrix} a^2A & 0 \\ BA^2 + CBA + a^2B & a^2C \end{bmatrix}$$

and

$$M^4 = \begin{bmatrix} a^2A^2 & 0 \\ 2a^2BA + CBA^2 + a^2CB & a^4I \end{bmatrix}.$$

Now, by equating the block in position $(2,1)$ of $M^4 + aM^3 - a^2M^2 - a^3M$ to zero, we have $a^2BA + CBA^2 + aBA^2 + aCBA = 0$, so $BA = 0$. Yet this implies that $B = 0$, because B and A are nonnegative and A has no zero row. Now, the same argument as before gives a contradiction.

We can conclude that the spectra Λ have each possible JCF allowed except $J_1(a) \oplus J_1(a) \oplus J_1(0) \oplus J_2(-a)$. □

Now we study the case $a > b > 0$ with repetitions on the nonpositive eigenvalues.

Theorem 3.5. *Let $\Lambda = \{a, b, c, d, -(a + b + c + d)\}$, with $a > b > 0 \geq c \geq d$ and $-\frac{a+b+c}{2} \leq d \leq -(b + c)$.*

i) If $c = d = -(a + b + c + d)$, then $\Lambda = \left\{a, b, -\frac{a+b}{3}, -\frac{a+b}{3}, -\frac{a+b}{3}\right\}$ is realizable if and only if $b \leq \frac{1+3\sqrt{6}-\sqrt{6\sqrt{6}-9}}{8}a$, and Λ is $\mathcal{UR}$ if and only if $b \leq \frac{a}{2}$.

ii) If $c > d = -(a + b + c + d)$, then $\Lambda = \left\{a, b, c, -\frac{a+b+c}{2}, -\frac{a+b+c}{2}\right\}$ is realizable if and only if Λ is $\mathcal{UR}$ ([7, Theorem 3.3]).

iii) If $c = d > -(a + b + c + d)$, then $\Lambda = \{a, b, c, c, -(a + b + 2c)\}$ is realizable if and only if Λ is $\mathcal{UR}$ ([7, Theorem 3.3]).

Proof. *i)* By Theorem 2.3, Λ is realizable if and only if

$$k_2 = -\frac{2a^2 + 2b^2 + ab}{3} \leq 0$$

$$k_3 = -\frac{(a+b)}{27}\left[8\left(a-\frac{11b}{16}\right)^2+\frac{135b^2}{32}\right] \leq 0$$

$$k_4 - \frac{k_2^2}{4} = -\frac{4(a^2+b^2)+(3\sqrt{6}-1)ab}{27}\left[b-\frac{1+3\sqrt{6}+\sqrt{6\sqrt{6}-9}}{8}\,a\right]$$

$$\times\left[b-\frac{1+3\sqrt{6}-\sqrt{6\sqrt{6}-9}}{8}\,a\right] \leq 0$$

$$\overset{a>b}{\Longleftrightarrow} b \leq \frac{1+3\sqrt{6}-\sqrt{6\sqrt{6}-9}}{8}\,a$$

$$k_5 - k_2k_3 = -\frac{(a+b)\left(32a^2+32b^2+(9\sqrt{17}-17)ab\right)}{162}\left[\left(a-\frac{(9\sqrt{17}+17)\,b}{64}\right)^2\right.$$

$$\left.+\frac{1215-153\sqrt{17}}{2048}\,b^2\right] \leq 0$$

$$k_5 - k_3\left(\frac{k_2}{2}-\sqrt{\frac{k_2^2}{4}-k_4}\right) \overset{\left\{\begin{smallmatrix} k_3 \leq 0 \\ k_4 \leq k_2^2/2\end{smallmatrix}\right\}}{\underset{\downarrow}{\leq}} k_5 - \frac{k_2k_3}{2}$$

$$\overset{(*)}{<} -\frac{a+b}{16}\left[16b^2\left(b-\frac{5}{8}a\right)^2+\frac{11}{4}a^2b^2\right] \leq 0$$

Note that $k_4 = -\dfrac{(a+b)^2\left(b-\frac{7+3\sqrt{5}}{2}\,a\right)\left(b-\frac{7-3\sqrt{5}}{2}\,a\right)}{27}\begin{cases} \leq 0 & \text{if} \quad b \leq \frac{7-3\sqrt{5}}{2}\,a \\ > 0 & \text{if} \quad b > \frac{7-3\sqrt{5}}{2}\,a \end{cases}.$

$$(*)\begin{cases} k_5 - \frac{k_2k_3}{2} = -\frac{a+b}{16}\left[16a^2\left(\frac{5}{8}b-a\right)^2+16b^2\left(b-\frac{5}{8}a\right)^2-\frac{7}{2}a^2b^2\right] \\ < -\frac{a+b}{16}\left[16a^2\left(\frac{5}{8}b\right)^2+16b^2\left(b-\frac{5}{8}a\right)^2-\frac{7}{2}a^2b^2\right] = -\frac{a+b}{16}\left[16b^2\left(b-\frac{5}{8}a\right)^2+\frac{11}{4}a^2b^2\right] \end{cases}.$$

In conclusion, Λ is realizable if and only if $b \leq \frac{1+3\sqrt{6}-\sqrt{6\sqrt{6}-9}}{8}\,a$.

