

Classification of numerical 3-semigroups by means of L-shapes

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Abstract We recall L-shapes, which are minimal distance diagrams, related to weighted 2-Cayley digraphs, and we give the number and the relation between minimal distance diagrams related to the same digraph. On the other hand, we consider some classes of numerical semigroups useful in the study of curve singularity. Then, we associate L-shapes to each numerical 3-semigroup and we describe some main invariants of numerical 3-semigroups in terms of their associated L-shapes. Finally, we give a characterization of the parameters of the L-shapes associated with a numerical 3-semigroup in terms of its generators, and we use it to classify the numerical 3-semigroups of interest in curve singularity.

Keywords Numerical semigroup · Symmetric · Free · Curve plane semigroup · Cayley digraph · L-shape · Frobenius number

1 Introduction

We are interested in numerical semigroups useful in the study of curve singularity, which receive the name of the corresponding classes of associated curves. Combinatorial objects, as simplicial complexes [3], have been introduced to study some

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geometrical properties of these numerical semigroups. But the particular numerical semigroups dealt with here will be introduced arithmetically and they will be considered as pure arithmetical objects. From now on, a numerical n -semigroup denotes a numerical semigroup generated by n generators.

The 2-Cayley digraphs have been widely used to study metrical applications of local area networks. Their generalization to weighted 2-Cayley digraphs, adding weights to the arcs, allows other applications to be studied. Looking for paths of minimum length in these digraphs, periodical plane tessellations with L-shaped tiles appear in the bibliography [5, 6]. Particular classes of L-shapes, the so called *minimal distance diagrams*, have been used to study certain distance properties in 2-Cayley digraphs, and have also been associated with numerical 3-semigroups [1, 9].

These particular L-shapes associated to a numerical 3-semigroup contain relevant information about the semigroup and, in fact, some main invariants of the semigroup, such as Apéry sets, Frobenius number, set of gaps, ... can be described in terms of the associated L-shapes.

Our goal in this work is to show the close relation between numerical 3-semigroups, which are interesting in the mentioned geometric context, and their associated L-shapes, in such a way that we can describe problems in a numerical 3-semigroup in terms of the associated L-shapes, and vice-versa.

Our last objective is to understand in depth the relations between semigroups and L-shapes to try to understand some complex problems in geometry by means of these combinatoric tools, which are not so complex as those used in geometry [3]. In particular, we try to understand *complete intersection numerical n -semigroups* [4, 8] which, for $n \geq 4$, are distinguished from symmetric semigroups. A thorough knowledge of the case $n = 3$ will allow us to approach the case $n = 4$.

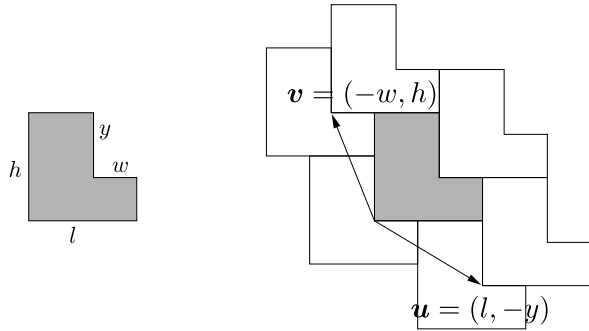
In Sect. 2, we introduce L-shapes related to weighted 2-Cayley digraphs and minimum distance diagrams in this context. In Sect. 3, we study the number and the relation between the minimum distance diagrams related to the same digraph. In Sect. 4, we introduce numerical semigroups interesting in the study of curve singularity and we describe some invariants of numerical 3-semigroups in terms of their associated L-shapes. Finally, in Sect. 5 we characterize these numerical 3-semigroups in terms of the number and the type of their associated L-shapes.

2 L-shaped minimum distance diagrams

In this section we introduce L-shapes related to certain weighted 2-Cayley digraphs and particular cases of them which are also minimum distance diagrams. These geometrical objects will be used to obtain the main information about their related semigroups.

For integers $g, n \in \mathbb{N}$, let us denote the equivalence class of g modulo n by $[g]_n$. Given integers $a, b, c \in \mathbb{N}$, $1 \leq a < b < c$, $\gcd(a, b, c) = 1$ and weights $W_a, W_b \in \mathbb{R}^+$, a *weighted 2-Cayley digraph* $G(c; a, b; W_a, W_b) = \text{Cay}(\mathbb{Z}_c; a, b; W_a, W_b)$ is a directed graph with sets of vertices $V(G) = \mathbb{Z}_c$ and arcs $A(G) = \{[g]_c \xrightarrow{W_a} [g + a]_c, [g]_c \xrightarrow{W_b} [g + b]_c : g = 0, \dots, c - 1\}$, where $\xrightarrow{W_a}$ and $\xrightarrow{W_b}$ stand for

Fig. 1 L-shape $L(l, h, w, y)$ and the related plane tessellation



weighted arcs with related weights W_a and W_b , respectively. In this context, we consider \mathbb{Z}_c generated by the set $\{a, b\}$. We denote the distance from vertex $[g]_c$ to vertex $[h]_c$ in G by $d_G([g]_c, [h]_c)$ (the weight of a minimal weighted path from $[g]_c$ to $[h]_c$). From now on, we simplify “weighted 2-Cayley digraph $G(c; a, b; W_a, W_b)$ ” to “digraph $G(c; a, b; W_a, W_b)$ ”. Metrical properties of these digraphs have been widely studied using minimum distance diagrams [5, 6, 10] that will be defined later.

Let us consider unit squares $\llbracket i, j \rrbracket = [i, j] \times [i + 1, j + 1] \subset \mathbb{R}^2$ with integer non negative coordinates $(i, j) \in \mathbb{N}^2$. Each square $\llbracket i, j \rrbracket$ is related to the equivalence class $[ia + jb]_c$, and we denote the weight of the unit square $\llbracket i, j \rrbracket$ by $\delta(i, j) = iW_a + jW_b$. Given any integer $0 \leq n < c$, let us consider the set of unit squares in the first quadrant related to $[n]_c$

$$Q_n = \{\llbracket i, j \rrbracket : [ia + ib]_c = [n]_c, 0 \leq i, j\},$$

and the weight of $[n]_c$

$$M_n = \min\{\delta(i, j) : \llbracket i, j \rrbracket \in Q_n\} = d_G([0]_c, [n]_c).$$

As it is well known, an L-shape related to the digraph $G(c; a, b; W_a, W_b)$ consists of c unit squares related to the equivalence classes $[n]_c, 0 \leq n < c$. L-shapes are denoted by the lengths of their sides, $L(l, h, w, y)$, with $0 \leq w < l$ and $0 \leq y < h$ as in the left hand side of Fig. 1. Rectangles are considered degenerated L-shapes, i.e., L-shapes with $wy = 0$. Conditions

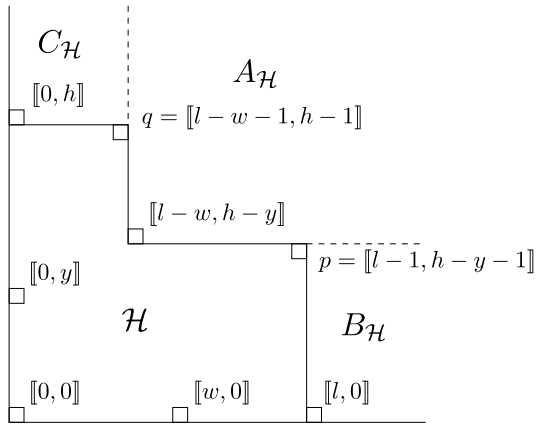
$$lh - wy = c \quad \text{and} \quad \gcd(l, h, w, y) = 1 \tag{1}$$

are necessary for an L-shape $L(l, h, w, y)$ to be related to a given digraph $G(c; a, b; W_a, W_b)$. It is also well known that an L-shape $L(l, h, w, y)$ periodically tessellates the plane by translation through the vectors $\mathbf{u} = (l, -y)$ and $\mathbf{v} = (-w, h)$ (right hand side of Fig. 1). In terms of equivalence classes, this fact results in $[ia + jb]_c = [(i + l)a + (j - y)b]_c$ and $[ia + jb]_c = [(i - w)a + (j + h)b]_c$, that is

$$la \equiv yb \pmod{c} \quad \text{and} \quad hb \equiv wa \pmod{c}. \tag{2}$$

Fiol *et al.* [6] showed that an L-shape $L(l, h, w, y)$ is related to the digraph $G(c; a, b; W_a, W_b)$ if and only if (1) and (2) hold.

