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A map of sufficient conditions for the symmetric nonnegative inverse eigenvalue problem ☆

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The symmetric nonnegative inverse eigenvalue problem (SNIEP) asks for necessary and sufficient conditions in order that a list of real numbers be the spectrum of a symmetric nonnegative real matrix. A number of sufficient conditions for the existence of such a matrix are known. In this paper, in order to construct a map of sufficient conditions, we compare these conditions and establish inclusion relations or independence relations between them.

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1. Introduction

The *real nonnegative inverse eigenvalue problem* (hereafter RNIEP) is the problem of characterizing all possible real spectra of entrywise nonnegative matrices. This problem remains unsolved. A complete solution is known only for spectra of size $n \leq 4$. A number of *realizability criteria* or sufficient conditions for the existence of a nonnegative matrix with a given real spectrum have been obtained, from different points of view. In [12] the authors construct a map of sufficient conditions for the RNIEP, in which they show inclusion or independence relations between these conditions.

If in the RNIEP we require that the nonnegative matrix be symmetric, we have the *symmetric nonnegative inverse eigenvalue problem* (hereafter SNIEP). For a long time it was thought that the RNIEP and the SNIEP were equivalent, but in [8] it was proved that both problems are different and in [5] that they are different for $n \geq 5$. Both problems, RNIEP and SNIEP, are equivalent for $n \leq 4$ and remain open for $n \geq 5$.

The first known sufficient condition for the SNIEP is due to Perfect and Mirsky [14] for doubly stochastic matrices and Fiedler [7] gave the first symmetric realizability criterion for nonnegative matrices. Several realizability criteria which were first obtained for the RNIEP have later been shown to be realizability criteria for the SNIEP as well. Fiedler [7], Radwan [15] and Soto [18] showed, respectively, that the Kellogg [9], Borobia [1] and Soto 2 [17] criteria are also symmetric realizability criteria. In [22,10,19] the authors propose, directly, symmetric realizability criteria.

There are in the literature some other criteria for the SNIEP based on a theorem from Rado, see Soto–Rojo–Moro–Borobia [21] and Soto–Rojo–Manzaneda [20]. These criteria trivially contain, by their own definition, any other sufficient condition, and for this reason we will leave them out of the analysis. Their interest lies in providing different procedures to realize certain lists.

The paper is organized as follows: Section 2 contains the list of all sufficient conditions that we shall consider, in chronological order, and some technical results that we will use in the next section. In Section 3 we construct a map of symmetric realizability criteria, establishing inclusion or independence relations between these criteria.

2. Sufficient conditions for the SNIEP

In this paper, by a *list* we understand a collection $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of real numbers with possible repetitions. By a *partition of a list* Λ we mean a family of sublists of Λ whose disjoint union is Λ. As is commonly accepted, we understand that a summatory is equal to zero when the index set of the summatory is empty.

We will say that a list Λ is (*symmetrically*) *realizable* if it is the spectrum of an entrywise (*symmetric*) nonnegative matrix A. In this case A is said to be a *realizing matrix*.

The RNIEP and the SNIEP have an obvious solution when only nonnegative real numbers are considered, so the interest of both problems is when there is at least one negative number in the list.

In what follows we list the sufficient conditions, or realizability criteria, that we are going to consider, in chronological order. The first result in this area was announced by Suleĭmanova in 1949 and proved by Perfect in 1953.

Theorem 1. (Suleĭmanova [23], 1949) *Let* $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ *satisfy*

$$\lambda_0 \geq |\lambda| \quad \textit{for} \quad \lambda \in \Lambda \quad \textit{and} \quad \lambda_0 + \sum_{\lambda_i < 0} \lambda_i \geq 0, \tag{1}$$

then Λ *is realizable.*

Theorem 2. (Suleĭmanova–Perfect [23,13], 1949, 1953) *Let* $\Lambda = \{\lambda_0, \lambda_{01}, \ldots, \lambda_{0t_0}, \lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}, \ldots, \lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}\}$ *satisfy*

$$\lambda_0 \geq |\lambda| \quad \textit{for} \quad \lambda \in \Lambda \quad \textit{and} \quad \lambda_j + \sum_{\lambda_{ji} < 0} \lambda_{ji} \geq 0 \quad \textit{for} \quad j = 0, 1, \ldots, r, \tag{2}$$

then Λ *is realizable.*

Theorem 3. (Perfect 1 [13], 1953) *Let*

$$\Lambda = \{\lambda_0, \lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}, \ldots, \lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}, \delta\},$$

where

$$\lambda_0 \geq |\lambda| \quad \textit{for} \quad \lambda \in \Lambda, \quad \sum_{\lambda \in \Lambda} \lambda \geq 0, \quad \delta \leq 0,$$

$$\lambda_j \geq 0 \quad \textit{and} \quad \lambda_{ji} \leq 0 \quad \textit{for} \quad j = 1, \ldots, r \quad \textit{and} \quad i = 1, \ldots, t_j.$$

If

$$\lambda_j + \delta \leq 0 \quad \textit{and} \quad \lambda_j + \sum_{i=1}^{t_j} \lambda_{ji} \leq 0 \quad \textit{for} \quad j = 1, \ldots, r, \tag{3}$$

then Λ *is realizable.*

Theorem 4. (Perfect–Mirsky [14], 1965) *Let* $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ *with* $\lambda_1 \geq |\lambda|$ *for* $\lambda \in \Lambda$ *and* $\lambda_i \geq \lambda_{i+1}$ *for* $i = 1, \ldots, n-1$*. If*

$$\frac{\lambda_1}{n} + \frac{\lambda_2}{n(n-1)} + \cdots + \frac{\lambda_n}{2 \cdot 1} \geq 0, \tag{4}$$

then Λ *is symmetrically realizable.*

Remark 1. In [14, Theorem 8], the previous result is given with $\lambda_1 = 1$ and asserts that Λ is realized by a doubly stochastic matrix. Reading the proof, one can see, as the authors point out, that in fact Λ is realized by a symmetric doubly stochastic matrix.

Theorem 5. (Ciarlet [4], 1968) *Let* $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ *satisfy*

$$|\lambda_j| \leq \frac{\lambda_0}{n}, \quad j = 1, \dots, n, \tag{5}$$

then Λ *is realizable.*

Theorem 6. (Kellogg [9], 1971) *Let* $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ *with* $\lambda_0 \geq |\lambda|$ *for* $\lambda \in \Lambda$ *and* $\lambda_i \geq \lambda_{i+1}$ *for* $i = 0, \dots, n-1$. *Let* M *be the greatest index* j $(0 \leq j \leq n)$ *for which* $\lambda_j \geq 0$ *and* $K = \{i \in \{1, \dots, \lfloor n/2 \rfloor\} \ / \ \lambda_i \geq 0,\ \lambda_i + \lambda_{n+1-i} < 0\}$. *If*

$$\lambda_0 + \sum_{i \in K,\ i<k} (\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-k} \geq 0 \quad \textit{for all} \ \ k \in K, \tag{6}$$

and

$$\lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{n+1-i}) + \sum_{j=M+1}^{n-M} \lambda_j \geq 0, \tag{7}$$

then Λ *is realizable.*

Theorem 7. (Salzmann [16], 1972) *Let* $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ *with* $\lambda_i \geq \lambda_{i+1}$ *for* $i = 0, \dots, n-1$. *If*

$$\sum_{0 \leq j \leq n} \lambda_j \geq 0, \tag{8}$$

and

$$\frac{\lambda_i + \lambda_{n-i}}{2} \leq \frac{1}{n+1} \sum_{0 \leq j \leq n} \lambda_j, \quad i = 1, \dots, \lfloor n/2 \rfloor, \tag{9}$$

then Λ *is realizable by a diagonalizable nonnegative matrix.*

Theorem 8. (Fiedler [7], 1974) *Let* $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ *with* $\lambda_i \geq \lambda_{i+1}$ *for* $i = 0, \dots, n-1$. *If*

$$\lambda_0 + \lambda_n + \sum_{\lambda \in \Lambda} \lambda \geq \frac{1}{2} \sum_{1 \leq i \leq n-1} |\lambda_i + \lambda_{n-i}|, \tag{10}$$

then Λ *is symmetrically realizable.*

Soules in 1983 gave two constructive sufficient conditions for symmetric realization. The inequalities that appear in these conditions are obtained by imposing the diagonal entries of the matrix $Rdiag(\lambda_1, \dots, \lambda_n)R^t$ to be nonnegative, where R is an orthogonal matrix with a certain pattern. For a particular R, this criterion is the Perfect–Mirsky criterion.