On the one hand, by Theorem 2.4, Λ is symmetrically realizable if and only if

$$\lambda_2+\lambda_5 = \frac{2b-a}{3} \leq 0 \Longleftrightarrow b \leq \frac{a}{2}$$
$$s_3 = -3k_3 \geq 0.$$

On the other hand, Λ can be scaled as $\frac{6}{a+b}\Lambda = \left\{\frac{6a}{a+b}, \frac{6b}{a+b}, -2, -2, -2\right\}$, which can be written in the well known form $\Lambda_{\pm t} = \{3+t, 3-t, -2, -2, -2\}$, with $t = \frac{3(a-b)}{a+b}$. In [4, Proposition 1], it is proved that if $\Lambda_{\pm t}$ is diagonally realizable, then $t \geq 1$. But this is equivalent to $b \leq \frac{a}{2}$. In conclusion, we have that Λ is diagonally realizable if and only if Λ is symmetrically realizable, and if and only if $b \leq \frac{a}{2}$.

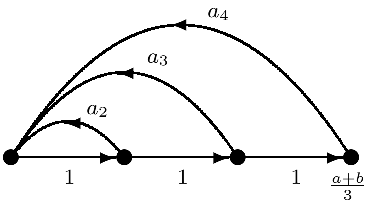
There exists a proof of the realizability of $\Lambda_{\pm t}$ due to Laffey and Meehan, but we think this new one is interesting in terms of the coefficients of the characteristic polynomial of Λ.

Theorem 2.1 or Theorem 2.3, see Remark 2.1, give JCF with maximal Jordan blocks and Theorem 2.4 the diagonal one for Λ. Therefore, we just need to find $J_1(a) \oplus J_1(b) \oplus J_2\left(-\frac{a+b}{3}\right) \oplus J_1\left(-\frac{a+b}{3}\right)$. We use Lemma 2.2 for this purpose. If $\left\{\frac{a+b}{3}, -\frac{a+b}{3}\right\}$ is the spectrum of

$$B = \begin{bmatrix} 0 & \frac{a+b}{3} \\ \frac{a+b}{3} & 0 \end{bmatrix} \sim \begin{bmatrix} \alpha = \frac{a+b}{3} & 0 \\ 0 & -\frac{a+b}{3} \end{bmatrix},$$

then, to realize Λ, we need to realize $\left\{a, b, -\frac{a+b}{3}, -\frac{a+b}{3}\right\}$ by a matrix A with diagonal entries $0, 0, 0, \frac{a+b}{3}$. Following the technique in [19], we look for

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \\ \hline a_4 & 0 & 0 & \frac{a+b}{3} \end{array}\right]$$



Since $|A - xI| = x^4 - \frac{a+b}{3}x^3 - a_2x^2 - (a_3 - a_2\frac{a+b}{3})x - a_4 + a_3\frac{a+b}{3} = (x-a)(x-b)\left(x + \frac{a+b}{3}\right)^2$, by identifying the coefficients, we have

$$\begin{aligned} a_2 &= \frac{5a^2 + 5b^2 + ab}{9} \geq 0 \\ a_3 &= \frac{(a+b)(8a^2 - 11ab + 8b^2)}{27} = -k_3 \geq 0 \\ a_4 &= \frac{4(a+b)^2(2a-b)(a-2b)}{81} \geq 0 \Longleftrightarrow b \leq \frac{a}{2} \end{aligned}$$

where the second inequality is due to the fact that Λ is realizable. Now, the application of Lemma 2.2 with $\mathbf{t}^T = (1/2, 1/2)$ and $\mathbf{s}^T = (1, 1)$ provides the matrix C, with the desired JCF for $b \leq \frac{a}{2}$,

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ a_2 & 0 & 1 & 0 & 0 \\ a_3 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ a_4 & 0 & 0 & 0 & \frac{a+b}{3} \\ a_4 & 0 & 0 & \frac{a+b}{3} & 0 \end{bmatrix}. \quad \square$$

Case of three positive eigenvalues

Let $\Lambda = \{a, b, c, -r, -(a+b+c-r)\}$, with $a \geq b \geq c > 0 > -r \geq -(a+b+c-r) \geq -a$. Note that, with three positive eigenvalues, the nonpositive eigenvalues must be negative.

The case $a = b$ is clearly impossible, because the Perron conditions $a \geq r$ and $a \geq 2a + c - r$ are contradictory with $c > 0$. If $b = c$, then $\Lambda = \{a, b, b, -r, -(a + 2b - r)\}$ with $a > b > 0 > -r \geq -(a + 2b - r)$ and we have:

Theorem 3.6. *Let $\Lambda = \{a, b, b, -r, -(a+2b-r)\}$, with $a > b > 0$ and $0 < r \leq a+2b-r < a$.*

i) If $r \neq a + 2b - r$, then Λ is realizable if and only if Λ is $\mathcal{UR}$.
ii) If $r = a + 2b - r$, then $\Lambda = \left\{a, b, b, -\left(\frac{a}{2} + b\right), -\left(\frac{a}{2} + b\right)\right\}$ is realizable if and only if $b \leq \frac{\sqrt{5}-1}{4}a$, and Λ is $\mathcal{UR}$ if and only if $b < \frac{\sqrt{5}-1}{4}a$.