Fig. 2 Regions $A_{\mathcal{H}}, B_{\mathcal{H}}, C_{\mathcal{H}}$ and some highlighted unit squares



Let $\Delta(i, j)$ be the set of unit squares in the first quadrant dominated by $[[i, j]]$, i.e.

$$\Delta(i, j) = \{[[u, v]] : 0 \leq u \leq i, 0 \leq v \leq j\}.$$

Definition 1 An L-shape related to $G(c; a, b; W_a, W_b)$, L , is a *minimum distance diagram* (MDD for short) if it satisfies the following two properties:

- (a) For each $[[s, t]] \in L$, identity $\delta(s, t) = M_{sa+tb}$ holds.
- (b) If $[[s, t]] \in L$ then $\Delta(s, t) \subset L$.

The following result is a geometrical characterization of minimum distance diagrams.

Theorem 1 ([2]) *Let $\mathcal{H} = L(l, h, w, y)$ be an L-shape related to the digraph $G = G(c; a, b; W_a, W_b)$. Then \mathcal{H} is an MDD related to G if and only if $lW_a \geq yW_b$, $hW_b \geq wW_a$ and both equalities do not hold at the same time.*

3 Properties of MDDs related to a weighed 2-Cayley digraph

In this section we study geometrical properties of MDDs related to the same weighed 2-Cayley digraph. In particular, we give their cardinal and geometrical links between them.

From now on, given an L-shape $\mathcal{H} = L(l, h, w, y)$, we assume the left lower square of \mathcal{H} is $[[0, 0]]$ by default. Set $\nabla(i, j) = \{[[s, t]] : s \geq i, t \geq j\}$, $A_{\mathcal{H}} = \nabla(l-w, h-y)$, $B_{\mathcal{H}} = \nabla(l, 0) \setminus \nabla(l, h-y)$ and $C_{\mathcal{H}} = \nabla(0, h) \setminus \nabla(l-w, h)$. Note the disjoint union $\mathbb{N}^2 = \mathcal{H} \cup A_{\mathcal{H}} \cup B_{\mathcal{H}} \cup C_{\mathcal{H}}$ as it is depicted in Fig. 2.

We use the notation $[[i, j]] \sim [[m, n]]$ whenever $[ia + jb]_c = [ma + nb]_c$.

Lemma 1 *Let $\mathcal{H} = L(l, h, w, y)$ be an MDD related to a digraph $G = G(c; a, b; W_a, W_b)$. Then,*

- (a) if $\llbracket i, j \rrbracket \sim \llbracket s, t \rrbracket$ with $\llbracket i, j \rrbracket \in \mathcal{H}$ and $\delta(i, j) < \delta(s, t)$, it follows that $\delta(m, n) > d_G(\llbracket 0 \rrbracket_c, \llbracket ma + nb \rrbracket_c)$ for all unit squares $\llbracket m, n \rrbracket \in \nabla(s, t)$,
- (b) no other MDD related to G intersects region $A_{\mathcal{H}}$,
- (c) if $lW_a > yW_b$, no other MDD related to G intersects region $B_{\mathcal{H}}$,
- (d) if $hW_b > wW_a$, no other MDD related to G intersects region $C_{\mathcal{H}}$,
- (e) if $lW_a = yW_b$, then $hW_b > wW_a$,
- (f) if $hW_b = wW_a$, then $lW_a > yW_b$.

Proof Item (a) is a consequence of the fact that any subpath of a minimum path in G is also a minimum path.

Item (b) comes from the equivalence $\llbracket 0, 0 \rrbracket \sim \llbracket l - w, h - y \rrbracket$, i.e. by conditions (2). Thus, applying (a), no unit square $\llbracket s, t \rrbracket \in A_{\mathcal{H}}$ satisfies minimality condition $\delta(i, j) = M_{sa+tb}$ and so, it can not belong to any MDD related to G .

Consider the unit squares $\llbracket 0, y \rrbracket \in \mathcal{H}$ and $\llbracket l, 0 \rrbracket \notin \mathcal{H}$. From the equivalence $\llbracket 0, y \rrbracket \sim \llbracket l, 0 \rrbracket$ (by (2)), we can apply again item (a) to prove (c). The same argument proves (d) by taking unit squares $\llbracket w, 0 \rrbracket \in \mathcal{H}$ and $\llbracket 0, h \rrbracket \notin \mathcal{H}$.

For (e) and (f), we have $hW_b > yW_b = lW_a > wW_a$ and $lW_a > wW_a = hW_b > yW_b$.

Note that if $w = 0$, then $hW_b \neq wW_a$, and if $y = 0$, then $lW_a \neq yW_b$.

Given any MDD related to the digraph $G(c; a, b; W_a, W_b)$, following Theorem 1 and items (e) and (f) of Lemma 1, we discuss the three possible cases: either $lW_a = yW_b$ or $hW_b = wW_a$ or $(lW_a - yW_b)(hW_b - wW_a) > 0$.

Theorem 2 *Let $\mathcal{H} = L(l, h, w, y)$ be an MDD related to a digraph $G = G(c; a, b; W_a, W_b)$. If $(lW_a - yW_b)(hW_b - wW_a) > 0$, then there is no other MDD related to G .*

Proof From Lemma 1(b), another MDD related to G , \mathcal{H}' , only intersects region $B_{\mathcal{H}} \cup C_{\mathcal{H}} \cup \mathcal{H}'$. From Lemma 1(c) and (d), the MDD \mathcal{H}' only intersects \mathcal{H} . As \mathcal{H} and \mathcal{H}' have the same area, it follows that $\mathcal{H} = \mathcal{H}'$. □

Theorem 3 *Let $\mathcal{H} = L(l, h, w, y)$ be an MDD related to a digraph $G = G(c; a, b; W_a, W_b)$. If $hW_b = wW_a$, then there is another MDD \mathcal{H}' related to G given by*

$$\mathcal{H}' = \begin{cases} L(w, 2h - y, 2w - l, h) & l < 2w, \\ L(w, (\lfloor l/w \rfloor + 1)h - y, w - r, h) & l > 2w > 0, l = \lfloor l/w \rfloor w + r, 0 < r < w, \\ L(w, lh/w - y, 0, h) & l \geq 2w > 0, l = \lfloor l/w \rfloor w. \end{cases}$$

Proof Equivalence between unit squares, defined by the compatibility equations (2), leads to the idea of equivalence between regions of the plane (those regions containing clusters of equivalent unit squares). This idea allows us to think of the L-shaped MDD \mathcal{H}' as a recomposition of pieces of \mathcal{H} , like a puzzle.

If $hW_b = wW_a$, from Lemma 1, another MDD related to G , $\mathcal{H}' = L(l', h', w', y') \neq \mathcal{H}$, intersects the region $C_{\mathcal{H}}$, that is $\mathcal{H}' \not\subset \mathcal{H}$ and $\mathcal{H}' \subset \mathcal{H} \cup C_{\mathcal{H}}$.

Note that $w \neq 0$. We consider the two cases, either $l < 2w$ or $l \geq 2w > 0$.

