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Ruling out certain 5-spectra for the symmetric nonnegative inverse eigenvalue problem ☆

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ABSTRACT

A method is developed to show that certain spectra cannot be realized for the S-NIEP. It is applied in the 5-by-5 case to rule out many spectra that were previously unresolved. These are all in the case of 3 positive and 2 negative eigenvalues as all other cases are now resolved. For spectra of the sort we discuss, a diagram is given of the spectra that are excluded here, as well as those trivially realizable, those realizable because of the trace 0 case and those that may also be excluded because of the J–L–L conditions. A small region remains unresolved; it is a very small fraction of the area of those spectra we consider.

The n-by-n symmetric nonnegative inverse eigenvalue problem, S-NIEP, asks which collections of n real numbers, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, occur as the spectrum of an n-by-n

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symmetric nonnegative matrix, counting multiplicities. Already for $n = 5$ this has proven a very challenging problem. In [6] a detailed discussion is given of many parts of the case $n = 5$, for positive trace. In the trace 0 case ($\sum_{i=1}^{n} \lambda_i = 0$) such spectra have recently been characterized for $n = 5$ [9], but the general (trace ≥ 0) case remains open. It is convenient to categorize by sub-cases, based upon the number of positive eigenvalues. In case that number is 1, 4, or 5 resolution of the 5-by-5 S-NIEP is straightforward. In [1], the unresolved cases were narrowed to some with 2 positive eigenvalues and most nontrivial ones with 3 positive eigenvalues. Here, we first note that all cases with 2 positive eigenvalues may be resolved, and then (principally) give a new method to rule out many unresolved spectra with 3 positive eigenvalues. Some ruled out spectra are known to be realizable for the 5-by-5 R-NIEP (which only requests a nonnegative, not necessarily symmetric matrix realizing the given eigenvalues).

For both the R-NIEP and S-NIEP, $\sum_{i=1}^{n} \lambda_i \geq 0$ is clearly necessary and by the Perron–Frobenius theory $\lambda_1 \geq |\lambda_n|$, *i.e.* λ_1 is the spectral radius, is also necessary. In case $n = 4$, these two conditions alone are necessary and sufficient for both the S-NIEP and R-NIEP (this is straightforward and may be found in [5], among other places). In addition, if $\lambda_1 = \lambda_2$, a "tie" for spectral radius, the matrix must be reducible, and the spectrum must be partitionable into (at least) 2 nonnegative spectra in lower dimensions. Another necessary condition, J–L–L [3,5], is based upon traces of powers:

$$\left(\sum_{i=1}^{n} \lambda_i^k\right)^m \leq n^{m-1} \sum_{i=1}^{n} \lambda_i^{km}, \quad k,\, m = 1, 2, \ldots. \tag{1}$$

Whenever there is just one positive eigenvalue, it is known that the trace condition is sufficient, as well as necessary for the S-NIEP [12,2,4]. When there are just 2 positive eigenvalues, it has been observed [8] that "partitioned majorization" is sufficient for the S-NIEP, *i.e.* if the nonpositive eigenvalues $\lambda_3 \geq \lambda_4 \geq \cdots \geq \lambda_n$ may be partitioned into 2 subsets such that the larger sum of the absolute values in one set is no more than λ_1 and $\lambda_1 + \lambda_2$ is at least $|\lambda_3| + \cdots + |\lambda_n|$. For $n \leq 5$, this condition is also necessary [5,7,6]. Note that for $n > 5$ this is not true is shown by the spectrum $7, 5, -1/2, -7/2, -4, -4$, which is symmetrically realizable [4, Example 8 with $\delta = 1/2$] but is not partitionable.

When $n = 5$, this leaves the unresolved cases for the S-NIEP (because the cases of 4 or 5 nonnegative eigenvalues are straightforward):

$$\begin{aligned} \lambda_1 > \lambda_2 \geq \lambda_3 > 0 > \lambda_4 \geq \lambda_5 \\ \lambda_1 + \lambda_5 \geq 0 \\ \sum_{i=1}^{5} \lambda_i > 0 \\ \text{and} \quad \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 < 0. \end{aligned}$$

The third inequality is strict, as [9] resolves all trace 0 cases and the last inequality may be assumed, as, otherwise, $\{\lambda_1, \lambda_2, \lambda_4, \lambda_5\}$ and $\{\lambda_3\}$ would be realizable. Thus far, none of these cases has been resolved, except those for which translation by $-\left(\frac{1}{5}\sum \lambda_i\right) I$ leads to a spectrum realizable according to [9]. Of course, some may also be ruled out by the J–L–L conditions, (1). An example that meets all known necessary conditions (and is realizable for the R-NIEP [11]) is

$$6, 3, 3, -5, -5.$$

We give here an argument, based upon the eigenvalue interlacing inequalities for symmetric matrices, that rules out this spectrum (among infinitely many others) and could be used to rule out others.