Proof. *i)* If $\Lambda = \{a, b, b, -r, -(a+2b-r)\}$ is realizable (in particular $s_3 \geq 0$), as $\lambda_2+\lambda_5 = -a-b+r < 0$, from Theorem 2.4, Λ has a symmetric realization, therefore Λ is diagonally realizable. For the nondiagonal JCF, a matrix can be obtained via Theorem 2.1 or Theorem 2.3. Thus Λ is realizable if and only if it is $\mathcal{UR}$.

ii) By Theorem 2.3, Λ is realizable if and only if

$$k_2 = -\frac{1}{4}\left(3a^2 + 4ab + 8b^2\right) \leq 0$$

$$k_3 = -\frac{a}{4}\left(a^2 - 2ab - 4b^2\right) \leq 0 \Longleftrightarrow b \leq \frac{\sqrt{5}-1}{4}a$$

$$k_4 - \frac{k_2^2}{4} = -\frac{a^2}{64}\left(9a^2 - 8ab - 16b^2\right) \leq 0 \Longleftrightarrow b \leq \frac{\sqrt{10}-1}{4}a$$

$$k_5 - k_3\left(\frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4}\right) = -\frac{a^2}{32}\Big(a\left(3a^2 - 2ab - 4b^2\right) + \left(a^2 - 2ab - 4b^2\right)\sqrt{9a^2 - 8ab - 16b^2}\Big) \leq 0$$

$$\text{(Note that } k_4 = \frac{b}{4}\left(2a^3 + 5a^2b + 4ab^2 + 4b^3\right) > 0\text{).}$$

Therefore Λ is realizable if and only if $b \leq \frac{\sqrt{5}-1}{4}a$.

We shall now check when Corollary 2.1 is applicable in order to achieve the result that Λ is $\mathcal{UR}$. From Lemma 2.3, there exists a symmetric circulant realization of Λ if and only if

$$\lambda_1 + (\lambda_3 - \lambda_2)\frac{\sqrt{5}-1}{2} - \lambda_2 = \frac{\sqrt{5}}{20}\left(\left(\sqrt{5}-1\right)a - 4b\right) \geq 0 \Longleftrightarrow b \leq \frac{\sqrt{5}-1}{4}a.$$

A symmetric circulant realization of Λ is for example

$$C = \begin{bmatrix} 0 & c_1 & c_2 & c_2 & c_1 \\ c_1 & 0 & c_1 & c_2 & c_2 \\ c_2 & c_1 & 0 & c_1 & c_2 \\ c_2 & c_2 & c_1 & 0 & c_1 \\ c_1 & c_2 & c_2 & c_1 & 0 \end{bmatrix} \quad \text{with} \quad c_1 = \frac{\left(5+\sqrt{5}\right)a + 4\sqrt{5}\,b}{20} \quad \text{and}$$

$$c_2 = \frac{\left(5-\sqrt{5}\right)a - 4\sqrt{5}\,b}{20}.$$

It is clear that C is ODP if and only if $c_2 > 0$, *i.e.*, if and only if $b < \frac{\sqrt{5}-1}{4}a$, in which case, since Λ is diagonally ODP realizable, by Corollary 2.1, Λ is $\mathcal{UR}$. See the next theorem for the case $b = \frac{\sqrt{5}-1}{4}a$. $\square$

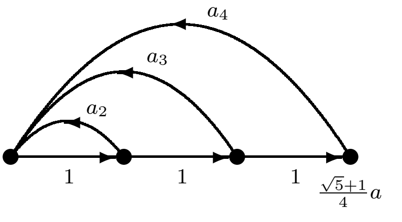
Theorem 3.7. *The spectra* $\Lambda = \left\{a, \frac{\sqrt{5}-1}{4}a, \frac{\sqrt{5}-1}{4}a, -\frac{\sqrt{5}+1}{4}a, -\frac{\sqrt{5}+1}{4}a\right\}$, *with* $a > 0$, *have each possible JCF allowed except* $J_1(a) \oplus J_1\left(\frac{\sqrt{5}-1}{4}a\right) \oplus J_1\left(\frac{\sqrt{5}-1}{4}a\right) \oplus J_2\left(-\frac{\sqrt{5}+1}{4}a\right)$, *and therefore are not* $\mathcal{UR}$.

Proof. From Theorem 2.1 or Theorem 2.3, we obtain the JCF with maximal Jordan blocks, from Theorem 2.4 the diagonal one and from Lemma 2.2 we find $J_1(a) \oplus J_2\left(\frac{\sqrt{5}-1}{4}a\right) \oplus J_1\left(-\frac{\sqrt{5}+1}{4}a\right) \oplus J_1\left(-\frac{\sqrt{5}+1}{4}a\right)$. If $\left\{\frac{\sqrt{5}+1}{4}a, -\frac{\sqrt{5}+1}{4}a\right\}$ is the spectrum of

$$B = \begin{bmatrix} 0 & \frac{\sqrt{5}+1}{4}a \\ \frac{\sqrt{5}+1}{4}a & 0 \end{bmatrix} \sim \begin{bmatrix} \alpha = \frac{\sqrt{5}+1}{4}a & 0 \\ 0 & -\frac{\sqrt{5}+1}{4}a \end{bmatrix},$$

then, to realize Λ, we need to realize $\left\{a, \frac{\sqrt{5}-1}{4}a, \frac{\sqrt{5}-1}{4}a, -\frac{\sqrt{5}+1}{4}a\right\}$ by a matrix A with diagonal entries $0, 0, 0, \frac{\sqrt{5}+1}{4}a$. Following the technique in [19], we look for

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \\ \hline a_4 & 0 & 0 & \frac{\sqrt{5}+1}{4}a \end{array}\right]$$



