Linear and Multilinear Algebra

ISSN: 0308-1087 (Print) 1563-5139 (Online) Journal homepage: http://www.tandfonline.com/loi/glma20

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To cite this article: C. R. Johnson, C. Marijuán & M. Pisonero (2017) Symmetric nonnegative realizability via partitioned majorization, Linear and Multilinear Algebra, 65:7, 1417-1426, DOI: 10.1080/03081087.2016.1242113

To link to this article: http://dx.doi.org/10.1080/03081087.2016.1242113

LINEAR AND MULTILINEAR ALGEBRA, 2017
VOL. 65, NO. 7, 1417–1426
http://dx.doi.org/10.1080/03081087.2016.1242113

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Symmetric nonnegative realizability via partitioned majorization

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ABSTRACT

A sufficient condition for symmetric nonnegative realizability of a spectrum is given in terms of (weak) majorization of a partition of the negative eigenvalues by a selection of the positive eigenvalues. If there are more than two positive eigenvalues, an additional condition, besides majorization, is needed on the partition. This generalizes observations of Suleĭmanova and Loewy about the cases of one and two positive eigenvalues, respectively. It may be used to provide insight into realizability of 5-element spectra and beyond.

ARTICLE HISTORY
Received 3 May 2016
Accepted 21 September 2016

COMMUNICATED BY
R. Loewy

KEYWORDS
(Weak) majorization; nonnegative inverse eigenvalue problem (NIEP); symmetric realizability

AMS SUBJECT CLASSIFICATIONS
15A18; 15A29; 15A42

We say that a collection of n real (complex) numbers (repeats allowed) is *realizable* if they occur as the spectrum of an n-by-n nonnegative matrix. The collection is further called *symmetrically realizable* if the nonnegative matrix may be taken to be symmetric. Of course, realizability requires the *Perron condition* that the largest of the absolute values of the numbers be in the collection and the *trace condition* that the sum of the numbers be nonnegative. These conditions are not generally sufficient and a complete description of the realizable real spectra or the symmetrically realizable spectra is far from known. Since they are necessary, we assume the Perron and trace conditions on a proposed spectrum throughout the following discussion.

In [1], it was pointed out (without proof) that when the collection contains just one positive number, the trace condition is equivalent to realizability and in [2], etc., that it is symmetrically realizable.

In the next case, in which there are just two positive eigenvalues, a sufficient condition for symmetric realizability was given. If the Perron and trace condition are met, and the negative eigenvalues may be partitioned into two parts, in each of which the sum of the absolute values is not more than the Perron root, then symmetric realizability follows. This statement, properly attributed to Loewy, was reported in [3] without proof, and in [4] with Loewy's proof. Notice that, in this case, because of the trace condition, the two

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positive eigenvalues weakly majorize the two partial absolute sums of the negative spectral elements. Notice also that additional nonnegative spectral elements could be appended and symmetric realizability would still hold.

Our purpose here is to note a broad generalization of the Suleĭmanova/Loewy observations when there are s positive eigenvalues. The focus is upon (weak) majorization (Theorem 3), which remains sufficient with a slight additional condition when there are more than two positive eigenvalues. First we need some prior results (Theorems 1 and 2), and then we use the ideas to make several observations, including about symmetric realizability in the 5-by-5 case. Examples are given that limit possible weakening of our conditions.

Theorem 1 (Kellogg [5], 1971): *Let $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ with $\lambda_0 \geq |\lambda|$ for $\lambda \in \Lambda$ and $\lambda_i \geq \lambda_{i+1}$ for $i = 0, \ldots, n-1$. Let M be the greatest index j $(0 \leq j \leq n)$ for which $\lambda_j \geq 0$ and $K = \{i \in \{1, \ldots, \lfloor n/2 \rfloor\} \ / \ \lambda_i \geq 0,\ \lambda_i + \lambda_{n+1-i} < 0\}$. If*

$$\lambda_0 + \sum_{i \in K,\ i<k} (\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-k} \geq 0 \quad \text{for all } k \in K, \tag{1}$$

and

$$\lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{n+1-i}) + \sum_{j=M+1}^{n-M} \lambda_j \geq 0, \tag{2}$$

then Λ is realizable.