Case $l < 2w$: Fig. 3 shows which piece of \mathcal{H} has to be considered to obtain \mathcal{H}' . The two shaded pieces are equivalent regions, in the sense mentioned above, and they

Fig. 3 Case $l < 2w$. Diagrams \mathcal{H} and \mathcal{H}' , respectively

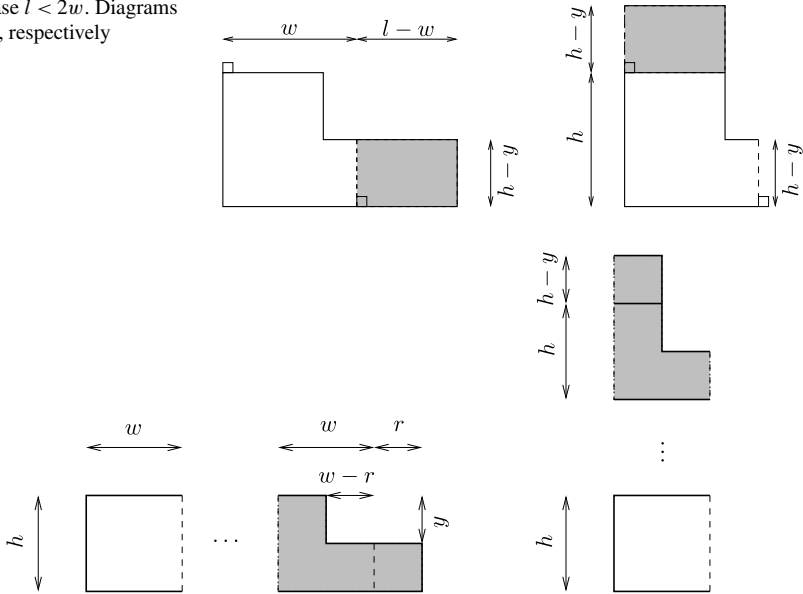


Fig. 4 Case $l > 2w > 0$ and $0 < r < w$. \mathcal{H} is divided into $\lfloor l/w \rfloor + 1$ pieces

follow the distribution given by the vectors $u = (l, -y)$ and $v = (-w, h)$ from (2). It follows that $l' = w, h' = 2h - y, w' = 2w - l$ and $y' = h$. So $\mathcal{H}' = L(w, 2h - y, 2w - l, h)$.

We check now that \mathcal{H}' fulfills compatibility conditions (1) and (2), and Theorem 1. Conditions (1): clearly $l'h' - w'y' = c$, and $\gcd(l', h', w', y') = \gcd(w, -y, 2w - l, h) = \gcd(w, y, l, h) = 1$.

Conditions (2) are also clear since $l'a \equiv_c y'b \iff wa \equiv_c hb$ and $w'a \equiv_c h'b \iff (2w - l)a \equiv_c (2h - y)b \iff la \equiv_c yb$.

Theorem 1: the first inequality $l'W_a \geq y'W_b \iff wW_a \geq hW_b$ is true because $hW_b = wW_a$; the second one is also true because $h'W_b \geq w'W_a \iff (2h - y)W_b \geq (2w - l)W_a \iff lW_a \geq yW_b$.

Case $l \geq 2w > 0$: Set $l = \lfloor l/w \rfloor w + r$, with $0 \leq r < w$.

In the case $0 < r < w$, see Fig. 4, it follows that $l' = w, h' = \lfloor l/w \rfloor h + (h - y), w' = w - r$ and $y' = h$. So $\mathcal{H}' = L(w, (\lfloor l/w \rfloor + 1)h - y, w - r, h)$.

Conditions (1):

$$\begin{aligned} l'h' - w'y' &= w[(\lfloor l/w \rfloor + 1)h - y] - (w - r)h \\ &= (w\lfloor l/w \rfloor + r)h - wy = lh - wy = c, \end{aligned}$$

$$\gcd(l', h', w', y') = \gcd(w, (\lfloor l/w \rfloor + 1)h - y, w - r, h) = \gcd(w, -y, -l, h) = 1.$$

Conditions (2): We have $l'a \equiv_c y'b \iff wa \equiv_c hb$, and

$$\begin{aligned} w'a \equiv_c h'b &\iff (w - r)a \equiv_c [(\lfloor l/w \rfloor + 1)h - y]b \iff yb \equiv_c \lfloor l/w \rfloor hb + ra, \\ &\text{since } wa \equiv_c hb \end{aligned}$$

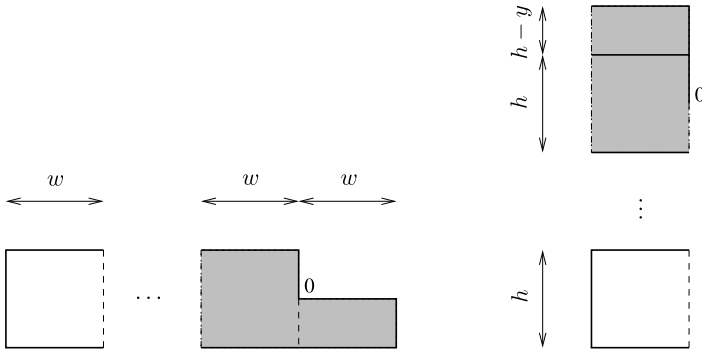


Fig. 5 Case $l \geq 2w > 0$ and $r = 0$. \mathcal{H} is divided into l/w pieces

$$\iff wa \equiv_c hb, \quad \text{since } yb \equiv_c la = (\lfloor l/w \rfloor w + r)a$$

Theorem 1: We have $l'W_a \geq y'W_b \iff wW_a \geq hW_b$. The second inequality is also true because $h'W_b \geq w'W_a \iff [(\lfloor l/w \rfloor + 1)h - y]W_b \geq (w - r)W_a \iff (\lfloor l/w \rfloor h - y)W_b + rW_a \geq 0$, since $\lfloor l/w \rfloor \geq 2$.

In the case $r = 0$, see Fig. 5, we have $l' = w, h' = (l/w - 1)h + (h - y) = lh/w - y, w' = 0$ and $y' = h$. So, $\mathcal{H}' = L(w, lh/w - y, 0, h)$ (note the position of 0 in the figure).

Conditions (1): $l'h' - w'y' = w(lh/w - y) - 0 = lh - wy = c$, and

$$\begin{aligned} \gcd(l', h', w', y') &= \gcd(w, lh/w - y, 0, h) = \gcd(w, -y, h) \\ &= \gcd(w, -y, h, l) = 1, \end{aligned}$$

Conditions (2): $l'a \equiv_c y'b \iff wa \equiv_c hb$, and

$$w'a \equiv_c h'b \iff 0 \equiv_c (lh/w - y)b \iff yb \equiv_c (l/w)hb = la,$$

follows from $hb \equiv_c wa$ and $la \equiv_c yb$.

Theorem 1: The first inequality, $l'W_a \geq y'W_b$, is fulfilled by the same reasons as in the previous case; for the second one, $h'W_b \geq w'W_a \iff (lh/w - y)W_b \geq 0$, since $l/w \geq 2$. □

Theorem 4 Let $\mathcal{H} = L(l, h, w, y)$ be an MDD related to a digraph $G = G(c; a, b; W_a, W_b)$. If $lW_a = yW_b$, then there is another MDD \mathcal{H}' related to G given by

$$\mathcal{H}' = \begin{cases} L(2l - w, y, l, 2y - h) & h < 2y, \\ L((\lfloor h/y \rfloor + 1)l - w, y, l, y - r) & h > 2y > 0, h = \lfloor h/y \rfloor y + r, 0 < r < y, \\ L(lh/y - w, y, l, 0) & h \geq 2y > 0, h = \lfloor h/y \rfloor y. \end{cases}$$

Proof We can proceed analogously to the proof of Theorem 3. The MDD \mathcal{H}' is now formed by unit squares from \mathcal{H} and the region $B_{\mathcal{H}}$. Note that if $lW_a = yW_b$, then $y \neq 0$. □

From now on, we denote the transformations given in Theorems 3 and 4 by $\mathcal{T}_1(\mathcal{H}) = \mathcal{H}'$ and $\mathcal{T}_2(\mathcal{H}) = \mathcal{H}'$, respectively.

Theorem 2 shows that there is a unique MDD $\mathcal{H} = L(l, h, w, y)$ related to a digraph $G = G(N; a, b; W_a, W_b)$ whenever $(lW_a - yW_b)(hW_b - wW_a) > 0$. On the contrary, Theorems 3 and 4 show that if $(lW_a - yW_b)(hW_b - wW_a) = 0$, then G has more than one related MDD. Moreover, in the latter case, we also see that an MDD can be obtained from another one by applying a suitable transformation, \mathcal{T}_1 or \mathcal{T}_2 , given by either Theorems 3 or 4. The last results in this section show that no more than two MDD are related to the same weighed 2-Cayley digraph.

Lemma 2 *Let $\mathcal{H} = L(l, h, w, y)$ be an MDD related to a digraph $G = G(c; a, b; W_a, W_b)$ with $(lW_a - yW_b)(hW_b - wW_a) = 0$. If $\mathcal{H}' = L(l', h', w', y')$ is the MDD related to G obtained from \mathcal{H} by the corresponding transformation \mathcal{T}_1 or \mathcal{T}_2 , then*

- $lW_a = yW_b \implies l'W_a > y'W_b$,
- $hW_b = wW_a \implies h'W_b > w'W_a$.