Theorem 9. (Soules 1 [22], 1983) *Let* $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ *with* $\lambda_i \geq \lambda_{i+1}$ *for* $i = 1, \dots, n-1$ *and let* $x = (x_1, \dots, x_n) > 0$. *If*

$$d_i = \frac{x_i^2 \lambda_1}{\sum\limits_{j=1}^{n} x_j^2} + \sum_{k=i+1}^{n} \frac{(x_i x_k)^2 \lambda_{n-k+2}}{\left(\sum\limits_{j=1}^{k-1} x_j^2\right)\left(\sum\limits_{j=1}^{k} x_j^2\right)} + \frac{\sum\limits_{j=1}^{i-1} x_j^2 \lambda_{n-i+2}}{\sum\limits_{j=1}^{i} x_j^2} \geq 0 \,, \tag{11}$$

for $i = 1, \dots, n$, *then there exists a symmetric nonnegative matrix with the ith diagonal entry* d_i, *spectrum* Λ *and* x *an eigenvector associated to* λ_1. *Further, if* $0 < x_1 \leq x_2 \leq \cdots \leq x_n$, *then* $d_1 \leq d_2 \leq \cdots \leq d_n$.

The next result is a generalization of Theorem 9.

Theorem 10. (Soules 2 [22], 1983) *Let* $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ *with* $\lambda_i \geq \lambda_{i+1}$ *for* $i = 1, \dots, n-1$ *and let* $x = (x_1, \dots, x_n) > 0$. *Let* $\{i_1, \dots, i_m\} \cup \{j_1, \dots, j_{n-m}\}$ *and* $\{k_1, \dots, k_{m-1}\} \cup \{l_1, \dots, l_{n-m-1}\}$ *be partitions of* $\{1, \dots, n\}$ *and* $\{3, \dots, n\}$, *respectively. If*

$$d_p = \frac{x_{i_p}^2 \lambda_1}{\sum\limits_{r=1}^{n} x_r^2} + \frac{x_{i_p}^2 \left(\sum\limits_{r=1}^{n-m} x_{j_r}^2\right) \lambda_2}{\left(\sum\limits_{r=1}^{n} x_r^2\right)\left(\sum\limits_{r=1}^{m} x_{i_r}^2\right)} + \sum_{t=p+1}^{m} \frac{\left(x_{i_p} x_{i_t}\right)^2 \lambda_{k_{m-t+1}}}{\left(\sum\limits_{r=1}^{t-1} x_{i_r}^2\right)\left(\sum\limits_{r=1}^{t} x_{i_r}^2\right)} + \frac{\sum\limits_{r=1}^{p-1} x_{i_r}^2 \lambda_{k_{m-p+1}}}{\sum\limits_{r=1}^{p} x_{i_r}^2} \geq 0$$

and

$$d_{m+q} = \frac{x_{j_q}^2 \lambda_1}{\sum\limits_{r=1}^{n} x_r^2} + \frac{x_{j_q}^2 \left(\sum\limits_{r=1}^{m} x_{i_r}^2\right) \lambda_2}{\left(\sum\limits_{r=1}^{n} x_r^2\right)\left(\sum\limits_{r=1}^{n-m} x_{j_r}^2\right)} + \sum_{t=q+1}^{n-m} \frac{\left(x_{j_q} x_{j_t}\right)^2 \lambda_{l_{n-m-t+1}}}{\left(\sum\limits_{r=1}^{t-1} x_{j_r}^2\right)\left(\sum\limits_{r=1}^{t} x_{j_r}^2\right)} + \frac{\sum\limits_{r=1}^{q-1} x_{j_r}^2 \lambda_{l_{n-m-q+1}}}{\sum\limits_{r=1}^{q} x_{j_r}^2} \geq 0$$

for $p = 1, \dots, m$ *and* $q = 1, \dots, n-m$, *then there exists a symmetric nonnegative matrix with the ith diagonal entry* d_i, *spectrum* Λ *and* x *an eigenvector associated to* λ_1.

Remark 2. If in the previous theorem we take $\{i_1, \ldots, i_m\} = \{1, \ldots, m\}$ and $\{j_1, \ldots, j_{n-m}\} = \{m+1, \ldots, n\}$ we have an equivalent condition, although in this case the realizing matrix has its columns permuted.

Remark 3. Note that in Theorems 9 and 10 we have $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \lambda_i$. So, if this sum is zero, then every d_i is zero.

Corollary 1. ([22], 1983) *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ with $\lambda_i \geq \lambda_{i+1}$ for $i = 1, \ldots, n-1$ and let $n = 2m+2$ (n even) or $n = 2m+1$ (n odd). If*

$$\frac{\lambda_1}{n} + \frac{(n-m-1)\lambda_2}{n(m+1)} + \sum_{k=1}^{m} \frac{\lambda_{n-2k+2}}{(k+1)k} \geq 0, \tag{12}$$

then Λ is symmetrically realizable.

Theorem 11. (Borobia [1], 1995) *Let $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ with $\lambda_i \geq \lambda_{i+1}$ for $i = 0, \ldots, n-1$ and let M be the greatest index j ($0 \leq j \leq n$) for which $\lambda_j \geq 0$. If there exists a partition $J_1 \cup \cdots \cup J_t$ of $\{\lambda_{M+1}, \ldots, \lambda_n\}$ such that*

$$\left\{\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_M > \sum_{\lambda \in J_1} \lambda \geq \cdots \geq \sum_{\lambda \in J_t} \lambda\right\} \tag{13}$$

satisfies the Kellogg condition, then Λ is realizable.

Theorem 12. (Soto 2 [17], 2003) *Let Λ be a list that admits a partition*

$$\{\lambda_{11}, \ldots, \lambda_{1t_1}\} \cup \cdots \cup \{\lambda_{r1}, \ldots, \lambda_{rt_r}\}$$

with $\lambda_{11} \geq |\lambda|$ for $\lambda \in \Lambda$, $\lambda_{ij} \geq \lambda_{i,j+1}$ and $\lambda_{i1} \geq 0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, t_i - 1$. For each list $\{\lambda_{i1}, \ldots, \lambda_{it_i}\}$ of the partition we define

$$S_{ij} = \lambda_{ij} + \lambda_{i,t_i-j+1} \quad \textit{for} \quad j = 2, \ldots, \lfloor t_i/2 \rfloor$$
$$S_{i,(t_i+1)/2} = \min\{\lambda_{i,(t_i+1)/2}, 0\} \quad \textit{if } t_i \textit{ is odd for} \quad i = 1, \ldots, r,$$

and

$$T_i = \lambda_{i1} + \lambda_{it_i} + \sum_{S_{ij}<0} S_{ij} \quad \textit{for} \quad i = 1, \ldots, r.$$

Let

$$L = \max\{-\lambda_{1t_1} - \sum_{S_{1j}<0} S_{1j}\ ,\ \max_{2 \leq i \leq r}\{\lambda_{i1}\}\}. \tag{14}$$

If

$$\lambda_{11} \geq L - \sum_{T_i<0,\ 2\leq i\leq r} T_i, \tag{15}$$

then Λ *is realizable.*

The following result gives a sufficient condition for the existence of an n-by-n symmetric nonnegative matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and diagonal entries $\omega_1, \ldots, \omega_n$.