Consider the possible S-NIEP spectrum

$$1, a, a, -(a+d), -(a+d)$$

in which

$$0 < a, d$$
$$a+d, 2d < 1 < a+2d.$$

We assume $a + d < 1$ because of the Perron condition (and we may, as well, assume irreducibility), $2d < 1$ because of the positive trace condition, and $a + 2d > 1$, as, otherwise, realizability occurs reducibly.

This is a parametrically described subset of the unresolved spectra mentioned. We give conditions on a and d that prevent this spectrum from being S-NIEP realizable.

Suppose the above spectrum is realizable by a symmetric, nonnegative 5-by-5 matrix $A = (a_{ij})$, and let $A(i)$ denote its 4-by-4 principal submatrix resulting from deletion of row and column i. Then, by interlacing, $A(i)$ has spectrum

$$p_i \geq a \geq q_i \geq -(a+d),$$

in which p_i is the spectral radius of $A(i)$. By interlacing and Perron–Frobenius, we obtain

$$1 \geq p_i \geq a+d.$$

Since $\mathrm{Tr}(A(i)) \geq 0$ and $\mathrm{Tr}(A) > 0$

$$p_i + q_i - d \geq 0 \quad \text{and} \quad 1 - 2d > 0.$$

Since $a_{ii} \geq 0$ and $a_{ii} = \mathrm{Tr}(A) - \mathrm{Tr}(A(i))$, we have

$$1 - p_i - d - q_i \geq 0$$

from which we conclude

$$d - p_i \leq q_i \leq 1 - p_i - d \quad (\leq 1 - a - 2d). \tag{2}$$

It follows that each $q_i < 0$.

From the identity $4\,\mathrm{Tr}(A) = \sum_{i=1}^{5} \mathrm{Tr}(A(i))$, we obtain

$$4 - 3d = \sum_{i=1}^{5} p_i + \sum_{i=1}^{5} q_i \,. \tag{3}$$

Note that $4\,\mathrm{Tr}(A^3) \geq \sum_{i=1}^{5} \mathrm{Tr}(A(i)^3)$ because cubing in A only contributes more non-negative summands than cubing in $A(i)$, making the corresponding diagonal entries no less. By cubing the eigenvalues, and algebra, we obtain

$$4 + 3a^3 - 3(a+d)^3 \geq \sum_{i=1}^{5} p_i^3 + \sum_{i=1}^{5} q_i^3 \,. \tag{4}$$

With these constraints in the record our idea is to minimize

$$\sum_{i=1}^{5} p_i^3 + \sum_{i=1}^{5} q_i^3 \,.$$

If that minimum violates the above inequality (4), then we may conclude that the assumed A does not exist, ruling out the indicated spectrum. Fortunately, the necessary optimization can be carried out analytically. First, for a given total weight $\sum_{i=1}^{5} p_i$, the value $\sum_{i=1}^{5} p_i^3$ will be minimized when all p_i's are the same, since $p_i \geq 0$, for $i = 1, \ldots, 5$. No variation in the p_i's can help decrease $\sum_{i=1}^{5} q_i^3$. Since $p_i \in [a+d, 1]$, we may write

$$p_i = a + d + t \,, \quad t \in [0, 1 - a - d] \,.$$

Then from (3)

$$\sum_{i=1}^{5} q_i = 4 - 3d - 5a - 5d - 5t = 4(1 - 2d) - 5a - 5t \,, \tag{5}$$

and

$$\sum p_i^3 = 5(a + d + t)^3 \,.$$

Now, as each $q_i < 0$, the value $\sum_{i=1}^{5} q_i^3$ is minimized when, subject to the constraints on q_i, the q_i vary as much as possible, *i.e.* as many of them are as small as possible. We have from (2)

$$d - (a + d + t) \leq q_i \leq 1 - (a + d + t) - d$$

or

$$-(a + t) \leq q_i \leq 1 - 2d - (a + t)\,.$$

If we let $q_i = -(a + t) + s_i$, $0 \leq s_i \leq 1 - 2d$, from (5) we obtain $\sum_{i=1}^{5} s_i = 4(1 - 2d)$. Then, a minimizing allocation of the s_i weight, subject to constraints is

$$s_1 = 0, s_2 = s_3 = s_4 = s_5 = 1 - 2d\,.$$

This gives $\sum_{i=1}^{5} q_i^3 = -(a + t)^3 + 4(1 - 2d - a - t)^3$ and

$$\sum p_i^3 + \sum q_i^3 = 5(a + d + t)^3 - (a + t)^3 - 4(1 - 2d - a - t)^3\,.$$

Thus, we have a one variable (t) minimization problem in which the objective function turns out to be quadratic in t (parametrized by a and d)

$$\min_{0 \leq t \leq 1-(a+d)} 5(a + d + t)^3 - (a + t)^3 - 4(1 - 2d - a - t)^3\,.$$

The derivative of the objective function is

$$18\left(\frac{4}{3} - d\right) t + 3d^2 + 18\left(\frac{2}{3} - d\right)(a + 2d) + 12(a + 2d - 1)\,,$$

which is positive on the interval, so that the objective function is increasing as a function of t, and the minimum is always at the left hand end-point ($t = 0$). This gives

$$\sum_{i=1}^{5} p_i^3 + \sum_{i=1}^{5} q_i^3 \geq 5(a + d)^3 - a^3 - 4(a + 2d - 1)^3\,.$$

Comparing to the upper bound (4) gives a contradiction to the existence of a realizing matrix when

$$2(a + d)^3 > 1 + a^3 + (a + 2d - 1)^3\,.$$

This gives our main result.