Note that if Λ verifies the Kellogg condition, then the spectrum obtained from Λ taking out the eigenvalues involved in the set $\{\lambda_i + \lambda_{n+1-i} = 0,\ i \in \{1, \ldots, \lfloor n/2 \rfloor\}\}$, if there is any, also verifies the Kellogg condition.

Theorem 2 (Borobia [6], 1995): *Let $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ with $\lambda_i \geq \lambda_{i+1}$ for $i = 0, \ldots, n-1$ and let M be the greatest index j $(0 \leq j \leq n)$ for which $\lambda_j \geq 0$. If there exists a partition $J_1 \cup \cdots \cup J_t$ of $\{\lambda_{M+1}, \ldots, \lambda_n\}$ such that*

$$\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_M > \sum_{\lambda \in J_1} \lambda \geq \cdots \geq \sum_{\lambda \in J_t} \lambda \tag{3}$$

satisfies the Kellogg condition, then Λ is realizable.

Fiedler [2] proves that the Kellogg condition guarantees symmetric realizability, and Radwan [3] proves that the Borobia condition also guarantees symmetric realizability.

Suppose that we have s positive real numbers $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s > 0$ and t nonpositive real numbers $0 \geq -\beta_1 \geq -\beta_2 \geq \cdots \geq -\beta_t$, repeats allowed. If we partition the β's into k nonempty parts $(k \leq s)$ $\mathcal{P}_1, \ldots, \mathcal{P}_k$, we refer to the sum of β's in part $\mathcal{P}_i$ as P_i, $i = 1, \ldots, k$. We adopt the convention that the parts of the partition are ordered so that $P_1 \geq P_2 \geq \cdots \geq P_k \geq 0$. Recall that real numbers $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k \geq 0$ are said to *weakly majorize* $P_1 \geq P_2 \geq \cdots \geq P_k$ if

$$\begin{aligned} \gamma_1 &\geq P_1 \\ \gamma_1 + \gamma_2 &\geq P_1 + P_2 \\ &\vdots \\ \gamma_1 + \cdots + \gamma_k &\geq P_1 + \cdots + P_k \,. \end{aligned}$$

If the last inequality is an equality, the term *majorization* is used. We may now give a sufficient condition for symmetric realizability that includes the observations of both Suleĭmanova and Loewy.

Theorem 3: *The collection of real numbers*

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s \,, \qquad \alpha_s > 0 \,,$$

together with

$$-\beta_1 \geq -\beta_2 \geq \cdots \geq -\beta_t \,, \qquad 0 \geq -\beta_1 \,,$$

is symmetrically realizable if there is a selection of k of the α's

$$\alpha_{i_1} \geq \alpha_{i_2} \geq \cdots \geq \alpha_{i_k} > 0$$

and a partition of the β's into k parts with

$$P_1 \geq P_2 \geq \cdots \geq P_k \geq 0$$

so that $\alpha_{i_1}, \ldots, \alpha_{i_k}$ *weakly majorize* $P_1, \ldots, P_k$ *and whenever* $\alpha_{i_{j_0}} - P_{j_0-1} > 0$ *for* $j_0 \in \{2, \ldots, k\}$ *then* $\alpha_{i_j} - P_{j-1} > 0$ *for all j with* $j_0 < j \leq k$.

***Proof*:** First note that any collection of real numbers verifying Theorem 2 is symmetrically realizable. Second note that if the collection of real numbers

$$\alpha_{i_1} \geq \cdots \geq \alpha_{i_k} > -P_k \geq \cdots \geq -P_1$$

verifies the Kellogg condition, then

$$\{\alpha_{i_1}, \ldots, \alpha_{i_k}\} \cup \cup_{j=1}^{k} \{-\beta_i \;/\; \beta_i \in \mathcal{P}_j\}$$

verifies Theorem 2. Third, we can assume $\alpha_{i_j} - P_{j-1} \neq 0$, for $j = 1, \cdots, k$, because of the comment after Theorem 1. So it is enough to prove that

$$\alpha_{i_1} \geq \cdots \geq \alpha_{i_k} > -P_k \geq \cdots \geq -P_1, \quad \text{with} \;\; \alpha_{i_j} - P_{j-1} \neq 0, \; j = 1, \cdots, k$$

verifies the Kellogg condition. For

$$K = \{j \geq 2 \;/\; \alpha_{i_j} - P_{j-1} < 0\},$$

and under our hypothesis, it can happen that either:

- $K = \emptyset$, in which case, the inequality (2)

$$\alpha_{i_1} - P_k \geq 0$$

  is true; or
- $K = \{2, \ldots, r\} \neq \emptyset$, in which case, the inequalities (1)

$$\alpha_{i_1} + \sum_{j=2}^{q-1} (\alpha_{i_j} - P_{j-1}) - P_{q-1} \geq 0 \qquad 2 \leq q \leq r$$

  are true because of the weak majorization hypothesis. Inequality (2) is also true

$$\alpha_{i_1} + \sum_{j=2}^{r} (\alpha_{i_j} - P_{j-1}) - P_k \geq 0$$

  because of $-P_k \geq -P_r$ and of the weak majorization hypothesis.

□

Theorem 3 gives a sufficient condition for symmetric realizability in terms of (weak) majorization of partitioned sums of the negative eigenvalues by a selection of positive eigenvalues. However, there is an extra condition (besides majorization). When $s = 1$ or 2, the extra condition disappears, so that the Suleĭmanova and Loewy observations are special cases. When $k \geq 3$ for the theorem to apply in the trace 0 case, the hypothesis of the extra condition must never be met as the consequent of the extra condition cannot hold, because of trace 0 and majorization ($\alpha_k > P_{k-1} \geq P_k$). Furthermore in the interesting case in which the P_j's are relatively large (i.e. $\alpha_{i_j} \leq P_{j-1}$), the extra conditions are vacuously met. We note that, for s positive eigenvalues, it is always possible to partition the negative eigenvalues into at most $2s - 1$ parts for which the P_j's are $\leq$ the Perron root: to obtain the P_j's add the two smallest β's if their sum is $\leq \alpha_1$ and reorder them; repeat this procedure and stop when this is not possible. The number of elements of this partition is $\leq 2s - 1$, other way we would go against the trace condition. Note that even with this partition, the positive eigenvalues may not majorize the P_j's (*e.g.* $\{10, 5, 5, -3, -8, -8\}$).

When two positive and several negative eigenvalues, meeting the trace and Perron conditions, are given, it is not difficult to see that a negative eigenvalue may be subdivided into two negative eigenvalues, whose sum is the original eigenvalue, in such a way that the partition condition of Theorem 3 is met:

Corollary 4: *If the eigenvalues $\alpha_1 \geq \alpha_2 > 0 \geq -\beta_1 \geq -\beta_2 \geq \cdots \geq -\beta_t$ meet the Perron and the trace conditions, then there is an index q, $1 \leq q \leq t$, and a subdivision of β_q into $\beta_{q_1} + \beta_{q_2} = \beta_q$, $\beta_{q_1}, \beta_{q_2} \geq 0$, so that the spectrum : $\alpha_1, \alpha_2, -\beta_1 \ldots, -\beta_{q-1}, -\beta_{q_1}, -\beta_{q_2}, -\beta_{q+1}, \ldots, -\beta_t$ is symmetrically realizable. Moreover, there are also indices $j \neq q$, $1 \leq j, q \leq t$ and a shift δ, $0 \leq \delta \leq \beta_q$, such that the proposed spectrum, with β_j and β_q replaced by $\beta_j + \delta, \beta_q - \delta$ is symmetrically realizable.*

Proof: Let suppose the set

$$\left\{ j \in \{1, \cdots, t\} : \alpha_1 - \sum_{i=1}^{j} \beta_i < 0 \right\}$$

is not empty, other way the result is clear. For the first statement take

$$q = \max \left\{ j \in \{1, \cdots, t\} : \alpha_1 - \sum_{i=1}^{j} \beta_i < 0 \right\},$$
$$\beta_{q_1} = \alpha_1 - \sum_{i=1}^{q-1} \beta_i \quad \text{and} \quad \beta_{q_2} = \sum_{i=1}^{q} \beta_i - \alpha_1.$$

For the second statement take the same q, $j = q + 1$ and $\delta = \sum_{i=1}^{q} \beta_i - \alpha_1$. □

In case $s = 1$ (Suleĭmanova), and $s = 2$ and $t = 2$, the condition given is known to be necessary and sufficient, see [7,8]. We also point out that it is necessary and sufficient when $s = 2$ and $t = 3$ (see comments previous to Example 6). Recall that symmetric realizability of five-element spectra with trace 0 were characterized in [4] (and realizability with trace 0 in [9]). No characterization of symmetric realizability, or of (real) realizability, is known in a richer case.

