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Comparison on the spectral radii of weighted digraphs that differ in a certain subdigraph

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Abstract

Let D_S be a weighted digraph of order n with a subdigraph S of order k, $M(D_S)$ its adjacency weight matrix and $\rho(D_S)$ its spectral radius. We consider the class $\mathcal{C}_k$ of weighted digraphs of order k and we study the preorder in $\mathcal{C}_k$ given by $D_{S'} \precsim D_S$ if and only if $\rho(D_{S'}) \leq \rho(D_S)$. We obtain that this order is equivalent to the entry-wise order $M(D_{S'}) \leq M(D_S)$. Several points of view are taken, under varying regularity conditions, and k polynomial conditions for the comparison are presented.

Keywords: Weighted digraphs, Induced subdigraphs, Comparing spectral radii.

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It is well known that if the weight of an arc of a weighted digraph D is increased, then the spectral radius $\rho(D)$, *i.e.* the maximal absolute value of the eigenvalues of D, does not decrease, and it increases when D is strongly connected. Suppose now that instead of changing the weight of an arc we change the weight function of a subdigraph S of $D = D_S$. It is easy to see that a larger spectral radius of the changed digraph S' can result in a smaller spectral radius of the full digraph $D_{S'}$. We question under which conditions if $\rho(S') \leq \rho(S)$ we have $\rho(D_{S'}) \leq \rho(D_S)$. We denote by $M(D)$ the adjacency (weight) matrix of D. Let $\mathcal{C}_k$ be the class of weighted digraphs of order k and let $\precsim$ be the preorder in $\mathcal{C}_k$ given by

$$S' \precsim S \iff \rho(D_{S'}) \leq \rho(D_S)\,.$$

We write $S' \prec S$ if $\rho(D_{S'}) < \rho(D_S)$.

Our main result is that this order and the domination order between the corresponding adjacency matrices are equivalent.

Theorem 1 *If $S', S \in \mathcal{C}_k$, then $S' \precsim S$ if and only if $M(S') \leq M(S)$.*

Of course, if $M(S') \leq M(S)$, then $S' \precsim S$. In the remainder of this work, we present the necessary analytical results about the asymptotic behaviour of the spectral radius (which may be of independent interest) that are necessary to prove the reverse implication, as well as polynomial conditions for the comparison.

Let $S', S \in \mathcal{C}_k$ with adjacency matrices

$$M(S') = \begin{pmatrix} A_{11} & a_{12} \\ a_{21}^T & a_{22} \end{pmatrix} \qquad \text{and} \qquad M(S) = \begin{pmatrix} B_{11} & b_{12} \\ b_{21}^T & b_{22} \end{pmatrix},$$

where A_{11} and B_{11} are $(k-1)$-by-$(k-1)$, a_{12}, a_{21}, b_{12} and b_{21} are $(k-1)$-by-1, and a_{22} and b_{22} are scalars. Let x and y be nonnegative scalars and $D_{S'}$ and D_S the weighted digraphs with adjacency matrices

$$F(x,y) = \begin{pmatrix} A_{11} & a_{12} & 0 \\ a_{21}^T & a_{22} & x \\ 0 & y & 0 \end{pmatrix} \qquad \text{and} \qquad G(x,y) = \begin{pmatrix} B_{11} & b_{12} & 0 \\ b_{21}^T & b_{22} & x \\ 0 & y & 0 \end{pmatrix}.$$

Our goal is to first show that for a weighted digraph of the form $D_{S'}$, the spectral radius is arbitrarily approximated by that of the digraph corresponding to

$$\begin{pmatrix} a_{22} & x \\ y & 0 \end{pmatrix}$$

for sufficiently large x and y. Given the explicit value of the latter spectral radius, this will mean that $\rho(D_{S'}) < \rho(D_S)$ whenever $a_{22} < b_{22}$ and x and y are sufficiently large. The spectral radius of the matrix $F(x, y)$ is eventually (as x and y increase) approximated by that of its lower right 2-by-2 submatrix.