Lemma 1. (Fiedler [7], 1974) *Let* $\lambda_1 \geq \cdots \geq \lambda_n$, *with* $\lambda_1 \geq |\lambda_n|$, *and* $\omega_1 \geq \cdots \geq \omega_n \geq 0$ *satisfy*

i) $\sum_{i=1}^{s} \lambda_i \geq \sum_{i=1}^{s} \omega_i$ *for* $s = 1, \ldots, n-1$;
ii) $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \omega_i$;
iii) $\lambda_i \leq \omega_{i-1}$ *for* $i = 2, \ldots, n-1$.

Then there exists an n*-by-*n *symmetric nonnegative matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$ *and diagonal entries* $\omega_1, \ldots, \omega_n$.

Remark 4. Note that if conditions i), ii) and iii) of the previous lemma are satisfied for $\lambda_1 \geq \cdots \geq \lambda_n$ and for a family of ω's unordered, they are also satisfied for the sequence of ω's ordered. Let suppose $\omega_k < \omega_{k+1}$ for a certain k and that conditions i), ii) and iii) are satisfied for $\omega_1, \ldots, \omega_n$. It is clear that conditions ii) and iii) and condition i) but for $s = k$ are also satisfied for $\omega_1, \ldots, \omega_{k-1}, \omega_{k+1}, \omega_k, \omega_{k+2}, \ldots, \omega_n$. If condition i) for $s = k$ were not true, that is $\lambda_1 + \cdots + \lambda_k < \omega_1 + \cdots + \omega_{k-1} + \omega_{k+1}$, we will reach a contradiction:

$$\left.\begin{array}{rl} \lambda_1 + \cdots + \lambda_k & < \omega_1 + \cdots + \omega_{k-1} + \omega_{k+1} \\ \lambda_{k+1} & \leq \omega_k \end{array}\right\} \Rightarrow \sum_{i=1}^{k+1} \lambda_i < \sum_{i=1}^{k+1} \omega_i \ .$$

Theorem 13. (Laffey–Šmigoc [10], 2007) *Let* $\Lambda_1 = \{\lambda_1, \ldots, \lambda_n\}$ *and* $\Lambda_2 = \{\mu_1, \ldots, \mu_m\}$ *with* $\lambda_1 \geq |\lambda|$ *for* $\lambda \in \Lambda_1$ *and* $\mu_1 \geq |\mu|$ *for* $\mu \in \Lambda_2$. *Suppose that* Λ_1 *is the spectrum of an irreducible nonnegative symmetric matrix with a diagonal element* c *and* Λ_2 *is the spectrum of a nonnegative symmetric matrix.*

(1) *If* $\mu_1 \leq c$, *then* $\{\lambda_1, \ldots, \lambda_n, \mu_2, \ldots, \mu_m\}$ *is symmetrically realizable.*
(2) *If* $c \leq \mu_1$, *then* $\{\lambda_1 + \mu_1 - c, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m\}$ *is symmetrically realizable.*

Remark 5. In order to apply this sufficient condition we will assume $n > 1$. Otherwise every symmetrically realizable spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$, with $\lambda_1 \geq \lambda_j$ for $j = 2, \ldots, m$, would be realizable by this criterion: with $\Lambda_1 = \{\lambda_1\}$, $c = \lambda_1$ and

$\Lambda_2 = \{\mu_1 = \lambda_1, \lambda_2, \dots, \lambda_m\}$. This also means that every spectrum realizable by this criterion should have at least three elements.

Note that lists with only one nonnegative element and satisfying Suleǐmanova are always realized by irreducible nonnegative symmetric matrices (the Frobenius normal form of any symmetric nonnegative realization has only one irreducible block matrix on the diagonal).

Note also that lists with the biggest element repeated and the other elements negative cannot be realized by Laffey–Šmigoc: Suppose $\{\lambda_1, \dots, \lambda_n\}$, with $\lambda_1 = \lambda_2 \geq 0 > \lambda_3 \geq \dots \geq \lambda_n$ and $n \geq 3$, is realized by Laffey–Šmigoc, then there exist $\Lambda_1 = \{\tilde{\lambda}_1, \lambda_{i_1}, \dots, \lambda_{i_p}\}$ realized by an irreducible nonnegative symmetric matrix with a diagonal element c (note $c < \tilde{\lambda}_1$) and $\Lambda_2 = \{\mu_1, \lambda_2, \lambda_{j_1}, \dots, \lambda_{j_q}\}$ with $\mu_1 \geq \lambda_2$ symmetrically realizable such that one of the next two situations is satisfied:

(1) If $\mu_1 \leq c$, then $\{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\} = \{\tilde{\lambda}_1, \lambda_{i_1}, \dots, \lambda_{i_p}, \lambda_2, \lambda_{j_1}, \dots, \lambda_{j_q}\}$ is symmetrically realizable with $\tilde{\lambda}_1 = \lambda_1$. This is not possible because $\lambda_1 = \lambda_2 \leq \mu_1 \leq c < \tilde{\lambda}_1 = \lambda_1$.

(2) If $c \leq \mu_1$, then $\{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\} = \{\tilde{\lambda}_1 + \mu_1 - c, \lambda_{i_1}, \dots, \lambda_{i_p}, \lambda_2, \lambda_{j_1}, \dots, \lambda_{j_q}\}$ is symmetrically realizable with $\tilde{\lambda}_1 + \mu_1 - c = \lambda_1$. This is not possible because $\lambda_1 = \tilde{\lambda}_1 + \mu_1 - c \geq \tilde{\lambda}_1 + \lambda_2 - c = \tilde{\lambda}_1 + \lambda_1 - c$ implies $c \geq \tilde{\lambda}_1$.

This will be a source of examples not satisfying Laffey–Šmigoc for Section 3.

The following criterion requires the concepts of negativity and realizability margin. Let $\mathcal{K}$ be a realizability criterion. If a list of real numbers satisfies the sufficient condition $\mathcal{K}$ we say that the list is $\mathcal{K}$ *realizable*. We denote the set of $\mathcal{K}$ realizable lists as

$$\mathcal{R}_{\mathcal{K}} = \{\Lambda \subset \mathbb{R} \ : \ \Lambda \text{ is } \mathcal{K} \text{ realizable}\}.$$

In this paper $\mathcal{K}$ will be the surname of an author(s). For example, a list satisfying Theorem 1 will be said to be Suleǐmanova realizable. Following the definitions in [2, Section 4] we define the $\mathcal{K}$ *negativity* of a list $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of real numbers, with $\lambda_1 \geq \lambda_j$ for $j = 2, \dots, n$, as:

$$\mathcal{N}_{\mathcal{K}}(\Lambda) = \begin{cases} +\infty & \begin{array}{l}\text{if } \{\lambda_1 + \delta, \lambda_2, \dots, \lambda_n\} \text{ is} \\ \text{not } \mathcal{K} \text{ realizable } \forall \delta \geq 0\end{array} \\ \min\{\delta \geq 0 : \{\lambda_1 + \delta, \lambda_2, \dots, \lambda_n\} \text{ is } \mathcal{K} \text{ realizable}\} & \text{otherwise} \end{cases}$$

and when Λ is $\mathcal{K}$ realizable we define the $\mathcal{K}$ *realizability margin* of Λ as the number:

$$\mathcal{M}_{\mathcal{K}}(\Lambda) = \max\left\{\epsilon \geq 0 : \begin{array}{l} \{\lambda_1 - \epsilon, \lambda_2, \dots, \lambda_n\} \text{ is } \mathcal{K} \text{ realizable} \\ \text{and } \ \lambda_1 - \epsilon \geq |\lambda_j| \ \text{ for } j = 2, \dots, n \end{array}\right\}.$$

Note that the $\mathcal{K}$ negativity of a list measures, in a certain sense, how far the list is from being $\mathcal{K}$ realizable. A similar interpretation can be given for the concept of $\mathcal{K}$ realizability margin of a $\mathcal{K}$ realizable list. For properties, closed expressions or bounds of these concepts see [11].