Theorem 5 (Spector [4], 2011): *Let* $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ *and* $s_k = \sum_{i=1}^{5} \lambda_i^k$. *Suppose* $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \geq -\lambda_1$ *and* $s_1 = 0$. *Then* σ *is symmetrically realizable if and only if the following conditions hold:*

(1) $\lambda_2 + \lambda_5 \leq 0$,
(2) $s_3 \geq 0$.

In case there is one positive eigenvalue (Suleĭmanova) $\lambda_2 + \lambda_5 \leq 0$ is guaranteed, but symmetric realizability follows without any condition besides trace 0. In the case of two positive eigenvalues, $\lambda_2 + \lambda_5 \leq 0$ is equivalent to $\lambda_1 + \lambda_3 + \lambda_4 \geq 0$ and is thus (along with the Perron condition) equivalent to Loewy's condition (or Theorem 3 with $s = 2$). This means that in the symmetric, trace 0 case, with one or two positive eigenvalues, only the condition $\lambda_1 + \lambda_3 + \lambda_4 \geq 0$ (equivalently $\lambda_2 + \lambda_5 \leq 0$) is necessary and sufficient for symmetric realizability; the additional condition $s_3 \geq 0$ comes for free. And, the condition of Theorem 3 (which is just Suleĭmanova/Loewy in this case) is necessary, as well as sufficient. When there are 3 positive eigenvalues (4 or 5 cannot occur), only the condition $s_3 \geq 0$ is relevant; $\lambda_2 + \lambda_5 \leq 0$ comes for free. Note also that in the n-by-n case, with two positive eigenvalues, $\lambda_1 + \lambda_3 + \cdots + \lambda_{n-1} \geq 0$, along with the Perron and the trace conditions, is sufficient for symmetric realizability.

For $n = 5$, not necessarily trace 0, $\lambda_2 + \lambda_5 > 0$ may happen, but using observations in [10], also mentioned in [11], $\lambda_1 + \lambda_3 + \lambda_4 \geq 0$ remains necessary and sufficient for symmetric realizability when there are no more than two positive eigenvalues. Four or five positive eigenvalues are trivially symmetrically realizable, which means that only the case of three positive eigenvalues need be characterized in order to complete characterization of symmetric realizability when $n = 5$. Then, the question is what replaces $s_3 \geq 0$ from the trace 0 case?

We now mention examples that limit possible generalizations of what we have said.

Example 6: For $s \geq 3$, the weak majorization condition, without qualification, of Theorem 3 is not sufficient. For $n = 6$ and the spectrum: $\{4, 4, 1, -3, -3, -3\}$, majorization holds with each part of the partition consisting of one negative eigenvalue. However, the spectrum is neither symmetrically realizable, nor realizable, as any realizing matrix would have to be reducible into two blocks, each with Perron root 4. But, this cannot happen, as two of the -3's would have to be paired with either 4, or 4 and 1. The difficulty is that $\alpha_2 > P_1$, but $\alpha_3 < P_2$.

Example 7: The partition condition of Theorem 3, or Loewy's condition, is not necessary for real, not necessarily symmetric, realizability, even when $s = 2$ and $t = 3$. The spectrum: $\{7, 5, -4, -4, -4\}$ is realizable by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 61/2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 566 & 73/4 & 92 & 61/2 & 0 \end{pmatrix}$$

and, of course, meets the nonsymmetric conditions of [9], but is not partitionable for majorization as in Theorem 3. Of course, this spectrum, which is not symmetrically realizable, illustrates the difference between the real and symmetric cases (first noted in [12]) when $n = 5$. There is no difference for $n = 4$.

Example 8: In spite of the above example, the spectrum: $\Lambda_\delta = \{7, 5, -\delta, -4+\delta, -4, -4\}$ is symmetrically realizable for each $\delta \in [0, 2]$. Since the Loewy condition (the case $s = 2$ of Theorem 3) is not met for $\delta \in [0, 1)$, this shows that it is not necessary when $t > 3$ (as it is when $t = 3$).