Proof Let us assume $hW_b = wW_a$. From Lemma 1(f) it follows $lW_a > yW_b$. Then,

- If $l < 2w$, we have $\mathcal{H}' = L(w, 2h - y, 2w - l, h)$ and so $h'W_b > w'W_a \iff lW_a > yW_b$.
- If $l \geq 2w > 0$ with $l = \lfloor l/w \rfloor w + r$ and $0 < r < w$, we have $\mathcal{H}' = L(w, (\lfloor l/w \rfloor + 1)h - y, w - r, h)$. Therefore, $h'W_b > w'W_a \iff \lfloor l/w \rfloor hW_b + rW_a > yW_b \iff (\lfloor l/w \rfloor w + r)W_a > yW_b \iff lW_a > yW_b$.
- If $l \geq 2w > 0$ with $l = \lfloor l/w \rfloor w$, we obtain $\mathcal{H}' = L(w, lh/w - y, 0, h)$ and so

$$h'W_b > w'W_a \iff (lh/w - y)W_b > 0 \iff lW_a - yW_b > 0.$$

If $lW_a = yW_b$, the proof is analogous to the previous case, now using Lemma 1(e). \square

By Lemma 2, transformations \mathcal{T}_1 and \mathcal{T}_2 can not be applied in any order. There are only two allowed compositions, $\mathcal{T}_2 \circ \mathcal{T}_1$ or $\mathcal{T}_1 \circ \mathcal{T}_2$, and both result in the identity transformation.

Theorem 5 *Let $\mathcal{H} = L(l, h, w, y)$ be an MDD related to a digraph $G = G(N; a, b; W_a, W_b)$ with $(lW_a - yW_b)(hW_b - wW_a) = 0$. Then, there are just two MDD related to G .*

Proof We first prove that if $\mathcal{H}' = L(l', h', w', y')$ is the MDD related to G obtained from \mathcal{H} by the corresponding transformation \mathcal{T}_1 or \mathcal{T}_2 , then any other MDD related to G is contained in the region $\mathcal{H} \cup \mathcal{H}'$. Let us assume $hW_b = wW_a$. Then, from Lemma 1(f) we have $lW_a > yW_b$ and, by Lemma 2, $h'W_b > w'W_a$. Hence, by Lemma 1(a), the region $\nabla(0, h') = \nabla(0, 2h - y)$ can not contain any piece of some other MDD related to G . By items (b), (c) and (d) of Lemma 1, we also know that none of the regions $A_{\mathcal{H}}$ and $B_{\mathcal{H}}$ can intersect any other MDD related to G . Thus, any other MDD related to G is contained in $\mathbb{N}^2 \setminus (C_{\mathcal{H}'} \cup A_{\mathcal{H}} \cup B_{\mathcal{H}}) = \mathcal{H} \cup \mathcal{H}'$. The case $lW_a = yW_b$ can be proved by analogy.

Now we prove that there are just two MDD related to G . If $hW_b = wW_a$, then $\mathcal{H}' = \mathcal{T}_1(\mathcal{H})$ follows from Theorem 3. Clearly $\mathcal{H}' \neq \mathcal{H}$. Analogously, if $lW_a = yW_b$ we have $\mathcal{T}_2(\mathcal{H}) \neq \mathcal{H}$. Therefore, we have at least two MDD related to G . We have to prove that there is no other MDD, $\mathcal{H}'' \neq \mathcal{H}, \mathcal{H}'$, related to G .

Let us assume $hW_b = wW_a$ ($\mathcal{H}' = \mathcal{T}_1(\mathcal{H})$) and the subcase $l < 2w$ (see Fig. 3). Denote the rectangle $\{[m, n]: i \leq m \leq s, j \leq n \leq t\}$ by $\rho[(i, j), (s, t)]$ whenever $i \leq s$ and $j \leq t$. Define the regions

$$\begin{aligned} \mathcal{E}_1 &= \rho[(l', 0), (l - 1, h - y - 1)] = \rho[(w, 0), (l - 1, h - y - 1)], \\ \mathcal{E}_2 &= \rho[(0, h), (l' - w' - 1, h' - 1)] = \rho[(0, h), (l - w - 1, 2h - y - 1)], \\ \mathcal{F} &= \mathcal{H} \setminus \mathcal{E}_1 = \Delta(w - 1, h - y - 1) \cup \Delta(l - w - 1, h - 1). \end{aligned}$$

Regions \mathcal{E}_1 and \mathcal{E}_2 are the shadowed pieces appearing in Fig. 3, and the region \mathcal{F} is the lighter piece in the same figure.

Clearly $C_{\mathcal{H}} = \mathcal{E}_2 \cup C_{\mathcal{H}'}$ and $\mathcal{E}_2 = \mathcal{E}_1 + \mathbf{v}$. So, when applying \mathcal{T}_1 to \mathcal{H} to obtain \mathcal{H}' , region \mathcal{F} remains fixed and region \mathcal{E}_1 is transformed into \mathcal{E}_2 . Then, any other MDD \mathcal{H}'' related to G , different from \mathcal{H} and \mathcal{H}' , is contained in the region $\mathcal{R} = \mathcal{H} \cup \mathcal{H}' = \mathcal{E}_1 \cup \mathcal{F} \cup \mathcal{E}_2$.

Let us consider the unit squares $[[w, 0]]$ and $[[0, h]] = [[w, 0]] + \mathbf{v}$, which represent the same equivalence class $[wa]_c = [hb]_c$. The class $[wa]_c$ is only represented by these two unit squares in the region \mathcal{R} . Therefore, the MDD \mathcal{H}'' contains either $[[w, 0]]$ or $[[0, h]]$. If $[[w, 0]] \in \mathcal{H}''$, then $\nabla(0, h) \cap \mathcal{H}'' = \emptyset$ because \mathcal{H}'' is an MDD. So, \mathcal{H}'' contains no unit square located in the region \mathcal{E}_2 . Hence, $\mathcal{H}'' = \mathcal{F} \cup \mathcal{E}_1 = \mathcal{H}$. On the contrary, if $[[0, h]] \in \mathcal{H}''$, then $\mathcal{H}'' = \mathcal{H}'$. In any case, we conclude that the MDD \mathcal{H}'' is not different from either \mathcal{H} or \mathcal{H}' .

The other subcases, $l > 2w > 0$ with $r > 0$ and $l \geq 2w > 0$ with $r = 0$, lead to the same conclusion with similar arguments.

If $lW_a = yW_b$ and $\mathcal{H}' = \mathcal{T}_2(\mathcal{H})$, the statement can be proved by analogous reasonings. □

4 L-shapes associated with numerical 3-semigroups

In this section, we first introduce some general classes of numerical semigroups useful in the study of curve singularities. Second, we associate L-shapes with numerical 3-semigroups and we describe how the main invariants of these semigroups are recognizable in their associated L-shapes.

4.1 Numerical semigroups

A numerical semigroup S is an additive subsemigroup of \mathbb{N} with $0 \in S$. We can suppose that $\text{gcd}(S) = 1$, otherwise we would take the quotient $S' = S/\text{gcd}(S)$. This condition is equivalent to the fact that the complementary of S in \mathbb{N} is finite, and we denote the set and the number of gaps in the semigroup S by $\bar{S} = \mathbb{N} - S$ and $|\bar{S}|$, respectively. They are also equivalent to the existence of a conductor element

$c(S) = \min\{x \in S : n \in S \text{ for any } n \geq x\}$ in S . The number $f(S) = c(S) - 1$ is called the *Frobenius number* of S .

Numerical semigroups are finitely generated and any set of generators includes the minimal set of generators $S = \langle b_0, \dots, b_g \rangle$, where $b_0 = \min(S - \{0\})$ and $b_{i+1} = \min(S - \langle b_0, \dots, b_i \rangle)$, $i = 0, \dots, g - 1$. The *Apéry set* of S with respect to the element $m \in S$ is $\text{Ap}(S, m) = \{s \in S : s - m \notin S\}$. Then we have $\max \text{Ap}(S, m) = f(S) - m$. For an up-to-date information about numerical semigroups see [11] and [8].

The classes of numerical semigroups that we will consider in what follows receive the name of their corresponding classes of associated curves.