Theorem 2 *For any $\epsilon > 0$, there are numbers $X, Y > 0$ such that for all $x \geq X$ and $y \geq Y$, we have*

$$0 \leq \rho(D_{S'}) - \rho\left(\begin{bmatrix} a_{22} & x \\ y & 0 \end{bmatrix}\right) \leq \epsilon\ .$$

Now, if we consider the digraphs $D_{S'}$ and D_S in terms of the relationship between the weights a_{22} and b_{22}, importantly the spectral radii will eventually (as x and y grow) follow the relationship between a_{22} and b_{22} irrespective of the relative sizes of the remaining weights a_{ij} , b_{ij}.

Corollary 3 *If $a_{22} < b_{22}$, then there are numbers $X, Y > 0$ such that for all $x \geq X$ and $y \geq Y$, we have $\rho(D_{S'}) < \rho(D_S)$.*

Applying Corollary 3 to our order, we may conclude:

Corollary 4 *Let $S', S \in \mathcal{C}_k$. If $S' \precsim S$, then $a_{ii} \leq b_{ii}$, for $1 \leq i \leq k$.*

Now we consider $S', S \in \mathcal{C}_k$ with adjacency matrices

$$M(S') = \left(\begin{array}{c|cc} A_{11} & \multicolumn{2}{c}{A_{12}} \\ \hline A_{21}^T & \begin{array}{cc} a_{22} & a_{23} \\ a_{32} & a_{33} \end{array} \end{array}\right) \qquad \text{and} \qquad M(S) = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21}^T & \begin{array}{cc} b_{22} & b_{23} \\ b_{32} & b_{33} \end{array} \end{array}\right),$$

where A_{11} and B_{11} are $(k-2)$-by-$(k-2)$, A_{12}, A_{21}, B_{12} and B_{21} are $(k-2)$-by-2 and a_{ij}, b_{ij} with $i, j \in \{2, 3\}$ are scalars. Let x and y be nonnegative scalars, and $D_{S'}$ and D_S the weighted digraphs with adjacency matrices

$$F(x,y) = \left(\begin{array}{c|cc|c} A_{11} & \multicolumn{2}{c|}{A_{12}} & 0 \\ \hline A_{21}^T & \begin{array}{cc} a_{22} & a_{23} \\ a_{32} & a_{33} \end{array} & \begin{array}{c} 0 \\ x \end{array} \\ \hline 0 & y \;\; 0 & 0 \end{array}\right) \qquad \text{and} \qquad G(x,y) = \left(\begin{array}{c|c|c} B_{11} & B_{12} & 0 \\ \hline B_{21}^T & \begin{array}{cc} b_{22} & b_{23} \\ b_{32} & b_{33} \end{array} & \begin{array}{c} 0 \\ x \end{array} \\ \hline 0 & y \;\; 0 & 0 \end{array}\right).$$

In this case, $a_{23} < b_{23}$ if and only if $\rho(D_{S'}) < \rho(D_S)$ for sufficiently large x and y, irrespective of the values of other weights besides x and y.

Theorem 5 *Assume $a_{23}, b_{23} > 0$. Then*

i) For any given $\delta > 0$, there is a constant C and numbers $X, Y > 0$ such that

$$\rho(D_{S'}) \leq C + \left(a_{23}^{1/3} + \delta\right)(xy)^{1/3}, \quad \forall x > X, \forall y > Y.$$

ii) There are numbers $X, Y > 0$ such that

$$\rho(D_S) \geq b_{23}^{1/3}(xy)^{1/3}, \quad \forall x > X, \forall y > Y.$$

iii) If $a_{23} < b_{23}$, then there exist numbers $X, Y > 0$ such that

$$\rho(D_{S'}) < \rho(D_S), \quad \forall x > X, \forall y > Y.$$

The next corollary completes the proof of Theorem 1.