Based on a Brauer's result, Soto [19] gives a family of symmetric realizability criteria denoted by Soto p, $p = 1, 2, \dots$. These criteria are defined recursively, starting from $p = 1$, which is equivalent to the Fiedler condition given in Theorem 8, see [12].

Theorem 14. (Soto p [19], 2013) *Let p be an integer with $p \geq 2$. Let Λ be a list that admits a partition*

$$\{\lambda_{11}, \dots, \lambda_{1t_1}\} \cup \dots \cup \{\lambda_{r1}, \dots, \lambda_{rt_r}\}$$

with $\lambda_{11} \geq |\lambda|$ for $\lambda \in \Lambda$, $\lambda_{ij} \geq \lambda_{i,j+1}$ and $\lambda_{i1} \geq 0$ for $i = 1, \dots, r$ and $j = 1, \dots, t_i - 1$, and $\{\lambda_{11}, \dots, \lambda_{1t_1}\}$ Soto p-1 realizable. Let $\mathcal{N}_{Sp-1}(\Lambda_i)$ be the Soto p-1 negativity of $\Lambda_i = \{\lambda_{i1}, \dots, \lambda_{it_i}\}$ and $\mathcal{M}_{Sp-1}(\Lambda_i)$ the Soto p-1 realizability margin of Λ_i. Let

$$\gamma = \max\{\lambda_{11} - \mathcal{M}_{Sp-1}(\Lambda_1), \ \max_{2 \leq i \leq r}\{\lambda_{i1}\}\}. \tag{16}$$

If

$$\lambda_{11} \geq \gamma + \sum_{\Lambda_i \notin \mathcal{R}_{Sp-1}} \mathcal{N}_{Sp-1}(\Lambda_i), \tag{17}$$

then Λ is (symmetrically) realizable.

Note that the Soto 2 condition given in Theorem 12, is equivalent to Theorem 14 with $p = 2$, see [19].

In practice, it is not necessary to know the margin of realizability of a list to use the previous theorem. It is enough to know a nonnegative lower bound of it, under certain circumstances, as it shows the following result:

Lemma 2. *Let Λ be a list that admits a partition $\Lambda = \{\lambda_{11}, \dots, \lambda_{1t_1}\} \cup \dots \cup \{\lambda_{r1}, \dots, \lambda_{rt_r}\}$ with $\lambda_{11} \geq |\lambda|$ for $\lambda \in \Lambda$, $\Lambda_i = \{\lambda_{i1}, \dots, \lambda_{it_i}\}$, $\lambda_{ij} \geq \lambda_{i,j+1}$ and $\lambda_{i1} \geq 0$ for $i = 1, \dots, r$ and $j = 1, \dots, t_i - 1$, and Λ_1 Soto p-1 realizable with $p \geq 3$. Let $0 \leq \epsilon \leq \mathcal{M}_{Sp-1}(\Lambda_1)$ and*

$$\widehat{\gamma} = \max\{\lambda_{11} - \epsilon, \ \max_{2 \leq i \leq r}\{\lambda_{i1}\}\}$$

such that

$$\lambda_{11} \geq \widehat{\gamma} + \sum_{\Lambda_i \notin \mathcal{R}_{Sp-1}} \mathcal{N}_{Sp-1}(\Lambda_i),$$

then Λ is Soto p and $\mathcal{M}_{Sp}(\Lambda) \geq \widehat{\gamma} + \sum_{\Lambda_i \notin \mathcal{R}_{Sp-1}} \mathcal{N}_{Sp-1}(\Lambda_i)$.

Proof. Let us see that $\Lambda = \bigcup_{i=1}^{r} \Lambda_i$ satisfies inequality (17). Depending on the values of γ and $\widehat{\gamma}$ we have the following cases:

$$\widehat{\gamma} = \lambda_{11} - \epsilon \geq \lambda_{11} - \mathcal{M}_{Sp-1}(\Lambda_1) = \gamma,$$
$$\widehat{\gamma} = \lambda_{11} - \epsilon \geq \max_{2\leq i\leq r}\{\lambda_{i1}\} = \gamma \quad \text{or} \quad \widehat{\gamma} = \max_{2\leq i\leq r}\{\lambda_{i1}\} = \gamma,$$

and in all of them it is clear that (17) is satisfied. □

3. A map of sufficient conditions for the SNIEP

It is well known that the Suleĭmanova and Kellogg criteria are symmetric realizability criteria, see [7], as well as the Borobia criterion, see [15]. Then Suleĭmanova–Perfect, Ciarlet, Salzmann and Perfect 1 are sufficient conditions for the SNIEP too, because all of them imply Borobia, see [12]. We need to know what the relations are between Perfect–Mirsky, Soules conditions, Laffey–Šmigoc and Soto conditions.

In the next results we will use the following equality, which can be easily proved by induction,

$$\sum_{k=1}^{n-1} \frac{1}{(k+1)k} = \frac{n-1}{n}. \tag{18}$$

Lemma 3. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ with $\lambda_1 \geq |\lambda|$ for $\lambda \in \Lambda$, $\lambda_i \geq \lambda_{i+1}$ for $i = 1, \ldots, n-1$ and let p be the greatest index j $(1 \leq j \leq n)$ for which $\lambda_j \geq 0$. If Λ satisfies Perfect–Mirsky, then $\{\lambda_1, \lambda_{p+1}, \ldots, \lambda_n\}$ also satisfies Perfect–Mirsky.*

Proof. Let us see that

$$\frac{\lambda_1}{n} + \frac{\lambda_2}{n(n-1)} + \cdots + \frac{\lambda_n}{2\cdot 1} \leq \frac{\lambda_1}{n-p+1} + \frac{\lambda_{p+1}}{(n-p+1)(n-p)} + \cdots + \frac{\lambda_n}{2\cdot 1}.$$

To prove this inequality is equivalent to prove

$$\frac{\lambda_2}{n(n-1)} + \cdots + \frac{\lambda_p}{(n-p+2)(n-p+1)} \leq \frac{(p-1)\lambda_1}{n(n-p+1)}. \tag{19}$$

On the one hand we have

$$\begin{aligned}&\frac{\lambda_2}{n(n-1)} + \cdots + \frac{\lambda_p}{(n-p+2)(n-p+1)}\\ &\quad\leq \lambda_1\left(\frac{1}{n(n-1)} + \cdots + \frac{1}{(n-p+2)(n-p+1)}\right)\end{aligned}$$

and on the other hand, the repeated use of equality (18) gives

$$\frac{1}{n(n-1)} + \cdots + \frac{1}{(n-p+2)(n-p+1)} = \frac{n-1}{n} - \frac{n-p}{n-p+1} = \frac{p-1}{n(n-p+1)},$$

and the inequality (19) is proved. □

The following result relates Perfect–Mirsky with other symmetric criteria.

Theorem 15.