In general, any numerical semigroup S verifies that $m \in S \implies c(S) - 1 - m \notin S$. A numerical semigroup S is *symmetric* if it satisfies $m \in S \iff c(S) - 1 - m \notin S$. In this case, the conductor verifies $c(S) = 2|\bar{S}|$ [7].

A numerical semigroup $S = \langle b_0, \dots, b_g \rangle$ is *free* if it satisfies

$$N_i b_i \in \langle b_0, \dots, b_{i-1} \rangle, \quad \text{for } i = 1, \dots, g$$

where $N_i = e_{i-1}/e_i$ with $e_i = \gcd(b_0, \dots, b_i)$, $i = 0, 1, \dots, g$.

A particular case of free semigroups are the so called *plane curve* semigroups arising as semigroups of values of analytically irreducible plane curve singularities. For the former, one has in addition

$$N_i b_i < b_{i+1}, \quad i = 1, \dots, g - 1.$$

In general, plane curve implies free and free implies symmetric.

Lemma 3 *A numerical semigroup with two generators $S = \langle a, b \rangle$ is a plane curve semigroup, it has the Apéry set $\text{Ap}(S, a) = \{0, b, 2b, \dots, (a - 1)b\}$ and conductor $c(S) = (a - 1)(b - 1)$.*

A numerical semigroup with three generators $S = \langle a, b, c \rangle$ is a free semigroup if and only if $c \gcd(a, b) \in \langle a, b \rangle$ and it is a plane curve semigroup if furthermore $c > \text{lcm}(a, b)$.

Lemma 4 *Let $S = \langle a, b, c \rangle$ be a minimally generated numerical semigroup with $1 < a < b < c$ and $\gcd(a, b) = p > 1$. Considering $S_p = \langle a/p, b/p \rangle$ and $S' = \langle a/p, b/p, c \rangle$ we have that:*

- (a) *S is symmetric if and only if S' is symmetric.*
- (b) *S is free if and only if $c \in S_p$.*
- (c) *S is plane curve if and only if $c \in S_p$ and $c > \text{lcm}(a, b)$.*

Proof Item (a) is the Proposition 8 of [7], and (b) and (c) follow from Lemma 3. \square

4.2 L-shapes associated with numerical 3-semigroups

The digraph $G^* = G(c; a, b; a, b)$ (i.e., the particular case $W_a = a$ and $W_b = b$) is closely related to the numerical 3-semigroup $S = \langle a, b, c \rangle$. Let L be an MDD related to G^* then, for each $[[i, j]] \in L$, identity $M_{ia+jb} = ia + jb$ holds. In other words, the value $ia + jb$ is the minimum element of the equivalence class $[ia + jb]_c$ that

belongs to the semigroup S . That is $ia + jb \in \text{Ap}(c, S)$. This observation leads to the identity

$$\{d_{G^*}([0]_c, [ia + jb]_c) : \llbracket i, j \rrbracket \in L\} = \{ia + jb : \llbracket i, j \rrbracket \in L\} = \text{Ap}(c, S). \tag{3}$$

Definition 2 An L-shape L is associated with the numerical 3-semigroup $\langle a, b, c \rangle$ if L is an MDD related to the digraph $G(c; a, b; a, b)$.

Many properties of a numerical semigroup $S = \langle a, b, c \rangle$ can be derived from a related Apéry set. Note that, by (3), the set of unit squares gives the Apéry set $\text{Ap}(c, S)$. Thus, some computations can be efficiently done from the associated L-shape. See [1] for how the sets of gaps of S can be obtained from an associated L-shape. The following result gives the Frobenius number and the number of gaps of a numerical 3-semigroup from an associated L-shape.

Theorem 6 ([1]) *Let $S = \langle a, b, c \rangle$ be a numerical semigroup and $\mathcal{H} = L(l, h, w, y)$ an L-shape associated with S . Then*

$$f(S) = \max\{(l - 1)a + (h - y - 1)b, (l - w - 1)a + (h - 1)b\} - c \tag{4}$$

$$\begin{aligned} 2|\bar{S}| &= \frac{l(h - y)}{c} [(l - 1)a + (h - y - 1)b] \\ &\quad + \frac{y(l - w)}{c} [(l - w - 1)a + (2h - y - 1)b] - c + 1 \end{aligned} \tag{5}$$

Note that the numbers $(l - 1)a + (h - y - 1)b$ and $(l - w - 1)a + (h - 1)b$ are the weights of the unit squares $\llbracket l - 1, h - y - 1 \rrbracket$ and $\llbracket l - w - 1, h - 1 \rrbracket$, i.e., the “dominant right interior corners” of the L-shape \mathcal{H} .

In the next section we will use the following result about the weight of the *convex corner*, i.e., the unit square $\llbracket l - w, h - y \rrbracket$, of an L-shape $L(l, h, w, y)$ associated to a semigroup.

Lemma 5 *Let $\mathcal{H} = L(l, h, w, y)$ be an L-shape associated with the numerical semigroup $S = \langle a, b, c \rangle$, with $1 < a < b < c$ and $\text{gcd}(a, b, c) = 1$. Then, if $hb = wa$ (or $la = yb$), the weight of the convex corner of \mathcal{H} is ac/h (or bc/h) and this weight does not change with the transformation \mathcal{T}_1 (or \mathcal{T}_2).*

Proof If $hb = wa$, the weight of the convex corner of \mathcal{H} is $(l - w)a + (h - y)b = la - yb = ac/h$. Now we compute the weight of the convex corner of $\mathcal{T}_1(\mathcal{H}) = \mathcal{H}'$ by cases.

- If $l < 2w$, the convex corner preserves its position and its weight, since $\mathcal{H}' = L(w, 2h - y, 2w - l, l)$ and its convex corner is $\llbracket w - (2w - l), 2h - y - l \rrbracket = \llbracket l - w, h - y \rrbracket$.
- If $l \geq 2w$ and w divides l , we have $\mathcal{H}' = L(w, lh/w - y, 0, h)$ and its convex corner is $\llbracket w, lh/w - y - h \rrbracket$ with weight $wa + (lh/w - y - h)b = la - yb$.

- If $l \geq 2w$ and $l = \lfloor l/w \rfloor w + r$ with $0 < r < w$, we have $\mathcal{H}' = L(w, (\lfloor l/w \rfloor + 1)h - y, w - r, h)$ and its convex corner is $[[w - (w - r), (\lfloor l/w \rfloor + 1)h - y - h]] = [[r, \lfloor l/w \rfloor h - y]]$ with weight $ra + (\lfloor l/w \rfloor h - y)b = ra + \lfloor l/w \rfloor wa - yb = la - yb$. □

Lemma 6 *Let $\mathcal{H} = L(l, h, w, y)$ be an L-shape associated with the numerical semigroup $S = \langle a, b, c \rangle$, with $1 < a < b < c$ and $\gcd(a, b, c) = 1$. Consider $m, n \in \mathbb{N}$ with $\gcd(m, n) = \gcd(m, c) = 1, n|a, n|b$ and $mb < nc$. Then \mathcal{H} is also related to the 3-numerical semigroup $S' = \langle a \frac{m}{n}, b \frac{m}{n}, c \rangle$.*

Proof Obviously, conditions (1) and Theorem 1, taking $W_a = a$ and $W_b = b$, are fulfilled by \mathcal{H} and S' .

Consider the congruence $la \equiv yb \pmod{c}$ in (2), then there is some $\lambda \in \mathbb{Z}$ with $la - yb = \lambda c$. From $n|(la - yb)$ and $\gcd(n, c) = 1$, we have $n|\lambda$ and then $m(l \frac{a}{n} - y \frac{b}{n}) = m \frac{\lambda}{n} c$ and so $la \frac{m}{n} \equiv yb \frac{m}{n} \pmod{c}$ holds. The second congruence $hb \frac{m}{n} \equiv wa \frac{m}{n} \pmod{c}$ follows from the same argument. Thus, conditions (2) hold for \mathcal{H} and S' . □

5 Classification of numerical 3-semigroups by means of L-shapes

In this section, we characterize the numerical 3-semigroups considered in the previous section, in terms of the parameters of their associated L-shapes.

The next theorem gives a first characterization of symmetric 3-semigroups by means of the parameters of an associated L-shape.