To realize Λ_δ for $\delta \in [0, 1]$, a method similar to one used in [13] is used to construct the matrix

$$\begin{pmatrix} 0 & 0 & x & 4 & 0 & 0 \\ 0 & 0 & y & 0 & z & z \\ x & y & 0 & x & 0 & 0 \\ 4 & 0 & x & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 4 \\ 0 & z & 0 & 0 & 4 & 0 \end{pmatrix}, \quad \begin{aligned} x &= \frac{\sqrt{-\delta^2 + 4\delta + 23 - \sqrt{\delta^4 - 8\delta^3 - 18\delta^2 + 136\delta + 145}}}{2} \\ y &= \sqrt{2(\delta - 3)(\delta - 1)} \\ z &= \frac{\sqrt{-\delta^2 + 4\delta + 23 + \sqrt{\delta^4 - 8\delta^3 - 18\delta^2 + 136\delta + 145}}}{2} \end{aligned}$$

with spectrum Λ_δ.

To realize Λ_δ for $\delta \in [1, 2]$ we can use Fiedler's method [2, Lemma 2.2], the one used in the proof of the Loewy condition. First we realize the spectra $\{8 - \delta, -4 + \delta, -4\}$ and $\{4 + \delta, -\delta, -4\}$ by the matrices

$$A = \begin{pmatrix} 0 & \sqrt{16-2\delta} & \sqrt{16-2\delta} \\ \sqrt{16-2\delta} & 0 & 4-\delta \\ \sqrt{16-2\delta} & 4-\delta & 0 \end{pmatrix} \quad \text{and}$$

$$B = \begin{pmatrix} 0 & \sqrt{8+2\delta} & \sqrt{8+2\delta} \\ \sqrt{8+2\delta} & 0 & \delta \\ \sqrt{8+2\delta} & \delta & 0 \end{pmatrix},$$

respectively, and respective normalized Perron eigenvectors

$$u^T = \left(\frac{2}{\sqrt{12-\delta}}, \sqrt{\frac{8-\delta}{24-2\delta}}, \sqrt{\frac{8-\delta}{24-2\delta}} \right) \quad \text{and}$$

$$v^T = \left(\frac{2}{\sqrt{8+\delta}}, \sqrt{\frac{4+\delta}{16+2\delta}}, \sqrt{\frac{4+\delta}{16+2\delta}} \right).$$

We obtain the eigenvalues 7, 5 as the spectrum of the matrix

$$\begin{pmatrix} 8-\delta & \rho \\ \rho & 4+\delta \end{pmatrix}, \quad \rho = \sqrt{(3-\delta)(\delta-1)}\,.$$

Then the spectrum Λ_δ is symmetrically realized by

$$\begin{pmatrix} A & \rho u v^T \\ \rho v u^T & B \end{pmatrix}.$$

By symmetry, we obtain that Λ_δ is also symmetrically realizable for $\delta \in [2, 4]$.

Example 9: It might be asked whether Loewy's condition, as stated, generalizes to the case in which there are more than two positive eigenvalues. If the negative eigenvalues may be partitioned into two sets, each beaten in absolute sum by the Perron root, is symmetric realizability guaranteed. The answer is "no" as $\{6, 2, 2, -5, -5\}$ is not symmetrically realizable by the second condition of Theorem 5 ($216 + 8 + 8 \not\geq 125 + 125$), but 6 beats both 5 and 5.

We offer two remarks, applying Theorem 3 to get general conditions for symmetric realizability when $s = 2$.

Remark 10: With two positive eigenvalues, $s = 2$ in Theorem 3, only the cases $t \geq 3$ are of interest; if $t = 2$, realizability is guaranteed by the Perron and trace conditions, see [7,8]. When $t \geq 3$, partitionability, a la Theorem 3, is most subtle in the trace 0 case. It occurs if and only if there is a subset of the β's whose sum lies between α_2 and α_1. When $t = 3$, this happens if and only if $\alpha_2 \leq \beta_3 \leq \alpha_1$. For $t > 3$, it is more complicated and we may only give a simple sufficient condition: the sum of the top $\frac{t}{2}$ (n even) or the top $\frac{t-1}{2}$ (n odd) β's lies between α_2 and α_1. In the even case, it suffices that $\beta_1 \geq \frac{2\alpha_2}{t}$ and in the odd case that $\beta_1 \geq \frac{2\alpha_2}{t-1}$ or $\beta_t \leq \frac{2\alpha_1}{t+1}$. These conditions guarantee symmetric realizability.