Since $|A - xI| = x^4 - \frac{\sqrt{5}+1}{4}ax^3 - a_2x^2 - (a_3 - a_2\frac{\sqrt{5}+1}{4}a)x - a_4 + a_3\frac{\sqrt{5}+1}{4}a = (x - a)\left(x - \frac{\sqrt{5}-1}{4}a\right)^2\left(x + \frac{\sqrt{5}+1}{4}a\right)$, by identifying the coefficients, we have

$$\begin{aligned} a_2 &= \frac{7-\sqrt{5}}{8}a^2 \geq 0 \\ a_3 &= 0 \geq 0 \\ a_4 &= \frac{\sqrt{5}-1}{16}a^4 \geq 0. \end{aligned}$$

Now, the application of Lemma 2.2 with $\mathbf{t}^T = (1/2, 1/2)$ and $\mathbf{s}^T = (1, 1)$ provides the matrix C, with the desired JCF,

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ a_2 & 0 & 1 & 0 & 0 \\ a_3 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ a_4 & 0 & 0 & 0 & \frac{\sqrt{5}+1}{4}a \\ a_4 & 0 & 0 & \frac{\sqrt{5}+1}{4}a & 0 \end{bmatrix}.$$

Now we consider $J_1(a) \oplus J_1\left(\frac{\sqrt{5}-1}{4}a\right) \oplus J_1\left(\frac{\sqrt{5}-1}{4}a\right) \oplus J_2\left(-\frac{\sqrt{5}+1}{4}a\right)$ with minimal polynomial

$$Q(x) = (x-a)\left(x - \frac{\sqrt{5}-1}{4}a\right)\left(x + \frac{\sqrt{5}+1}{4}a\right)^2.$$

We shall now prove that, if B is a realization of Λ such that $Q(B) = 0$, then B is diagonalizable, *i.e.*, also $R(B) = 0$, where

$$R(x) = (x-a)\left(x - \frac{\sqrt{5}-1}{4}a\right)\left(x + \frac{\sqrt{5}+1}{4}a\right)$$

is the minimal polynomial of any diagonalizable realization of Λ. First, we use Theorem 2.2 to obtain the general expression of the realizations B of Λ such that $Q(B) = 0$. The characteristic polynomial of Λ is

$$P(x) = x^5 + k_1x^4 + k_2x^3 + k_3x^2 + k_4x + k_5 = x^5 - \frac{5}{4}a^2x^3 + \frac{5}{16}a^4x - \frac{1}{16}a^5.$$

Since $k_1 = k_3 = 0$, any weighted digraph D realizing Λ has no loops or 3-cycles. As the coefficient k_4 is positive, then there are at least two disjoint 2-cycles in D. The independent term of $P(x)$ is obtained from cyclic structures over the five vertices $\{1, 2, 3, 4, 5\}$, so there is at least one 5-cycle in D. Without loss of generality, we can suppose the existence of the 5-cycle 123451, which we call the *base* 5-cycle of the digraph D.

As there are no 3-cycles, there can be no arcs of the 5-cycle 142531 in D, because each of these arcs closes a 3-cycle with two consecutive arcs of the base 5-cycle. In consequence, the 2-cycles of D can only be over the base 5-cycle. We analyze the possible realizing digraphs D in terms of the number of 2-cycles included in it.

If there are just two disjoint 2-cycles in D, without loss of generality, we can suppose that they are 121 and 343. The partial digraph D' of D, formed by the base 5-cycle and these two 2-cycles, is shown in Fig. 3.

There may also be some arcs of the 5-cycle 135241 in D, but there cannot be two consecutive arcs xy, yz because they would form a 3-cycle with the arc zx of the base 5-cycle 123451. Thus, there can be at most one of the following pairs of non consecutive

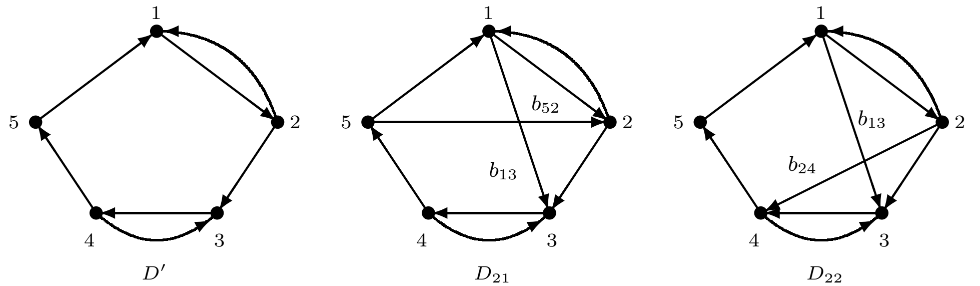


Fig. 3. Realizing digraphs D', D_{21} and D_{22}.