1. *Ciarlet implies Perfect–Mirsky and the inclusion is strict.*
2. *Perfect–Mirsky implies Suleĭmanova and the inclusion is strict.*
3. *Salzmann and Perfect 1 are independent of Perfect–Mirsky.*

Proof. *1.* Let $\lambda_1 \geq \cdots \geq \lambda_n$ satisfy Ciarlet: $\dfrac{\lambda_1}{n-1} \geq |\lambda_j|$, $j = 2, \ldots, n$. We have

$$\frac{\lambda_1}{n} + \frac{\lambda_2}{n(n-1)} + \cdots + \frac{\lambda_n}{2 \cdot 1} \geq \frac{\lambda_1}{n} - \frac{|\lambda_2|}{n(n-1)} - \cdots - \frac{|\lambda_n|}{2 \cdot 1} \geq$$

$$\frac{\lambda_1}{n} - \max_{2 \leq j \leq n} |\lambda_j| \left(\frac{1}{n(n-1)} + \cdots + \frac{1}{2 \cdot 1} \right) = \frac{\lambda_1}{n} - \max_{2 \leq j \leq n} |\lambda_j| \left(\frac{n-1}{n} \right)$$

where the equality is due to (18). Finally

$$\frac{\lambda_1}{n} - \max_{2 \leq j \leq n} |\lambda_j| \left(\frac{n-1}{n} \right) = \frac{n-1}{n} \left(\frac{\lambda_1}{n-1} - \max_{2 \leq j \leq n} |\lambda_j| \right)$$

which is nonnegative because of the Ciarlet condition and then the list satisfies Perfect–Mirsky. The inclusion is strict as shows the list $\{1, 1, -1\}$.

2. Because of the previous lemma, it is enough to prove the result for lists with only one nonnegative element. Let $\{\lambda_1, \ldots, \lambda_n\}$, with $\lambda_1 \geq 0 > \lambda_2 \geq \cdots \geq \lambda_n$, satisfy Perfect–Mirsky:

$$\frac{\lambda_1}{n} + \frac{\lambda_2}{n(n-1)} + \cdots + \frac{\lambda_n}{2 \cdot 1} \geq 0.$$

Let us see that

$$\frac{\lambda_1}{n} + \frac{\lambda_2}{n(n-1)} + \cdots + \frac{\lambda_n}{2 \cdot 1} \leq \frac{\lambda_1 + \cdots + \lambda_n}{n},$$

or equivalently

$$\sum_{j=2}^{n} \left(\frac{1}{(n-j+2)(n-j+1)} - \frac{1}{n} \right) \lambda_j \leq 0 \, . \tag{20}$$

Note that the coefficients of the λ_j's in (20) increase with j and use (18) to see that their sum is zero. Let

$$q = \max\left\{ j \ : \ \frac{1}{(n-j+2)(n-j+1)} - \frac{1}{n} < 0 \right\}.$$

We have

$$\sum_{j=2}^{q}\left(\frac{1}{(n-j+2)(n-j+1)} - \frac{1}{n}\right) = -\sum_{j=q+1}^{n}\left(\frac{1}{(n-j+2)(n-j+1)} - \frac{1}{n}\right).$$

Then

$$\begin{aligned}\sum_{j=2}^{q}\left(\frac{1}{(n-j+2)(n-j+1)} - \frac{1}{n}\right)\lambda_j \leq \lambda_q \sum_{j=2}^{q}\left(\frac{1}{(n-j+2)(n-j+1)} - \frac{1}{n}\right) = \\ -\lambda_q \sum_{j=q+1}^{n}\left(\frac{1}{(n-j+2)(n-j+1)} - \frac{1}{n}\right)\end{aligned}$$

and the inequality (20) is satisfied because

$$\begin{aligned}\sum_{j=2}^{n}\left(\frac{1}{(n-j+2)(n-j+1)} - \frac{1}{n}\right)\lambda_j &\leq \sum_{j=q+1}^{n}\left(\frac{1}{(n-j+2)(n-j+1)} - \frac{1}{n}\right)(\lambda_j - \lambda_q) \\ &\leq 0.\end{aligned}$$

The inclusion is strict as shows the list $\{2, 0, -2\}$.

3. The list $\{1, 1, -1\}$ satisfies Perfect–Mirsky but not Salzmann nor Perfect 1. The list $\{5, 2, -2, -3\}$ satisfies Salzmann and Perfect 1 but not Perfect–Mirsky. □

Soules [22] proves that Perfect–Mirsky implies Soules 1 for $x = (1, \ldots, 1)$, and that the inclusion is strict: the list $\{5, 0, -2, -2\}$ is Soules 1 for $x = (2, 2, 2, 1)$ but not Perfect–Mirsky. In [22], it is also proved that Soules 1 implies Kellogg. The list $\{2, 0, -2\}$, given in the proof of the next theorem, proves that the inclusion is strict.

Theorem 16.

1. *Soules 1 implies Suleĭmanova and the inclusion is strict.*
2. *Perfect 1 and Salzmann are independent of Soules 1.*

Proof. *1.* Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ with $\lambda_i < 0$ for $i = 2, \ldots, n$. If Λ satisfies Soules 1, then the sum, for $i = 1, \ldots, n$, of the inequalities (11) gives $\lambda_1 + \cdots + \lambda_n \geq 0$ and so Λ is Suleĭmanova.

Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ with $\lambda_1 \geq \cdots \geq \lambda_p \geq 0 > \lambda_{p+1} \cdots \geq \lambda_n$. If Λ satisfies Soules 1, then the inequalities (11) are also true if we change λ_i, for $2 \leq i \leq p$, by λ_1. Now, the sum of these new inequalities gives $\lambda_1 + \lambda_{p+1} + \cdots + \lambda_n \geq 0$.

The list $\{2, 0, -2\}$ satisfies Suleĭmanova but not Soules 1: let $x = (x_1, x_2, x_3) > 0$, then

$$d_1 = 2\left(\frac{x_1^2}{x_1^2 + x_2^2 + x_3^2} - \frac{x_2^2}{x_1^2 + x_2^2}\right) \quad , \quad d_2 = 2\left(\frac{x_2^2}{x_1^2 + x_2^2 + x_3^2} - \frac{x_1^2}{x_1^2 + x_2^2}\right),$$

so

$$d_1 + d_2 = 2\left(\frac{x_1^2 + x_2^2}{x_1^2 + x_2^2 + x_3^2} - 1\right) < 0$$

which is impossible for $d_1, d_2 \geq 0$.

2. The list $\{1, 1, -1, -1\}$ satisfies Perfect 1 and Salzmann but not Soules 1 because it does not satisfy Suleĭmanova. The list $\{1, 1, -1\}$ satisfies Soules 1 because it satisfies Perfect–Mirsky but not Perfect 1 nor Salzmann. □

Soules [22] proves that Soules 2 does not imply Kellogg (so neither Suleĭmanova, Salzmann or Fiedler): the family of lists $\{3 - t, 1 + t, -1, -1, -1, -1\}$, $t \in (0, 1)$, satisfies Corollary 1 but not Kellogg. This list does not satisfy Suleĭmanova–Perfect or Perfect 1.

The next result relates Soules 2 with the other symmetric criteria.

Theorem 17.

1. *Soules 2 is independent of Suleĭmanova, Suleĭmanova–Perfect, Perfect 1, Salzmann, Fiedler and Kellogg.*
2. *Borobia does not imply Soules 2.*
3. *Corollary 1 is strictly contained in Soules 2.*

Proof. *1.* The list $\{2, 0, 0, -1, -1\}$ is Suleĭmanova (so Suleĭmanova–Perfect, Fiedler and Kellogg), Perfect 1 and Salzmann but not Soules 2: with the convention of Remark 2, for any vector $x = (x_1, x_2, x_3, x_4, x_5) > 0$ and any partition of $\{3, 4, 5\}$, we have

$$d_5 = \frac{2x_5^2}{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2} > 0$$

which contradicts Remark 3. This, together with the comments prior to the theorem, prove the independency.