Corollary 6 *Let $S', S \in \mathcal{C}_k$. If $S' \precsim S$, then $a_{ij} \leq b_{ij}$, for $i \neq j$.*

In general, it is difficult to compare the spectral radii of two given weighted digraphs of order n. Here, we study a special case in which both digraphs differ only in a certain subdigraph, which, without loss of generality, we take to be the one corresponding to the first k vertices. Let S' be the induced subdigraph of a weighted digraph $D = D_{S'}$ in its first k vertices, and let

$$M(S') = A \qquad \text{and} \qquad M(D_{S'}) = G(A) = \begin{pmatrix} A & G_{12} \\ G_{21} & G_{22} \end{pmatrix}.$$

We want to compare $\rho(D_{S'})$ and $\rho(D_S)$, and, in particular, to describe the set of $S' \in \mathcal{C}_k$ such that $\rho(D_{S'}) < \rho(D_S)$, for a fixed $S \in \mathcal{C}_k$.

If $G \geq 0$ is an n-by-n matrix, and a scalar $\rho > 0$ is given, by M-matrix theory, $\rho(G) < \rho$ if and only if $\rho I - G$ is an M-matrix. (When we say M-matrix, we mean a nonsingular M-matrix, otherwise we explicitly refer to a singular M-matrix). Now, because of the determinantal characterization of M-matrices among the Z-matrices (nonpositive off-diagonal entries), $\rho(G) < \rho$ if and only if any nested sequence of n principal minors (PMs) of $\rho I - G$ is positive. And, if G is irreducible, $\rho(G) = \rho$ if and only if any nested sequence of n PMs of $\rho I - G$ has sign sequence $+, +, \ldots, +, 0$. Thus, given a fixed subdigraph $S \in \mathcal{C}_k$ of the weighted strongly connected digraph D_S, with adjacency matrices $M(S) = B$ and $M(D_S) = G(B)$, the inequality $\rho(D_{S'}) < \rho(D_S)$ may be checked via k polynomial inequalities in the weights of the arcs of S'. The polynomials may be taken to be the last k trailing PMs of $\rho I - G(A)$, for $\rho = \rho(D_S)$, as the first $n - k$ trailing PMs of $\rho I - G(A)$ are the same as those of $\rho I - G(B)$, which are positive. According to Theorem

1, $S' \precsim S$ for all weighted digraph D if and only if $M(S') \leq M(S)$ in the entry-wise partial order, *i.e.*, this order is independent of the digraph D.

We may record these observations as one solution to our problem. We denote by $q_i(A)$ the $n-k+i^{th}$ trailing PM of $\rho(D_S)I - G(A)$, viewed as a polynomial in the weights of the arcs of S'.

Theorem 7 *Let $S', S \in \mathcal{C}_k$ and assume that D_S is strongly connected. Then*

i) $S' \prec S$ *if and only if* $q_i(A) > 0$, *for* $1 \leq i \leq k$;

ii) $S \precsim S'$ *if and only if there is* $i \in \{1, \ldots, k\}$ *such that* $q_i(A) \leq 0$;

iii) when $D_{S'}$ *is strongly connected, we have* $\rho(D_{S'}) = \rho(D_S)$ *if and only if* $q_i(A) > 0$, *for* $1 \leq i \leq k-1$, *and* $q_k(A) = 0$.

If $N_k \in \mathcal{C}_k$ is the null graph and D_{N_k} is strongly connected, so is $D_{S'}$ for any $S' \in \mathcal{C}_k$. We then have Corollary 8 as a consequence of Theorem 7.

Corollary 8 *Suppose D_{N_k} is strongly connected and $S', S \in \mathcal{C}_k$. Then*