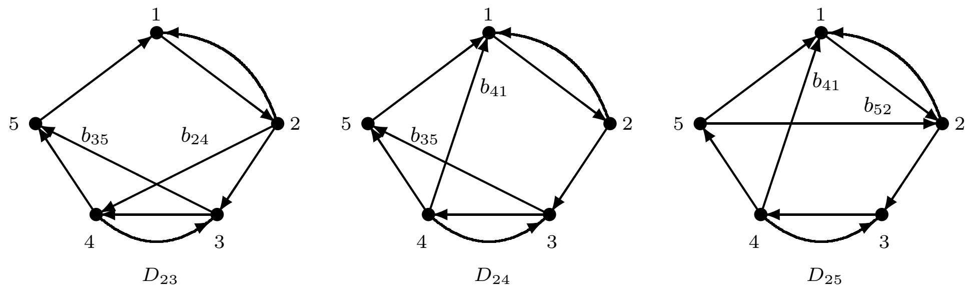


Fig. 4. Realizing digraphs D_{23}, D_{24} and D_{25}.

arcs in D: 13 and 52, 13 and 24, 35 and 24, 35 and 41 or 52 and 41. In this way, the possible realizing digraphs $D_{21}, D_{22}, D_{23}, D_{24}, D_{25}$ are those shown in Fig. 3 and Fig. 4.

The arcs of D' are also arcs of all these digraphs. Their adjacency matrices are respectively

$$B_{21} = \begin{bmatrix} 0 & b_{12} & b_{13} & 0 & 0 \\ b_{21} & 0 & b_{23} & 0 & 0 \\ 0 & 0 & 0 & b_{34} & 0 \\ 0 & 0 & b_{43} & 0 & b_{45} \\ b_{51} & b_{52} & 0 & 0 & 0 \end{bmatrix}, \; B_{22} = \begin{bmatrix} 0 & b_{12} & b_{13} & 0 & 0 \\ b_{21} & 0 & b_{23} & b_{24} & 0 \\ 0 & 0 & 0 & b_{34} & 0 \\ 0 & 0 & b_{43} & 0 & b_{45} \\ b_{51} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{23} = \begin{bmatrix} 0 & b_{12} & 0 & 0 & 0 \\ b_{21} & 0 & b_{23} & b_{24} & 0 \\ 0 & 0 & 0 & b_{34} & b_{35} \\ 0 & 0 & b_{43} & 0 & b_{45} \\ b_{51} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{24} = \begin{bmatrix} 0 & b_{12} & 0 & 0 & 0 \\ b_{21} & 0 & b_{23} & 0 & 0 \\ 0 & 0 & 0 & b_{34} & b_{35} \\ b_{41} & 0 & b_{43} & 0 & b_{45} \\ b_{51} & 0 & 0 & 0 & 0 \end{bmatrix}, \; B_{25} = \begin{bmatrix} 0 & b_{12} & 0 & 0 & 0 \\ b_{21} & 0 & b_{23} & 0 & 0 \\ 0 & 0 & 0 & b_{34} & 0 \\ b_{41} & 0 & b_{43} & 0 & b_{45} \\ b_{51} & b_{52} & 0 & 0 & 0 \end{bmatrix}.$$

Note that some of the entries b_{ij} can be 0, clearly the ones corresponding to the arcs that are not in D'. The evaluation of the minimal polynomial $Q(x)$ in each of these matrices is a matrix whose entry $(5, 5)$ is of the form $\left(\sqrt{5} + 3\right) a^4/128$ plus a nonnegative expression

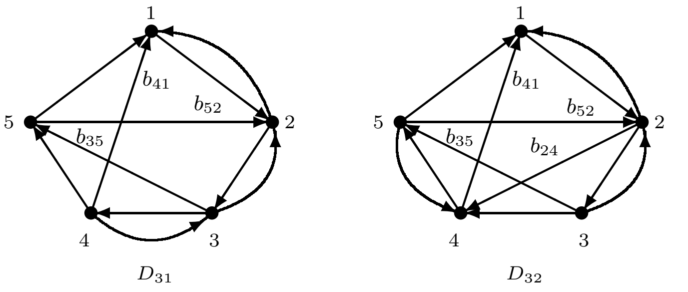


Fig. 5. Realizing digraphs D_{31} and D_{32}.

in the entries of each matrix. For instance, the entry $(5,5)$ of the matrix $Q(B_{21})$ is $\left(\sqrt{5}+3\right)a^4/128 + b_{34}b_{45}(b_{13}b_{51}+b_{23}b_{52}) \geq \left(\sqrt{5}+3\right)a^4/128 > 0$. Then $Q(B_{2i}) \neq 0$, $i = 1,\dots,5$.

If there are exactly three 2-cycles in a realizing digraph D, two of them being disjoint, they can be consecutive or not. If they are consecutive, for instance $121, 232$ and 343, the arcs 13 and 24 cannot be present, but the arc 41 and one of the arcs 35 or 52 can. If they are not consecutive, for instance $121, 232$ and 454, only one arc of each 2-paths 241, 352 and 524 can be present. These two possible realizing digraphs D_{31} and D_{32} are shown in Fig. 5. Their adjacency matrices are respectively

$$B_{31} = \begin{bmatrix} 0 & b_{12} & 0 & 0 & 0 \\ b_{21} & 0 & b_{23} & 0 & 0 \\ 0 & b_{32} & 0 & b_{34} & b_{35} \\ b_{41} & 0 & b_{43} & 0 & b_{45} \\ b_{51} & b_{52} & 0 & 0 & 0 \end{bmatrix}, \; B_{32} = \begin{bmatrix} 0 & b_{12} & 0 & 0 & 0 \\ b_{21} & 0 & b_{23} & b_{24} & 0 \\ 0 & b_{32} & 0 & b_{34} & b_{35} \\ b_{41} & 0 & 0 & 0 & b_{45} \\ b_{51} & b_{52} & 0 & b_{54} & 0 \end{bmatrix}.$$

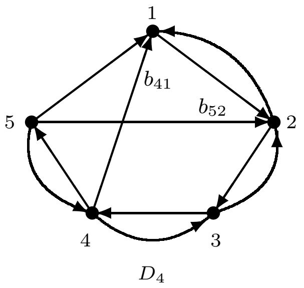
The entry $(1,4)$ of the matrix $Q(B_{31})$ is $b_{12}b_{23}b_{34}\left(\sqrt{5}-1\right)a/4 > 0$, and the entry $(1,5)$ of the matrix $Q(B_{32})$ is $b_{12}(b_{23}(b_{34}b_{45} + \left(\sqrt{5}-1\right)b_{35}a/4) + \left(\sqrt{5}-1\right)b_{24}b_{45}a/4) > 0$. Then $Q(B_{31}) \neq 0$ and $Q(B_{32}) \neq 0$.