Theorem 7 *Let $S = \langle a, b, c \rangle$ be a numerical semigroup and $\mathcal{H} = L(l, h, w, y)$ an L-shape associated to S . Then, S is symmetric if and only if $wy(la - yb)(wa - hb) = 0$.*

Proof We can write the expression (4) as

$$f(S) = (l - w - 1)a + (h - y - 1)b + \max\{wa, yb\} - c \tag{6}$$

Using (5), (6) and $c = lh - wy$ we have

- If $wa = yb$, then $[[w, 0]]$ and $[[0, y]]$ belong to \mathcal{H} and have the same weight, so $w = y = 0$.
- If $wa > yb$, then

$$\begin{aligned} S \text{ symmetric} &\iff 2|\bar{S}| = f(S) + 1 \\ &\iff (l - w)y(hb - wa) = 0 \\ &\iff y(hb - wa) = 0, \quad \text{since } 0 < w < l. \end{aligned}$$

- If $wa < yb$, in the same way, S is symmetric if and only if $w(la - yb) = 0$.

So, if S is symmetric, then $wy(la - yb)(wa - hb) = 0$. Reciprocally, note that $w = 0$ implies $wa < yb$ and also $la - yb = 0$ implies $wa < yb$. Analogously, $y(wa - hb) = 0$ implies $wa > yb$. □

As a consequence of this result, a symmetric 3-semigroup has associated either an L-shape $\mathcal{H} = L(l, h, w, y)$ with $wy = 0$, in which case \mathcal{H} is rectangular but not unique, or $(la - yb)(wa - hb) = 0$, in which case S has associated two L-shapes, where only one of them can be rectangular (Theorems 3 and 4).

Note that, in the characterization of symmetric 3-semigroups given in Theorem 7, it is not required that $\{a, b, c\}$ be a minimal system of generators for S . In this section, we classify numerical semigroups introduced in Sect. 4 in the 3-generated case, $S = \langle a, b, c \rangle$. This classification is done with respect to the number and the degeneration or not of their associated L-shapes. This classification basically depends on whether $\{a, b, c\}$ is a minimal system of generators for S or not, and on whether the terms a, b and c are coprime or not.

First of all, we consider the case $1 < a < b < c$ with $\gcd(a, b, c) = 1$ when $\{a, b, c\}$ is not a minimal system of generators of S . Thus, $\gcd(a, b) \in \{1, a\}$.

If $\gcd(a, b) = 1$, then $c = \lambda a + \mu b, \lambda, \mu \in \mathbb{N}$ and we have the following result for S .

Theorem 8 *Let $S = \langle a, b, c \rangle$ be a numerical semigroup with $1 < a < b < c$, $\gcd(a, b) = 1$ and $c = \lambda a + \mu b$ with $\lambda, \mu \in \mathbb{N} \cup \{0\}$. We have:*

- (a.1) *If $c = \lambda a$ with $\lambda \leq b$, then S has associated a unique rectangular L-shape $\mathcal{H} = L(\lambda, a, 0, 0)$ if λ divides b , and $\mathcal{H} = L(\lambda, a, \lambda - r, 0)$, with $\lambda - r \equiv b \pmod{\lambda}$, if not.*
- (a.2) *If $c = \lambda a$ with $\lambda > b$, then S has associated a rectangular L-shape $\mathcal{H} = L(\lambda, a, b, 0)$ and another, $\mathcal{H}' = \mathcal{T}_1(\mathcal{H})$, that can be rectangular or not (also $\mathcal{H} = \mathcal{T}_2(\mathcal{H}')$).*
- (b.1) *If $c = \mu b$ with $\mu \leq a$, then S has associated a unique rectangular L-shape $\mathcal{H} = L(b, \mu, 0, 0)$ if μ divides a , and $\mathcal{H} = L(b, \mu, 0, \mu - r)$, with $\mu - r \equiv a \pmod{\mu}$, if not.*
- (b.2) *If $c = \mu b$ with $\mu > a$, then S has associated a rectangular L-shape $\mathcal{H} = L(b, \mu, 0, a)$ and another, $\mathcal{H}' = \mathcal{T}_2(\mathcal{H})$, that can be rectangular or not (also $\mathcal{H} = \mathcal{T}_1(\mathcal{H}')$).*
- (c) *S has associated two rectangular L-shapes if and only if $c = \lambda a = \mu b > ab$.*
- (d) *If $c = \lambda a + \mu b$ such that a and b do not divide c , then S has associated two non rectangular L-shapes $\mathcal{H} = L(\lambda' + b, a, b, a - \mu')$, being $c = \lambda'a + \mu'b$ with $1 \leq \mu' < a$, and $\mathcal{H}' = \mathcal{T}_1(\mathcal{H})$ (also $\mathcal{H} = \mathcal{T}_2(\mathcal{H}')$).*

Proof In each case, we prove that the corresponding L-shapes verify the compatibility conditions (1) and (2), and Theorem 1.

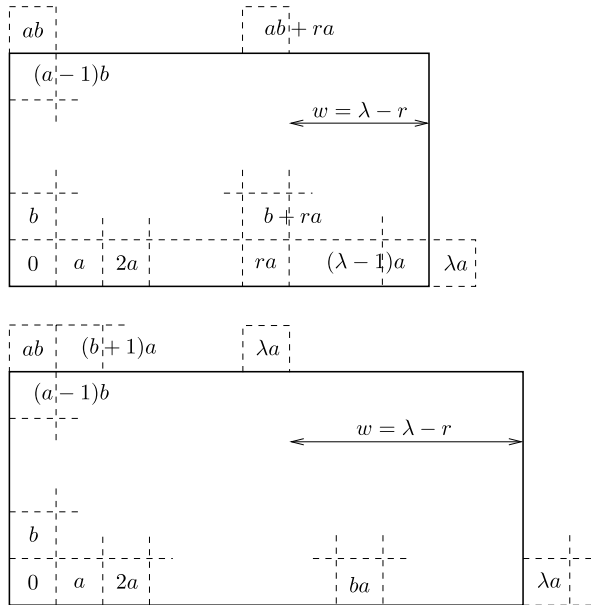
(a.1) There exists $r, 0 \leq r < \lambda$, such that $b + r \equiv 0 \pmod{\lambda}$ and so $ab + ar \equiv 0 \pmod{\lambda a}$ (see left hand side of Fig. 6). Note that $r = 0$ if and only if λ divides b .

For conditions (1), if $\gcd(\lambda, a, \lambda - r) = g > 1$, then $\gcd(\lambda, a, r) = g$ and g divides b , thus we obtain the contradiction $\gcd(a, b) \geq g > 1$.

For Theorem 1, we have $la - yb = \lambda a > 0$ and $hb - wa = a(b - \lambda + r) > 0$, since $0 < r < \lambda$, which also proves the unicity of this L-shape. Only a simple inspection is needed to verify the rest of the conditions.

(a.2) If $\lambda > b$, the situation is represented in the right hand side of Fig. 6, where the zero $[\lambda a]_{\lambda a}$ in the unit square $[[\lambda, 0]]$ is also obtained in the unit square $[[\lambda - b, a]]$. Hence, an associated L-shape is $\mathcal{H} = L(\lambda, a, b, 0)$.

Fig. 6 Diagrams of $L(\lambda, a, \lambda - r, 0)$ and $L(\lambda, a, b, 0)$



Conditions (1) and (2) are clear. Now we have $la - yb = \lambda a > 0$ and $hb - wa = ab - ba = 0$. Then the statement follows from Theorems 1 and 5.

In a similar way we can prove (b.1) and (b.2).

(d) We have $c = \lambda a + \mu b = (\lambda + kb)a + (\mu - ka)b$ for any $k \in \mathbb{Z}$. Since a does not divide c , there exists $k \in \mathbb{N} \cup \{0\}$ such that $0 < \mu - ka < a$. Denoting $\mu' = \mu - ka$ and $\lambda' = \lambda + kb$, we can write $c = \lambda'a + \mu'b$ with $1 \leq \mu' < a$.

Let us see that $\mathcal{H} = L(\lambda' + b, a, b, a - \mu')$ is a non rectangular ($\mu' < a$) L-shape associated with S , as shown in Fig. 7. Conditions (1) are clear. From $la - yb = (\lambda' + b)a - (a - \mu')b = \lambda'a + \mu'b \equiv 0 \pmod{c}$ and $hb - wa = ab - ba = 0 \equiv 0 \pmod{c}$ follow conditions (2). Since $la - yb > 0$, the L-shape \mathcal{H} verifies Theorem 1, and Theorem 3 gives the other L-shape $\mathcal{H}' = L(b, \mu'' + a, b - \lambda'', a)$, where $c = \lambda''a + \mu''b$ with $1 \leq \lambda'' < b$.