2. The list $\{2, 0, 0, -1, -1\}$ is Suleĭmanova (so Borobia) but not Soules 2, as was proved in *1.*.

3. The list $\{2, 1, -1, -2\}$ is Soules 2 with $x = (1, 1, 1, 1)$, $\{i_1, \dots, i_m\} = \{1, 2\}$, $\{j_1, \dots, j_{4-m-1}\} = \{3, 4\}$, $\{k_1, \dots, k_{m-1}\} = \{3\}$ and $\{\ell_1, \dots, \ell_{4-m-1}\} = \{4\}$, but not Corollary 1. □

Remark 6. In order to explain the map given at the end of this section we would like to enumerate some results relative to Corollary 1, the corollary of the Soules 2 condition. It can be proven:

1. Perfect–Mirsky implies Corollary 1 and the inclusion is strict.
2. Corollary 1 is independent of Suleĭmanova, Suleĭmanova–Perfect, Perfect 1, Salzmann, Fiedler, Kellogg and Soules 1.
3. Corollary 1 implies Borobia and the inclusion is strict.
4. Corollary 1 implies Soto 2 and the inclusion is strict.

We omit the details of this proof because the one we know is too long.

The next result relates Laffey–Šmigoc with the other symmetric criteria.

Theorem 18.

1. *Ciarlet implies Laffey–Šmigoc and the inclusion is strict.*
2. *Laffey–Šmigoc is independent of Suleĭmanova, Suleĭmanova–Perfect, Perfect 1, Perfect–Mirsky, Salzmann, Fiedler, Kellogg, Soules 1, Soules 2, Corollary 1 and Borobia.*

Proof. *1.* Let $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$, with $\lambda_1 \geq \cdots \geq \lambda_n$, satisfy Ciarlet:

$$\frac{\lambda_1}{n-1} \geq |\lambda_j| \,, \quad j = 2, \dots, n.$$

We assume $n \geq 3$, see Remark 5. It can happen that:

- $\lambda_2 \geq 0$ and $\lambda_n < 0$. In this case we take

$$\begin{aligned} \Lambda_1 &= \{\lambda_1\} \cup \{\lambda_j \;:\; \lambda_j < 0\} \\ \Lambda_2 &= \{\mu_1 = \lambda_2\} \cup \{\lambda_j \;:\; \lambda_j \geq 0,\, j \geq 2\} \\ c &= \lambda_1 + \sum_{\lambda_j < 0} \lambda_j. \end{aligned}$$

  Note that Λ_1 is the spectrum of an irreducible symmetric matrix, see Remark 5, and we can take the diagonal as $c, 0, \dots, 0$ because the sufficient conditions of Lemma 1 are satisfied. Also Λ_2 is symmetrically realizable. Let p be the number of negative elements in Λ. Note that $1 \leq p \leq n-2$. Let us see that $\mu_1 \leq c$:

$$c = \lambda_1 + \sum_{\lambda_j<0} \lambda_j = \lambda_1 - \sum_{\lambda_j<0} |\lambda_j| \geq \lambda_1 - \sum_{\lambda_j<0} \frac{\lambda_1}{n-1} = \frac{n-1-p}{n-1}\lambda_1$$
$$\geq (n-1-p)\lambda_2 \geq \lambda_2 = \mu_1.$$

Then Λ satisfies Laffey–Šmigoc.

- $\lambda_2 < 0$. In this case we take

$$\Lambda_1 = \{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}\}$$
$$\Lambda_2 = \{\mu_1 = -\lambda_n, \lambda_n\}$$
$$c = \lambda_1 + \sum_{j=2}^{n-1} \lambda_j.$$

Now we have

$$\lambda_1 + \sum_{j=2}^{n} \lambda_j \geq 0 \quad \Longrightarrow \quad c = \lambda_1 + \sum_{j=2}^{n-1} \lambda_j \geq -\lambda_n = \mu_1$$

and the same argument as before gives that Λ satisfies Laffey–Šmigoc.

- $\lambda_n \geq 0$. In this case we take

$$\Lambda_1 = \{\lambda_1, \lambda_2\}$$
$$\Lambda_2 = \{\mu_1 = \lambda_2, \lambda_3, \ldots, \lambda_n\}$$
$$c = \frac{\lambda_1}{n-1}.$$

Because of the Ciarlet condition, Λ_1 is the spectrum of the irreducible nonnegative symmetric matrix

$$\begin{pmatrix} \frac{\lambda_1}{n-1} & \sqrt{\frac{n-2}{n-1}\left(\frac{\lambda_1}{n-1} - \lambda_2\right)\lambda_1} \\ \sqrt{\frac{n-2}{n-1}\left(\frac{\lambda_1}{n-1} - \lambda_2\right)\lambda_1} & \frac{n-2}{n-1}\lambda_1 + \lambda_2 \end{pmatrix}$$

and $\mu_1 = \lambda_2 \leq \dfrac{\lambda_1}{n-1} = c$, so Λ satisfies Laffey–Šmigoc.

The list $\{7, 5, 0, -4, -4, -4\}$ satisfies Laffey–Šmigoc, see [10], but not Ciarlet.

2. The list $\{7, 5, 0, -4, -4, -4\}$ satisfies Laffey–Šmigoc, see [10], but not any of the other conditions. The list $\{2, 2, -1, -1\}$ satisfies Perfect–Mirsky and Salzmann (so Suleĭmanova, Suleĭmanova–Perfect, Fiedler, Kellogg, Soules 1, Soules 2, Corollary 1 and

Borobia) but not Laffey–Šmigoc, see Remark 5. Finally, the list $\{1, 1, -1, -1\}$ satisfies Perfect 1 but not Laffey–Šmigoc. $\square$

The following results relate the Kellogg and Borobia realizability criteria with the Soto p criteria.

Theorem 19. *If Λ is Kellogg realizable, then Λ is Soto p realizable for some p (p depends on Λ).*

Proof. Let $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ satisfy Kellogg: $\lambda_0 \geq \cdots \geq \lambda_n,\ \lambda_0 \geq |\lambda_n|,\ K = \{i \in \{1, \ldots, \lfloor n/2 \rfloor\} \ /\ \lambda_i \geq 0,\ \lambda_i + \lambda_{n+1-i} < 0\},\ M = \max\{j \in \{0, \ldots, n\} \ /\ \lambda_j \geq 0\}$ and the conditions

$$\lambda_0 + \sum_{i \in K,\ i<k} (\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-k} \geq 0 \quad \text{for all} \quad k \in K$$

and

$$\lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{n+1-i}) + \sum_{j=M+1}^{n-M} \lambda_j \geq 0.$$

Suppose $K = \{k_1, \ldots, k_p\}$, with $k_i \leq k_{i+1}$ for $i = 1, \ldots, p-1$, and define

$$\begin{aligned} \Lambda_0 &= \{\lambda_0, \lambda_{M+1}, \ldots, \lambda_{n-M}\}\ , \qquad S = -\sum_{j=M+1}^{n-M} \lambda_j\ , \\ \Lambda_i &= \{\lambda_{k_i}, \lambda_{n+1-k_i}\} \qquad \text{for} \qquad i = 1, \ldots, p\ , \\ \Lambda_{p+1} &= \Lambda - \Lambda_0 - \bigcup_{i=1}^{p} \Lambda_{k_i}\ , \end{aligned}$$

and

$$\Gamma_i = \Lambda_0 \cup \left(\bigcup_{j=p+1-i}^{p} \Lambda_j \right) \qquad \text{for} \qquad i = 1, \ldots, p\ .$$

Note that, with the notations of Theorems 12 and 14, we have

$$T_i = \lambda_{k_i} + \lambda_{n+1-k_i} = -\mathcal{N}_{Sj}(\Lambda_i) \qquad \text{for} \qquad i = 1, \ldots, p \qquad \text{and} \qquad \forall j \geq 1\ .$$