If there are exactly four 2-cycles in a realizing digraph D, they are necessarily consecutive, for instance $121, 232, 343$ and 454, in which case, the arcs 13, 24 and 35 cannot be present, but the arcs 41 and 52 can. This possible realizing digraph D_4 and its adjacency matrix are shown in Fig. 6. The entry $(1,4)$ of the matrix $Q(B_4)$ is $b_{12}b_{23}b_{34}(\sqrt{5}-1)a/4 > 0$. Then $Q(B_4) \neq 0$.

If there are five 2-cycles in a realizing digraph D, they are necessarily $121, 232, 343, 454$ and 515, and there cannot be more arcs in D. This possible realizing digraph D_5 is shown in Fig. 7 and its adjacency matrix is the matrix B_5 of the statement. The characteristic polynomial of B_5 is

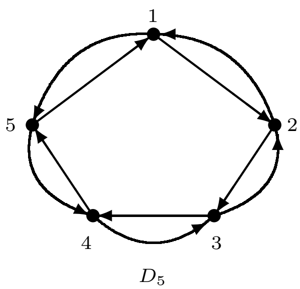
$$\begin{aligned} P(x) = x^5 &- (b_{12}b_{21} + b_{23}b_{32} + b_{34}b_{43} + b_{45}b_{54} + b_{15}b_{51})x^3 + (b_{12}b_{21}(b_{34}b_{43} + b_{45}b_{54}) \\ &+ b_{15}b_{51}(b_{23}b_{32} + b_{34}b_{43}) + b_{23}b_{32}b_{45}b_{54})x - b_{12}b_{23}b_{34}b_{43}b_{51} - b_{15}b_{54}b_{43}b_{32}b_{21}. \end{aligned}$$

Then we need to solve the system of equations



$$B_4 = \begin{bmatrix} 0 & b_{12} & 0 & 0 & 0 \\ b_{21} & 0 & b_{23} & 0 & 0 \\ 0 & b_{32} & 0 & b_{34} & 0 \\ b_{41} & 0 & b_{43} & 0 & b_{45} \\ b_{51} & b_{52} & 0 & b_{54} & 0 \end{bmatrix}.$$

Fig. 6. Realizing digraph D_4 and its adjacency matrix.



$$B_5 = \begin{bmatrix} 0 & b_{12} & 0 & 0 & b_{15} \\ b_{21} & 0 & b_{23} & 0 & 0 \\ 0 & b_{32} & 0 & b_{34} & 0 \\ 0 & 0 & b_{43} & 0 & b_{45} \\ b_{51} & 0 & 0 & b_{54} & 0 \end{bmatrix}.$$

Fig. 7. Realizing digraph D_5 and its adjacency matrix B_5.

$$\begin{aligned} & b_{12}b_{21} + b_{23}b_{32} + b_{34}b_{43} + b_{45}b_{54} = \tfrac{5}{4}a^2 \\ & b_{12}b_{21}(b_{34}b_{43} + b_{45}b_{54}) + b_{15}b_{51}(b_{23}b_{32} + b_{34}b_{43}) + b_{23}b_{32}b_{45}b_{54} = \tfrac{5}{16}a^4 \\ & b_{12}b_{23}b_{34}b_{43}b_{51} + b_{15}b_{54}b_{43}b_{32}b_{21} = \tfrac{1}{16}a^5. \end{aligned} \tag{1}$$

The evaluation of the minimal polynomial in B_5 gives the matrix $Q(B_5) = (c_{ij})$, whose entries $(2,3)$ and $(3,2)$ are respectively

$$\begin{aligned} c_{23} &= \tfrac{1-3\sqrt{5}}{16} b_{23}a^3 + \tfrac{\sqrt{5}-1}{4} b_{23}(b_{12}b_{21} + b_{23}b_{32} + b_{34}b_{43})a + b_{15}b_{54}b_{43}b_{21} \\ c_{32} &= \tfrac{1-3\sqrt{5}}{16} b_{32}a^3 + \tfrac{\sqrt{5}-1}{4} b_{32}(b_{12}b_{21} + b_{23}b_{32} + b_{34}b_{43})a + b_{12}b_{34}b_{45}b_{51}. \end{aligned}$$

If these entries are null, then the expression

$$\frac{b_{32}}{b_{23}}c_{23} - c_{32} = \frac{b_{15}b_{54}b_{43}b_{32}b_{21} - b_{12}b_{23}b_{34}b_{45}b_{51}}{b_{23}}$$

must also be null. This means that the two 5-cycles 123451 and 154321 have the same weight $b_{12}b_{23}b_{34}b_{45}b_{51} = b_{15}b_{54}b_{43}b_{32}b_{21} = \frac{a^5}{32}$, since the independent term of $P(x)$, $-\frac{a^5}{16}$, is the sum of the weights of these two 5-cycles.