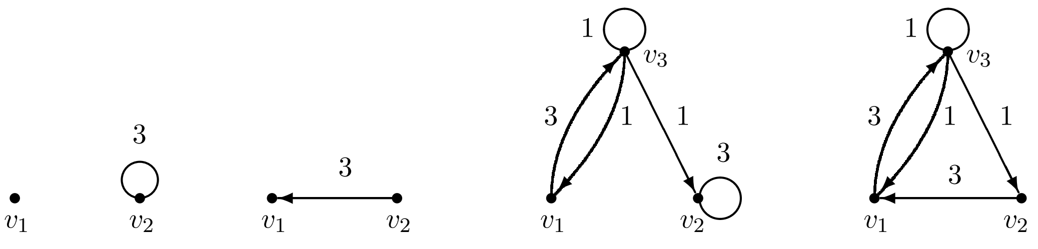
i) $S' \prec S$ *if and only if* $q_i(A) > 0$, *for* $1 \leq i \leq k$;

ii) $S' \precsim S$ *if and only if there is* $i \in \{1, \ldots, k\}$ *such that* $q_i(A) \leq 0$;

iii) $\rho(D_{S'}) = \rho(D_S)$ *if and only if* $q_i(A) > 0$, *for* $1 \leq i \leq k-1$, *and* $q_k(A) = 0$.

We note that the assumption in *iii)* of Theorem 7 that $D_{S'}$ is strongly connected is necessary and is not implied by the strong connectivity of D_S, as the next example shows:

Example 9 *Let $S', S \in \mathcal{C}_2$ and $D_{S'}$ and D_S be the weighted digraphs*



whose adjacency matrices are

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, \; G(A) = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix} \; \textit{and} \; G(B) = \begin{pmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Note that the digraph D_S is strongly connected while $D_{S'}$ is not. We have $\rho(D_{S'}) = \rho(D_S) = 3$; *however,* $q_1(A) = \det \begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix} = 0$.

We now turn to an alternate approach to the problem.

Theorem 10 *Let $S', S \in \mathcal{C}_k$ and assume that $\rho = \rho(D_S)$. Then the following statements are equivalent:*

i) $S' \prec S$,

ii) $\rho I - G(A)$ *is an* M*-matrix,*

iii) $\det(\rho I - G_{22}) \neq 0$ *and* $\rho(A + G_{12}(\rho I - G_{22})^{-1} G_{21}) < \rho$, *and*

iv) $\det(\rho I - G_{22}) \neq 0$ *and* $\rho I - (A + G_{12}(\rho I - G_{22})^{-1} G_{21})$ *is an* M*-matrix.*

We may now characterize $S' \in \mathcal{C}_k$ such that $\rho(D_{S'}) = \rho(D_S)$ for a fixed S, when $D_{S'}$ is strongly connected.

In what follows $H(1)$ denotes the matrix obtained from H by deleting the first row and the first column. Also, E_{11} denotes the matrix of appropriate size with all entries equal to 0 except the one in position $(1,1)$, which is 1.

Theorem 11 *Let $S', S \in \mathcal{C}_k$ with $M(S') = A = (a_{ij})$, and $D_{S'}$ strongly connected. Let $\rho = \rho(D_S)$, $H = G(A)$ and $H' = H - a_{11} E_{11}$. Then $\rho(H) \geq \rho$ if and only if*

i) $\rho(H') \leq \rho$, *and ii)* $\det(H(1) - \rho I) \neq 0$ *and* $a_{11} = -\dfrac{\det(H' - \rho I)}{\det(H(1) - \rho I)}$.

We may now characterize the $S' \in \mathcal{C}_k$ such that $\rho(D_{S'}) \geq \rho(D_S)$ for a fixed S, when $D_{S'}$ is strongly connected.

Corollary 12 *Let $S', S \in \mathcal{C}_k$ with $M(S') = A = (a_{ij})$, and $D_{S'}$ strongly connected. Let $\rho = \rho(D_S)$, $H = G(A)$ and $H' = H - a_{11} E_{11}$. Then $\rho(H) \geq \rho$ if and only if*

i) $\rho(H') > \rho$ *or*

i') $\rho(H') \leq \rho$ *and ii')* $\det(H(1) - \rho I) \neq 0$ *and* $a_{11} \geq -\dfrac{\det(H' - \rho I)}{\det(H(1) - \rho I)}$.

The previous results, with their proofs, appear in [1,2] in the context of nonnegative matrices.

References

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