Now, Kellogg's conditions become

$$\begin{aligned} \text{C1}: \quad & \lambda_0 + \sum_{i=1}^{t-1} T_i + \lambda_{n+1-k_t} \geq 0 \qquad \text{for} \qquad t = 1, \ldots, p \\ \text{C2}: \quad & \lambda_0 + \sum_{i=1}^{p} T_i - S \geq 0\ . \end{aligned}$$

We see first that Λ_0 is Fiedler (equivalent to Soto 1, see [12]):

$$\lambda_0 + \lambda_{n-M} + \sum_{\lambda \in \Lambda_0} \lambda - \frac{1}{2} \sum_{j=1}^{n-M-1} |\lambda_{M+j} + \lambda_{n-M-j}|$$
$$= 2(\lambda_0 - S) \geq 2\left(\lambda_0 + \sum_{i=1}^{p} T_i - S\right) \overset{\text{C2}}{\geq} 0 .$$

If $K = \emptyset$, that is $p = 0$, then $\Lambda = \Lambda_0 \cup \Lambda_1$ is Soto 2 because

$$\lambda_0 - \max\{S, \lambda_1\} = \begin{cases} \lambda_0 - S = \lambda_0 + \sum\limits_{j=M+1}^{n-M} \lambda_j \geq 0, \\ \text{or} \\ \lambda_0 - \lambda_1 \geq 0 . \end{cases}$$

In this case the result is proved.

Suppose now that $K \neq \emptyset$. We will proceed in the following steps:

1. $\Gamma_1 = \Lambda_0 \cup \Lambda_p$ is Soto 2 and $\mathcal{M}_{S2}(\Gamma_1) \geq \lambda_0 - L + T_p$ with $L = \max\{S, \lambda_{k_p}\}$. It is enough to prove the inequality $\lambda_0 - L + T_p \geq 0$. Depending on the values of L we have

$$\lambda_0 - L + T_p = \begin{cases} \lambda_0 - S + T_p \geq \lambda_0 + \sum\limits_{i=1}^{p} T_i - S \overset{\text{C2}}{\geq} 0 , \\ \lambda_0 - \lambda_{k_p} + T_p = \lambda_0 + \lambda_{n+1-k_p} \geq 0 , \end{cases}$$

   and the claim is proved.
2. $\Gamma_2 = \Gamma_1 \cup \Lambda_{p-1}$ is Soto 3 and $\mathcal{M}_{S3}(\Gamma_2) \geq \lambda_0 - \widehat{\gamma}_1 + T_{p-1}$ with $\widehat{\gamma}_1 = \max\{L - T_p, \lambda_{k_{p-1}}\}$. By Lemma 2, it is enough to prove $\lambda_0 - \widehat{\gamma}_1 + T_{p-1} \geq 0$. Depending on the values of $\widehat{\gamma}_1$ we have

$$\lambda_0 - \widehat{\gamma}_1 + T_{p-1} = \begin{cases} \lambda_0 - S + T_p + T_{p-1} \geq \lambda_0 + \sum\limits_{i=1}^{p} T_i - S \overset{\text{C2}}{\geq} 0 , \\ \lambda_0 - \lambda_{k_p} + T_p + T_{p-1} = \lambda_0 + T_{p-1} + \lambda_{n+1-k_p} \\ \quad \geq \lambda_0 + \sum\limits_{i=1}^{p-1} T_i + \lambda_{n+1-k_p} \overset{\text{C1}}{\geq} 0 , \\ \lambda_0 - \lambda_{k_{p-1}} + T_{p-1} = \lambda_0 + \lambda_{n+1-k_{p-1}} \geq 0 , \end{cases}$$

   and the claim is proved.
3. $\Gamma_3 = \Gamma_2 \cup \Lambda_{p-2}$ is Soto 4 and $\mathcal{M}_{S4}(\Gamma_3) \geq \lambda_0 - \widehat{\gamma}_2 + T_{p-2}$ with $\widehat{\gamma}_2 = \max\{\widehat{\gamma}_1 - T_{p-1}, \lambda_{k_{p-2}}\}$.
   Again, by Lemma 2, it is enough to prove $\lambda_0 - \widehat{\gamma}_2 + T_{p-2} \geq 0$. Depending on the values of $\widehat{\gamma}_2$, we have

$$\lambda_0 - \widehat{\gamma_2} + T_{p-2} = \begin{cases} \lambda_0 - S + T_p + T_{p-1} + T_{p-2} \geq \lambda_0 + \sum_{i=1}^{p} T_i - S \overset{\text{C2}}{\geq} 0 \,, \\ \lambda_0 + T_{p-1} + \lambda_{n+1-k_p} + T_{p-2} \geq \lambda_0 + \sum_{i=1}^{p-1} T_i + \lambda_{n+1-k_p} \overset{\text{C1}}{\geq} 0 \,, \\ \lambda_0 + \lambda_{n+1-k_{p-1}} + T_{p-2} \geq \lambda_0 + \sum_{i=1}^{p-2} T_i + \lambda_{n+1-k_{p-1}} \overset{\text{C1}}{\geq} 0 \,, \\ \lambda_0 - \lambda_{k_{p-2}} + T_{p-2} = \lambda_0 + \lambda_{n+1-k_{p-2}} \geq 0 \,, \end{cases}$$

and the claim is proved.

The same type of argument, with more combinatorics, proves that $\Gamma_{j-1} = \Gamma_{j-2} \cup \Lambda_{p-j}$ is Soto j and

$$\mathcal{M}_{Sj}(\Gamma_{j-1}) \geq \lambda_0 - \widehat{\gamma_{j-2}} + T_{p-(j-2)}$$

with

$$\widehat{\gamma_{j-2}} = \max\{\widehat{\gamma_{j-3}} - T_{p-(j-3)}, \lambda_{k_{p-j}}\} \qquad \text{for} \qquad j = 5, \dots, p+1.$$

In particular, $\Gamma_p = \Gamma_{p-1} \cup \Lambda_1$ is Soto $p+1$ and

$$\begin{aligned} \lambda_0 - L + T_1 \geq 0 \qquad & \text{if} \qquad p = 1, \\ \lambda_0 - \widehat{\gamma_{p-1}} + T_1 \geq 0 \qquad & \text{if} \qquad p \geq 2. \end{aligned}$$

Finally, we prove that Λ is also Soto $p+1$ if $k_1 = 1$ or Soto $p+2$ if $k_1 \neq 1$.

If $k_1 = 1$, it is clear that Λ, with the partition

$$\Lambda = \Gamma_{p-1} \cup \Lambda_1 \cup \left(\bigcup_{\substack{i \in \{1, \dots, \lfloor n/2 \rfloor\} \\ i \notin K}} \{\lambda_i, \lambda_{n+1-i}\} \right),$$

is Soto $p+1$ because the lists added are Soto p due to the fact that they are Suleĭmanova.

If $k_1 \neq 1$, that is $1 \notin K$, for the partition of Λ

$$\Gamma_p \cup \left(\bigcup_{\substack{i \in \{1, \dots, \lfloor n/2 \rfloor\} \\ i \notin K}} \{\lambda_i, \lambda_{n+1-i}\} \right),$$

we obtain (note the lists added are Soto $p+1$)

$$\lambda_0 - \max\{L - T_1, \lambda_1\} = \left\{ \begin{array}{l} \lambda_0 - L + T_1 \geq 0 \\ \text{or} \\ \lambda_0 - \lambda_1 \geq 0 \end{array} \right\} \quad \text{if} \quad p = 1,$$

$$\lambda_0 - \max\{\widehat{\gamma_{p-1}} - T_1, \lambda_1\} = \left\{ \begin{array}{l} \lambda_0 - \widehat{\gamma_{p-1}} + T_1 \geq 0 \\ \text{or} \\ \lambda_0 - \lambda_1 \geq 0 \end{array} \right\} \quad \text{if} \quad p \geq 2.$$

Now, by Lemma 2, we have that Λ is Soto $p+2$. $\square$

Remark 7. Note that we have proved that if Λ is Kellogg and $p = \#K$ with K the set defined in Theorem 6, then Λ is Soto $p+1$ if $1 \in K$ and Soto $p+2$ if $1 \notin K$.