The coefficients k_2 and k_4 of the characteristic polynomial $P(x)$ only depend on the weights of the five 2-cycles, but not on the factorization of these weights as weights of the corresponding arcs in each 2-cycle. Then, to solve the two first equations of the system (1), we can do $c_1 := b_{12}b_{21}, c_2 := b_{23}b_{32}, c_3 := b_{34}b_{43}, c_4 = b_{45}b_{54}$ and $c_5 := b_{51}b_{15}$. Thus we have

$$c_1 + c_2 + c_3 + c_4 + c_5 = \frac{5}{4}a^2, \quad c_1(c_3 + c_4) + c_5(c_2 + c_3) + c_2c_4 = \frac{5}{16}a^4.$$

Now, we consider the function $f(c_1, c_2, c_3, c_4, c_5) = c_1(c_3 + c_4) + c_5(c_2 + c_3) + c_2c_4$ and we prove that its maximum value, under the constraint $c_1 + c_2 + c_3 + c_4 + c_5 = \frac{5}{4}a^2$, is precisely $\frac{5}{16}a^4$. The corresponding Lagrange function is:

$$F(c_1, c_2, c_3, c_4, c_5) = c_1(c_3 + c_4) + c_5(c_2 + c_3) + c_2c_4 + \lambda(c_1 + c_2 + c_3 + c_4 + c_5 - \frac{5}{4}a^2),$$

and its partial derivatives are

$$\frac{\partial F}{\partial c_1} = c_3 + c_4 + \lambda = 0, \quad \frac{\partial F}{\partial c_2} = c_4 + c_5 + \lambda = 0, \quad \frac{\partial F}{\partial c_3} = c_1 + c_5 + \lambda = 0,$$
$$\frac{\partial F}{\partial c_4} = c_1 + c_2 + \lambda = 0, \quad \frac{\partial F}{\partial c_5} = c_2 + c_3 + \lambda = 0.$$

The unique solution of this system of equations is $c_1 = c_2 = c_3 = c_4 = c_5 = \frac{a^2}{4}$ and then $f\left(\frac{a^2}{4}, \frac{a^2}{4}, \frac{a^2}{4}, \frac{a^2}{4}, \frac{a^2}{4}\right) = \frac{5}{16}a^4$. Thus, the unique nonnegative solution of the original system (1) satisfies $b_{12}b_{21} = b_{23}b_{32} = b_{34}b_{43} = b_{45}b_{54} = b_{51}b_{15} = \frac{a^2}{4}$ and $b_{12}b_{23}b_{34}b_{45}b_{51} = b_{21}b_{32}b_{43}b_{54}b_{15} = \frac{a^5}{32}$, and the more general form of the matrix B_5 is

$$B_5 = \begin{bmatrix} 0 & b_{12} & 0 & 0 & \frac{8b_{12}b_{23}b_{34}b_{45}}{a^3} \\ \frac{a^2}{4b_{12}} & 0 & b_{23} & 0 & 0 \\ 0 & \frac{a^2}{4b_{23}} & 0 & b_{34} & 0 \\ 0 & 0 & \frac{a^2}{4b_{34}} & 0 & b_{45} \\ \frac{a^5}{32b_{12}b_{23}b_{34}b_{45}} & 0 & 0 & \frac{a^2}{4b_{45}} & 0 \end{bmatrix},$$

with $a, b_{12}, b_{23}, b_{34}, b_{45} > 0$.

Finally, note that the evaluation of the minimal polynomial $R(x)$ of the diagonalizable realizations of Λ in B_5 gives $R(B_5) = 0$.

Therefore, since the realizations of the form B_5 of Λ with $Q(B_5) = 0$ are diagonalizable, we have that Λ has each possible JCF allowed except $J_1(a) \oplus J_1\left(\frac{\sqrt{5}-1}{4}a\right) \oplus J_1\left(\frac{\sqrt{5}-1}{4}a\right) \oplus J_2\left(-\frac{\sqrt{5}+1}{4}a\right)$. □

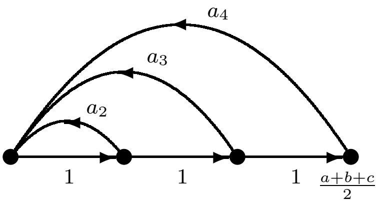
Theorem 3.8. *Let $\Lambda = \left\{a, b, c, -\frac{a+b+c}{2}, -\frac{a+b+c}{2}\right\}$, with $a > b > c > 0$. Then, Λ is realizable if and only if Λ is $\mathcal{UR}$.*

Proof. Theorem 2.1 or Theorem 2.3, see Remark 2.1, give JCF with maximal Jordan blocks and Theorem 2.4 the diagonal one for Λ. It is easy to check that Λ can be realizable without being symmetrically realizable, this happens when $c < 2b - a$. For these Λs, we use Lemma 2.2 to prove that they are diagonally realizable. If $\left\{\frac{a+b+c}{2}, -\frac{a+b+c}{2}\right\}$ is the spectrum of

$$B = \begin{bmatrix} 0 & \frac{a+b+c}{2} \\ \frac{a+b+c}{2} & 0 \end{bmatrix} \sim \begin{bmatrix} \alpha = \frac{a+b+c}{2} & 0 \\ 0 & -\frac{a+b+c}{2} \end{bmatrix},$$

then, to realize Λ, we need to realize $\left\{a, b, c, -\frac{a+b+c}{2}\right\}$ by a matrix A with diagonal entries $0, 0, 0, \frac{a+b+c}{2}$. Following the technique in [19], we look for