(c) As $c = \lambda a = \mu b > ab$, by exclusion, we are simultaneously in the cases (a.2) and (b.2). So S has associated the L-shapes $\mathcal{H} = L(\lambda, a, b, 0)$ and $\mathcal{H}' = L(b, \mu, 0, a)$. Obviously $\mathcal{T}_1(\mathcal{H}) = \mathcal{H}'$ and $\mathcal{T}_2(\mathcal{H}') = \mathcal{H}$. □

If $\gcd(a, b) = a$, then $b = ka, k \in \mathbb{N}$ and $\gcd(a, c) = 1$. In this case, we have the following result for S .

Theorem 9 *Let $S = \langle a, b, c \rangle$ be a numerical semigroup with $1 < a < b < c$, $\gcd(a, b, c) = 1$ and $b = ka$. Then:*

- (a) S has associated two L-shapes and at least one of them is the rectangular L-shape $\mathcal{H} = L(c, 1, k, 0)$.
- (b) S has associated two rectangular L-shapes if and only if k divides c . The other rectangular L-shape associated with S is $\mathcal{T}_1(\mathcal{H}) = L(k, c/k, 0, 1)$.

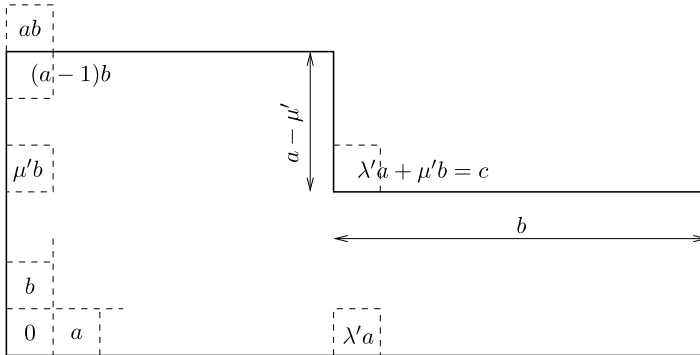


Fig. 7 Diagram of $\mathcal{H} = L(\lambda' + b, a, b, a - \mu')$

(c) If k does not divide c , the other non rectangular L-shape associated with S is $L(k, \lfloor c/k \rfloor + 1, k - r, 1)$, with $c = \lfloor c/k \rfloor k + r$ and $0 < r < k$.

Proof As $\gcd(a, c) = 1$, the zero $[ca]_c$ is located at unit squares $[[c, 0]]$ and $[[c - k, 1]]$. So $\mathcal{H} = L(c, 1, k, 0)$, verifying conditions (1) and (2) and Theorem 1, is an L-shape associated with S . Furthermore, $hb - wa = 0$ and Theorem 3 give the other L-shape $\mathcal{H}' = \mathcal{T}_1(\mathcal{H})$, which completes part (a).

If k divides c , then $\mathcal{T}_1(\mathcal{H}) = L(k, c/k, 0, 1)$. As $c > 2k$ in \mathcal{H} , the other unique possible $\mathcal{T}_1(\mathcal{H})$ corresponds to the case in which k does not divide c . If $c = \lfloor c/k \rfloor k + r$, with $0 < r < k$, then $\mathcal{T}_1(\mathcal{H}) = L(k, \lfloor c/k \rfloor + 1, k - r, 1)$. This new L-shape is non rectangular since $0 < r < k$. This proves (b) and (c). □

Remark 1 The geometric classification of numerical 3-semigroups whose system of generators is not minimal is irrelevant since, in fact, they are 2-generated semigroups and, in particular, they are plane curve semigroups (Lemma 3). Theorems 8 and 9 can be used as a classification of numerical 2-semigroups in terms of their associated L-shapes, and we next use them to classify minimally generated numerical 3-semigroups.

Now we consider numerical 3-semigroups S given by a minimal system of generators $\langle a, b, c \rangle$, and we discern the cases $\gcd(a, b) = 1$ and $\gcd(a, b) > 1$.

Remark 2 If $S = \langle a, b, c \rangle$ is a numerical semigroup with $1 < a < b < c$, $\gcd(a, b) = 1$ and $c \notin \langle a, b \rangle$, then c can only take values in the set of gaps of the semigroup $S' = \langle a, b \rangle$. In particular $b < c < (a - 1)(b - 1)$. Therefore, there are a finite number of numerical semigroups S under these conditions. The following theorem characterizes them in terms of their associated L-shapes.

Theorem 10 *Let $S = \langle a, b, c \rangle$ be a numerical semigroup with $1 < a < b < c$, $\gcd(a, b) = 1$ and $c \notin \langle a, b \rangle$. Then,*

(a) S is non symmetric if and only if S has associated a unique non rectangular L-shape.

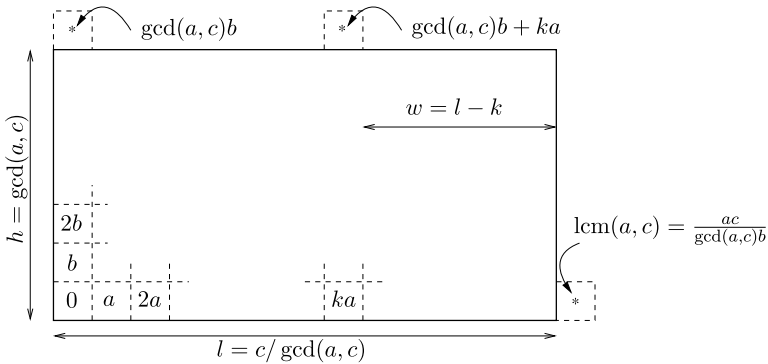


Fig. 8 $L(c/\gcd(a, c), \gcd(a, c), c/\gcd(a, c) - k, 0)$

- (b) S is symmetric if and only if S has associated a unique rectangular L-shape \mathcal{H} where:
 - (b.1) if $\gcd(a, c) = p > 1$ and $\gcd(b, c) = 1$, then $\mathcal{H} = L(\frac{c}{p}, p, 0, 0)$ or $\mathcal{H} = L(\frac{c}{p}, p, \frac{c}{p} - k, 0)$ with $0 < k < \frac{c}{p}$ and $b + k\frac{a}{p} \equiv 0 \pmod{\frac{c}{p}}$;
 - (b.2) if $\gcd(a, c) = 1$ and $\gcd(b, c) = q > 1$, then $\mathcal{H} = L(q, \frac{c}{q}, 0, 0)$ or $\mathcal{H} = L(q, \frac{c}{q}, 0, \frac{c}{q} - k)$ with $0 < k < \frac{c}{q}$ and $a + k\frac{b}{q} \equiv 0 \pmod{\frac{c}{q}}$;
 - (b.3) if $\gcd(a, c) = p > 1$ and $\gcd(b, c) = q > 1$, then \mathcal{H} is as in (b.1) if $aq < bp$ or as in (b.2) if $aq > bp$.
- (c) S is non free.

Proof

- (a) Follows from Theorem 7.
- (b) It is well known [7] that if $S = \langle a, b, c \rangle$ is a symmetric semigroup, then the generators a, b and c can not have any coprime pairs, thus $\gcd(a, c) > 1$ or $\gcd(b, c) > 1$.

If $\gcd(a, c) = p > 1$, then a/p and c/p are coprime and so, there exists k with

$$0 \leq k \leq \frac{c}{p} \quad \text{such that } b + k\frac{a}{p} \equiv 0 \pmod{\frac{c}{p}}$$

and so $pb + ka \equiv 0 \pmod{c}$. This means that we have a zero $[0]_c$ in the unit square $[[k, p]]$. On the other hand, since $\text{lcm}(a, c) = ac/p$, there is another zero $[0]_c$ in the unit square $[[c/p, 0]]$.

The rectangle $L(l, h, w, 0)$ of Fig. 8 with $l = c/p, h = p$ y $w = l - k$ is an L-shape associated with S . This L-shape is unique because $la - yb = la > 0$ and $hb - wa = pb + ka - \text{lcm}(a, c) > 0$ since $pb + ka$ is the zero $[0]_c$ in the unit square $[[k, p]]$ and $\text{lcm}(a, c)$ is the zero $[0]_c$ in the unit square $[[c/p, 0]]$, and both have distinct weight.