Theorem 20. *If Λ is Borobia realizable, then Λ is Soto p realizable for some p (p depends on Λ).*

Proof. Let $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ satisfy Borobia: $\lambda_0 \geq \cdots \geq \lambda_n$, $\lambda_0 \geq |\lambda_n|$, $M = \max\{j \in \{0, \ldots, n\} \,/ \lambda_j \geq 0\}$ and there exists a partition $J_1 \cup \cdots \cup J_t$ of $\{\lambda_{M+1}, \ldots, \lambda_n\}$ such that the list

$$\left\{ \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_M \geq \sum_{\lambda \in J_1} \lambda \geq \cdots \geq \sum_{\lambda \in J_t} \lambda \right\}$$

is Kellogg realizable. We can apply Theorem 19 to this new list.

Going through the proof of the previous theorem we have that the original list Λ is Soto $p+1$ or Soto $p+2$, with $p = \#K$, depending on the fact that $1 \in K$ or $1 \notin K$. This is possible because all the inequalities related to the Soto's criteria take the same value for both lists. Note that only the negative eigenvalues of the original list are modified by addition, the nonnegative eigenvalues remain equal. $\square$

Theorem 21.

1. *Soto $p-1$ is strictly contained in Soto p, for $p \geq 3$.*
2. *Kellogg and Borobia are independent of Soto p, for $p \geq 3$.*
3. *Soto p does not imply Soules 2, for $p \geq 2$.*
4. *Laffey–Šmigoc is independent of Soto p, for $p \geq 2$.*

Proof. *1.* Let

$$\Lambda_3 = \{9, 7, 4, -3, -3, -6, -8\}$$
$$\Lambda_4 = \{9.01, 8.1, 7, 4, -3, -3, -6, -8, -8.11\}$$

and for $p \geq 5$

$$\Lambda_p = \left(\Lambda_{p-1} - \left\{ 9 + \sum_{j=1}^{p-4} 10^{-2j} \right\} \right) \cup \left\{ 9 + \sum_{j=1}^{p-3} 10^{-2j}, 8 + \sum_{j=1}^{2p-7} 10^{-j}, -8 - \sum_{j=1}^{2p-6} 10^{-j} \right\}.$$

The list Λ_p satisfies Soto p and not Soto p−1, for p ≥ 3. For $p = 3$ see [19].

First we see that Λ_4 satisfies Soto 4 for the partition

$$\Lambda_4 = \Lambda_{41} \cup \Lambda_{42} \ \text{ with } \ \Lambda_{41} = \{9.01, -8.11\} \ \text{ and } \ \Lambda_{42} = \{8.1, 7, 4, -3, -3, -6, -8\}.$$

Clearly $\mathcal{M}_{S_3}(\Lambda_{41}) = 0.9$ and since $\Lambda_{42} = (\Lambda_3 - \{9\}) \cup \{8.1\}$ then $\mathcal{N}_{S_3}(\Lambda_{42}) = 9 - 8.1 = 0.9$. Therefore

$$\lambda_1 - \gamma - \mathcal{N}_{S_3}(\Lambda_{42}) = 9.01 - 8.1 - 0.9 \geq 0,$$

so Λ_4 is Soto 4 but does not satisfy Soto 3 for no partition.

Finally, for $p \geq 5$, we see that Λ_p satisfies Soto p with the partition

$$\Lambda_p = \Lambda_{p1} \cup \Lambda_{p2} \ \text{ with } \ \Lambda_{p1} = \left\{ 9 + \sum_{j=1}^{p-3} 10^{-2j}, -8 - \sum_{j=1}^{2p-6} 10^{-j} \right\} \ \text{ and}$$

$$\Lambda_{p2} = \left\{ 8 + \sum_{j=1}^{2p-7} 10^{-j} \right\}, \Lambda_{p-1} - \left\{ 9 + \sum_{j=1}^{p-4} 10^{-2j} \right\}$$

because

$$\gamma = \max\{\lambda_1 - \mathcal{M}_{S_{p-1}}(\Lambda_{p1})\} = -\lambda_n$$

and

$$\lambda_1 - \gamma - \mathcal{N}_{S_{p-1}}(\Lambda_{p2}) = \mathcal{M}_{S_{p-1}}(\Lambda_{p1}) - \mathcal{N}_{S_{p-1}}(\Lambda_{p2}) =$$

$$\left(9 + \sum_{j=1}^{p-3} 10^{-2j} - 8 - \sum_{j=1}^{2p-6} 10^{-j} \right) - \left(9 + \sum_{j=1}^{p-4} 10^{-2j} - 8 - \sum_{j=1}^{2p-7} 10^{-j} \right) = 0.$$

And Λ_p does not satisfy Soto p−1 for no partition.

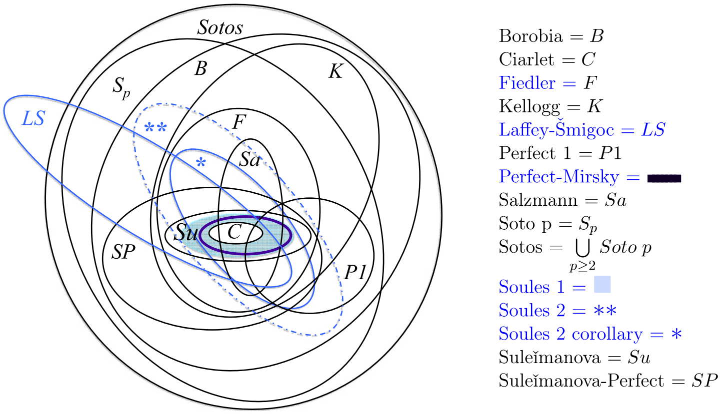
2. The list Λ_p just defined satisfies Kellogg but not Soto p. The list $\{25, 21, 18, 16, -10, -10, -10, -10, -10, -10, -10, -10\}$ is Soto 3 but not Kellogg nor Borobia, see [19] for details.

3. The list $\{2, 0, 0, -1, -1\}$ is Suleĭmanova so Soto p, for $p \geq 2$, but not Soules 2 (see the proof of Theorem 17).

4. The list $\{7, 5, 0, -4, -4, -4\}$ satisfies Laffey–Šmigoc, see [10], but not Soto p, for $p \geq 2$. The list $\{2, 2, -1, -1\}$ satisfies Soto p, for $p \geq 2$, but not Laffey–Šmigoc, see Remark 5. □

Conjecture. *Soules 2 is strictly contained in Borobia and Soto 2.*

Next we show a map with all the relations between the symmetric conditions studied.



The discontinuous line for Soules 2 in the map means that we only conjecture this position for this sufficient condition.

Recently, Ellard–Šmigoc [6], via a recursive approach to the SNIEP, have established the equivalence of several of the most general sufficient conditions for the SNIEP. They have modified the Laffey–Šmigoc condition (Ellard–Šmigoc, see [6, Section 3]) and the Soules 2 condition (*piecewise Soules*, see [6, Definition 2.8]). They also relate these criteria with Sotos criteria and C-realizability (see [3]). Explicitly they prove, [6, Theorem 4.1]:

$$\text{piecewise Soules} \iff \text{Ellard–Šmigoc} \iff \text{C-realizability} \iff \text{Sotos.}$$

As a consequence of this result it is obtained that the C-realizability is also a symmetric sufficient condition, something which was not previously known.

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