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \\ \hline a_4 & 0 & 0 & \frac{a+b+c}{2} \end{array}\right]$$



Since $|A - xI| = x^4 - \frac{a+b+c}{2}x^3 - a_2x^2 - \left(a_3 - a_2\frac{a+b+c}{2}\right)x - a_4 + a_3\frac{a+b+c}{2} = (x-a)(x-b)(x-c)\left(x + \frac{a+b+c}{2}\right)$, by identifying the coefficients, we have

$$\begin{aligned} a_2 &= \frac{a^2+b^2+c^2}{2} \geq 0 \\ a_3 &= \frac{a^3}{4} - \frac{b+c}{4}\left[a^2 + a(b+c) - (b-c)^2\right] = -k_3 \geq 0 \\ a_4 &= \frac{a+b+c}{2}(abc - k_3) \geq 0 \end{aligned}$$

where the second and third inequalities are due to the fact that Λ is realizable. Now, the application of Lemma 2.2, with $\mathbf{t}^T = (1/2, 1/2)$ and $\mathbf{s}^T = (1, 1)$, provides the matrix C, with diagonal JCF in the whole region of realizability,

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ a_2 & 0 & 1 & 0 & 0 \\ a_3 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ a_4 & 0 & 0 & 0 & \frac{a+b+c}{2} \\ a_4 & 0 & 0 & \frac{a+b+c}{2} & 0 \end{bmatrix}. \qquad \square$$

As a conclusion of the whole Subsection 3.3, we have the next result that characterizes the universal realizability of real 5-spectra of trace 0:

Theorem 3.9. *Let $\Lambda = \{a, b, c, d, -(a+b+c+d)\}$ with $a \geq b \geq c \geq d$, $a > 0$ and $b + c + d \leq 0$. Then, the following statements are equivalent:*

i) *Λ is $\mathcal{UR}$;*
ii) *Λ is realizable and it is not one of the next spectra*

$$\{a, a, 0, -a, -a\},$$
$$\left\{a, b, -\frac{a+b}{3}, -\frac{a+b}{3}, -\frac{a+b}{3}\right\} \text{ with } a \neq b > \frac{a}{2}$$

$$or \left\{a, \frac{\sqrt{5}-1}{4}a, \frac{\sqrt{5}-1}{4}a, -\frac{\sqrt{5}+1}{4}a, -\frac{\sqrt{5}+1}{4}a\right\}.$$

Declaration of competing interest

None declared.

Appendix A

In prior paper [7], we omitted an hypothesis required for a result to be true. The result is [7, Theorem 2.1] and the hypothesis "the matrix A with its basic component being a final component". The notation $\mathcal{CS}_a$ means the set of square nonnegative matrices with constant row sum a, M_n the set of square real matrices of size n and $\rho(A)$ the spectral radius of a matrix A.

The confusion comes from a wrong consideration of the normal Frobenius form of a reducible matrix. Explicitly, if A is a reducible matrix of order n with $\rho(A)$ simple and its basic component being a final component, then A is permutationally similar to

$$\widetilde{A} = \begin{bmatrix} A_{11} & & & & & & \\ 0 & A_{22} & & & & & \\ \vdots & \ddots & \ddots & & & & \\ 0 & \cdots & 0 & A_{kk} & & & \\ A_{k+1,1} & \cdots & \cdots & A_{k+1,k} & A_{k+1,k+1} & & \\ \vdots & & & \vdots & & \ddots & \\ A_{k+r,1} & \cdots & \cdots & A_{k+r,k} & \cdots & \cdots & A_{k+r,k+r} \end{bmatrix}$$

where each block A_{ii} is square of order n_i and is either irreducible or a 1×1 null matrix, and $[A_{k+i,1} \ \ldots \ A_{k+i,k+i-1}]$ is nonzero, for $i = 1, \ldots, r$. Without loss of generality, it can be assumed that $\lambda_1 = \rho(A)$ is an eigenvalue of $A_{11} \in \mathcal{CS}_{\lambda_1}$, and $A_{ii} \in \mathcal{CS}_{\rho(A_{ii})}$, $i = 2, \ldots, k+r$.

In the proof of [7, Theorem 2.1] we considered $[A_{i,1} \ \ldots \ A_{i,i-1}]$ nonzero, for $i = 2, \ldots, k$; and $[A_{k+i,1} \ \ldots \ A_{k+i,k+i-1}] = 0$, for $i = 1, \ldots, r$. Note that, in general, it is not true that a reducible matrix A with $\rho(A)$ simple is permutationally similar to a matrix of this form and that we can not assume that $\lambda_1 = \rho(A)$ is an eigenvalue of $A_{11} \in \mathcal{CS}_{\lambda_1}$ as shows the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}.$$

To guarantee this we need to assure that the basic component A_{ii} with spectral radius $\rho(A)$ is a final component. Consequently, the correct result should be:

Theorem 2.1. *Let $A \in M_n$ be a nonnegative matrix with $\lambda_1 = \rho(A)$ simple and its basic component being a final component. Then there exists a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ similar to A.*

The proof is the same but changing the order of applications of Lemma 2.1 and Lemma 2.2.

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