If $\gcd(b, c) = q > 1$ we obtain analogously the unique L-shape $L(l, h, 0, y)$ with $l = q, h = c/q, y = h - k$ and

$$0 \leq k \leq \frac{c}{q} \quad \text{such that } a + k\frac{b}{q} \equiv 0 \pmod{\frac{c}{q}}.$$

In the hypothesis (b.3) we have $\gcd(p, q) = 1$ and $aq \neq bp$, since if $aq = bp$ then identities $\gcd(a, b) = a/p = b/q = 1$ imply $a = p$ and $b = q$. Therefore we have $c \geq pq$, a contradiction with $b < c < (a - 1)(b - 1)$ (Remark 2).

(c) Follows from Lemma 3 and the hypothesis $c \notin \langle a, b \rangle$. □

We illustrate Theorems 8 and 10 in the following example.

Example 1 Let $S_c = \langle 6, 7, c \rangle$ with $c > 7$. Then, the term c takes values in the set $R \cup T$, where $R = \{8, 9, 10, 11, 15, 16, 17, 22, 23, 29\}$ is the set of gaps of the semigroup $\langle 6, 7 \rangle$ that are greater than 7, and T is the set of terms of the semigroup $\langle 6, 7 \rangle$ greater than 7.

- If $c \in T$, then S_c is ruled by Theorem 8:
 - (a.1) If $c = 6\lambda$ with $\lambda \leq 7$, then $\mathcal{H}_c = L(\lambda, 6, \lambda - r, 0)$ with $\lambda - r \equiv 7 \pmod{\lambda}, 0 \leq r < \lambda$.
 - (a.2) If $c = 6\lambda$ with $\lambda > 7$ and $\lambda = 7p + r, p \geq 1, 0 < r < 7$, then $\mathcal{H}_c = L(\lambda, 6, 7, 0)$ and $\mathcal{T}_1(\mathcal{H}_c) = L(7, 6(p + 1), 7 - r, 6)$.
 - (b.1) If $c = 7\mu$ with $\mu \leq 6$, then $\mathcal{H}_c = L(7, \mu, 0, 7 - r)$ with $\mu - r \equiv 6 \pmod{\mu}, 0 \leq r < \mu$.
 - (b.2) If $c = 7\mu$ with $\lambda > 6$ and $\mu = 6p + r, p \geq 1, 0 < r < 6$, then $\mathcal{H}_c = L(7, \mu, 0, 6)$ and $\mathcal{T}_2(\mathcal{H}_c) = L(7(p + 1), 6, 7, 6 - r)$.
 - (c) If $c = 42k$ with $k > 1$, then $\mathcal{H}_c = L(7k, 6, 7, 0)$ and $\mathcal{H}'_c = L(7, 6k, 0, 6)$.
 - (d) If $c = 6\lambda + 7\mu$ with $1 \leq \mu < 6$ and $\lambda = 7k + r$ with $0 < r < 7$, then $\mathcal{H}_c = L(\lambda + 7, 6, 7, 6 - \mu)$ and $\mathcal{T}_1(\mathcal{H}_c) = L(7, 6(k + 1) + \mu, 7 - r, 6)$.
- If $c \in R$ then S_c is ruled by Theorem 10: the semigroups $\langle 6, 7, 8 \rangle, \langle 6, 7, 9 \rangle$ and $\langle 6, 7, 15 \rangle$ have, respectively, a unique rectangular L-shape $L(4, 2, 1, 0), L(3, 3, 2, 0)$ and $L(5, 3, 1, 0)$ and, so, they are symmetric.

For the values $c = 10, 11, 16, 17, 22, 23$ and 29 we have, respectively, unique non rectangular L-shapes $L(4, 4, 3, 2), L(3, 4, 1, 1), L(5, 4, 2, 2), L(4, 5, 3, 1), L(6, 4, 1, 2), L(5, 5, 2, 1)$ and $L(6, 5, 1, 1)$ and, so, they are non symmetric.

Theorem 9 is illustrated in the following example.

Example 2 Each numerical semigroup $S_c = \langle 4, 12, c \rangle$, with $\gcd(4, 12, c) = 1$ and $c > 12$, has associated a rectangular L-shape $\mathcal{H}_c = L(c, 1, 3, 0)$. The other L-shape depends on whether 3 divides c or not.

- (b) If 3 divides c , then $c = 3q$ with $q \geq 5$ odd, i.e., $c = 15, 21, 27, \dots$, and the other L-shape is $\mathcal{T}_1(\mathcal{H}_c) = L(3, q, 0, 1)$.

(c) If 3 does not divide c , we discern the cases:

$c = 3q + 1$ with $q \geq 6$ even, i.e., $c = 19, 25, 31, \dots$, and the other L -shape is $\mathcal{T}_1(\mathcal{H}_c) = L(3, q + 1, 2, 1)$.

$c = 3q + 2$ with $q \geq 5$ odd, i.e., $c = 17, 23, 29, \dots$, and the other L -shape is $\mathcal{T}_1(\mathcal{H}_c) = L(3, q + 1, 1, 1)$.

Theorem 11 *Let $S = \langle a, b, c \rangle$ be a minimally generated numerical semigroup with $1 < a < b < c$ and $\gcd(a, b) = p > 1$. Considering $S_p = \langle a/p, b/p \rangle$ and $S' = \langle a/p, b/p, c \rangle$ we have that:*

- (a) S is non symmetric if and only if S' has associated a unique non rectangular L -shape.
- (b) S is symmetric but not free if and only if S' has associated a unique rectangular L -shape as described in Theorem 10(b).
- (c) S is free if and only if the associated L -shapes to S' are ruled by Theorem 8.
- (d) S is plane curve if and only if S is free and the weight of the convex corner $(l - w, h - y)$ of any associated L -shape with S' is greater than $\text{lcm}(a, b)$.

Furthermore, in all cases, the semigroups S and S' have associated the same L -shapes. In each case, these L -shapes are given by the corresponding Theorems 10 and 8.

Proof Clearly $1 < a/p < b/p < c$ and $\gcd(a/p, b/p) = 1$. If S is not free, then $c \notin S_p$ and, so, S' verifies the hypothesis of Theorem 10. It proves the parts (a) and (b).

On the contrary, if S is free, then $c \in S_p$ by Lemma 4, and hence S' is described by Theorem 8. Finally, note that the weight of the convex corners of the associated L -shapes with the semigroups S and S' are cp and c , respectively, and from Lemma 4 we conclude $c > \text{lcm}(a, b)$.

That S and S' have the same L -shapes follows from Lemma 6. □

Note that, from Theorems 10 and 11, all minimally generated numerical 3-semigroups, excepting a finite number, are free.

The following example illustrates Theorem 11.

Example 3 Let $S_c = \langle 12, 14, c \rangle$, with $c > 14$ and $\gcd(12, 14, c) = 1$, and let $S'_c = \langle 6, 7, c \rangle$. The semigroup S_c is ruled by Theorem 11:

- (a) S_c is non symmetric if and only if $c = 17, 23, 29$, with respective L -shapes $L(4, 5, 3, 1), L(5, 5, 2, 1), L(6, 5, 2, 1)$.
- (b) S_c is symmetric but not free if and only if $c = 15$, with L -shape $L(5, 3, 1, 0)$.
- (c) Following Theorem 8 and Example 1, S_c is free:
 - (b1) if $c = 21, 35$, with respective L -shapes $L(7, 3, 0, 7), L(7, 5, 0, 1)$.
 - (b2) if $c = 7\mu$ with $\mu \geq 7$ odd and $\mu = 6p + r$ with $p \geq 1$ and $0 < r < 6$, with two L -shapes $\mathcal{H}_c = L(7, \mu, 0, 6)$ and $\mathcal{T}_2(\mathcal{H}_c) = L(7(p + 1), 6, 7, 6 - r)$.
 - (d) if $c = 6\lambda + 7\mu$ with $\mu = 1, 3, 5$ and $\lambda = 7k + r$ with $0 < r < 7$, with two L -shapes $\mathcal{H}_c = L(\lambda + 7, 6, 7, 6 - \mu)$ and $\mathcal{T}_1(\mathcal{H}_c) = L(7, 6(k + 1) + \mu, 7 - r, 6)$.
- (d) The free semigroups of (c) are plane curve if and only if $c \geq 85$.

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