Remark 11: Let α_1 and α_2 be fixed. Assuming a uniform distribution on the β's, subject to $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_t$ in Theorem 3, the probability of partitionability meeting the conditions of Theorem 3 approaches 1 as $t \to \infty$. This means that with $s = 2$ in Theorem 3, the probability of symmetric realizability approaches 1 as $t \to \infty$.

Is it worth asking if the last two remarks remain valid for $s > 2$? The likely answer is yes.

We close with an observation that gives further insight into a classical example and natural questions raised about it.

Remark 12: Since at least the 1970's it has been recognized that the spectrum

$$\{3, 3, -2, -2, -2\}$$

is not even realizable, despite meeting all known general necessary conditions. The obstacle is the implied reducibility due to the tie for Perron root. (Of course, symmetric realizability is ruled out by the result of [4] or the necessity of partitioned majorization for $n = 5$, observed herein for two positive eigenvalues.)

Shortly after this spectrum was first discussed, author Johnson raised the question of the minimum $\varepsilon > 0$ for which

$$\{3 + \varepsilon, 3 - \varepsilon, -2, -2, -2\}$$

is realizable in an AMS talk. (Larger ε will also work.) This is, of course, of interest in both the case of realizability and symmetric realizability. Call the minimum ε, $\varepsilon_{\min}$ or $\varepsilon_{S\,\min}$, depending upon the case of realizability, and $\varepsilon_{\min}(k)$ or $\varepsilon_{S\,\min}(k)$ if k 0's are appended to the mentioned spectrum. Of course $\varepsilon_{\min}(k) \le \varepsilon_{S\,\min}(k)$, for $k = 0, 1, \ldots$. The case of realizability attracted the attention of several people and was fully resolved, $k = 0$, in [9]:

$$\varepsilon_{\min} = \sqrt{16\sqrt{6} - 39} \approx 0.438.$$

It follows from the observations herein, or earlier work that

$$\varepsilon_{S\,\min} = 1,$$

again showing that the difference between the S(ymmetric)NIEP and the R(eal)NIEP begins at $n = 5$, and that the difference is substantial. Of course, in the realizability case, the observation [14] shows that appending 0's will decrease $\varepsilon_{\min}$ and that $\lim_{k\to\infty} \varepsilon_{\min}(k) = 0$ (also shown constructively in [15]). Less obviously, as [14] does not apply, $\varepsilon_{S\,\min}(1) < \varepsilon_{S\,\min}$. Since $\{7, 5, -4, -4, -4, 0\}$ is symmetrically realizable (Example 8), then so is

$$\left\{3 + \frac{1}{2}, 2 + \frac{1}{2}, -2, -2, -2, 0\right\},$$

giving $\varepsilon_{S\min}(1) \leq \frac{1}{2}$. In fact, it is smaller. The 6-by-6 example

$$\begin{pmatrix} 0 & 0 & x & 2 & 0 & 0 \\ 0 & 0 & x^2 & 0 & x & x \\ x & x^2 & 0 & x & 0 & 0 \\ 2 & 0 & x & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 2 \\ 0 & x & 0 & 0 & 2 & 0 \end{pmatrix},$$

with $x = \frac{2}{\sqrt{3}}$, is nonnegative, symmetric, irreducible and has eigenvalues

$$\left\{ \frac{10}{3}, \frac{8}{3}, -2, -2, -2, 0 \right\}.$$

Thus, $\varepsilon_{S\min}(1) \leq \frac{1}{3}$, and $\varepsilon_{S\min}(1) < \varepsilon_{\min}$, even. In [15], estimates for $\varepsilon_{\min}(k)$ are given via explicit matrices, and $\frac{1}{3}$, our estimate for $\varepsilon_{S\min}(1)$, is already less that the estimate given for $\varepsilon_{\min}(5)$. However, their estimates of $\varepsilon_{\min}(k)$ converge rapidly to 0 in k.

We do not know if $\varepsilon_{S\min}(1) = \frac{1}{3}$, and it seems that $\varepsilon_{\min}(1)$ is also not known. Whether $\varepsilon_{S\min}(k)$ decreases for some positive k (it is obviously nonincreasing) and $\lim_{k\to\infty} \varepsilon_{S\min}(k)$, would also be of interest.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was partially supported by MTM2015-365764-C-1-P(MINECO/FEDER) and MTM2010-19281-C03-01.

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