Theorem 1. *Let $0 < a, d$ satisfy $a + d, 2d < 1 < a + 2d$. If $2(a+d)^3 \geq 1 + a^3 + (a + 2d - 1)^3$, then $1, a, a, -(a + d), -(a + d)$ are not the eigenvalues of a 5-by-5 symmetric nonnegative matrix.*

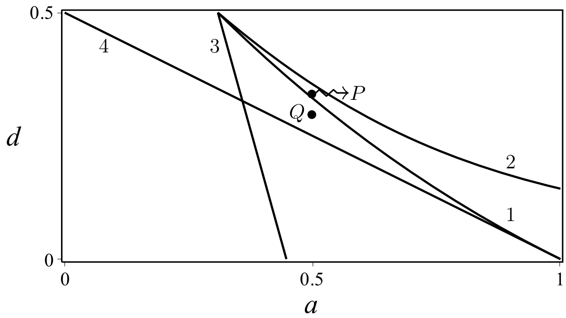

Fig. 1. Curve 1 $\equiv 2(a+d)^3 - a^3 - (a+2d-1)^3 = 1$, curve 2 $\equiv 50(a+d)^3 + (1-2d)^2 - 50a^3 = 25$, line 3 $\equiv 10a - (\sqrt{5}-5)d = 2\sqrt{5}$, line 4 $\equiv a + 2d = 1$, $P = (1/2, 1/3)$ and $Q = (1/2, 7/24)$.

Example 2. The mentioned spectrum $6, 3, 3, -5, -5$ corresponds to $a = \frac{1}{2}$ and $d = \frac{1}{3}$, in which case

$$2\left(\frac{5}{6}\right)^3 \geq 1 + \frac{1}{8} + \frac{1}{216}$$

and the condition of the theorem is satisfied. So this spectrum is not realizable.

In a, d-space, Fig. 1 depicts what may be said about realizability as a result of this (and other) work.

The slightly curved line 1 is the exclusionary curve given by Theorem 1. No spectrum corresponding to points above or on it is realizable for the 5-by-5 S-NIEP. The point $P = (1/2, 1/3)$, corresponds to the spectrum $6, 3, 3, -5, -5$, which is not S-NIEP realizable, though it is R-NIEP realizable. Curve 2 is an exclusionary boundary deduced from a cubic J–L–L condition. Other J–L–L conditions were also investigated. Though they tended to rule out somewhat different regions, all were comfortably inside our excluded region, and the one we have depicted ruled out perhaps the most. Of course, these curves rule out R-NIEP spectra (as well as S-NIEP), so that a gap is to be expected.

We also depicted the inclusionary line 3 of points corresponding to constant diagonal S-NIEP realizable spectra, deduced from the result of [9]. The equation of this line in $[0, 1] \times [0, 1/2]$ is

$$(\sqrt{5} - 5)d - 10a + 2\sqrt{5} = 0.$$

It happens to coincide with the line indicating nonnegative symmetric circulant realizability (on and to the left) which may be deduced from [10, Lemma 1], or directly. The line 4 is the line (on and below which) indicating realizability due to the fact that 2 of the positive and the two negative eigenvalues are 4-by-4 S-NIEP realizable. Its equation is $a + 2d = 1$. So all points on and below this line are (trivially) symmetrically realizable.

This leaves a small, nearly triangular region of unresolved spectra in the middle. For example, the point $Q = (1/2, 7/24)$, corresponding to the spectrum $\{24, 12, 12, -19, -19\}$

lies in this region. We tried all known methods of realization, as well as a number of *ad hoc* methods, to realize a spectrum in this region, without success. We especially investigated just east of line 3, above line 4, as well as just above line 4 to the right of line 3. It is likely that the true exclusionary boundary is to the west of our curve 1. We would be quite interested if someone could realize a spectrum in the interior of this triangle, such as the point Q, especially if it displayed some interesting structure. The points to the west of (and on) line 3, above line 4, correspond to the only spectra of our type, that we know of, for which the trace is less than a (*i.e.* the two larger positive eigenvalues, plus the negative ones, are less than 0).

We note that the method used to prove the main theorem involved the cubes of the eigenvalues of the principal submatrices. In fact any other positive integer power might be tried (and is no more difficult to carry out). It is worth mentioning that even powers, while they do rule out a portion of the (a, d) pairs and do improve, as they increase, subject to an asymptote, are never competitive with the cube. On the other hand, after the third power, odd powers rule out less. So the cubic argument that we used produces the strongest exclusionary result. That said, we suspect that spectra inside the unresolved triangle, but near to the exclusionary boundary given by our result, are also not realizable. This is because there is a positive gap between the two sides of the trace inequality we used (display (4), and above). This also means that points along the exclusionary curve are also not realizable.

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