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On Sufficient Conditions for the RNIEP and their Margins of Realizability

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Abstract

The Real Nonnegative Inverse Eigenvalue Problem (RNIEP) is that of characterizing all possible real spectra of nonnegative matrices. In this work we list some inclusion relations between several sufficient conditions and we study the negativity and the realizability margin of a spectrum with respect to these conditions.

Keywords: Nonnegative matrices, RNIEP, Sufficient conditions.

1 Preface

In the context of Spectral Graph Theory, we are interested in inverse spectral problems about digraphs. In particular, we look for necessary and sufficient conditions for a family of real numbers to be the spectrum of the adjacency matrix of a weighted digraph. This problem is known in the context of nonnegative matrices as the Real Nonnegative Inverse Eigenvalue Problem (RNIEP) and it has only been solved for real spectra of sizes lower than or equal to 4. If the family $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}$ is the spectrum of a nonnegative matrix (weighted digraph) of size n, we write $\sigma \in Spec_n$, or simply $\sigma \in Spec$.

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In what follows we list some sufficient conditions (named after their authors) so that σ will be the spectrum of a nonnegative matrix:

- **Suleǐmanova**, 1949: If $\sigma = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$:
$$\left.\lambda_0 \geq |\lambda|\ ,\ \lambda \in \sigma\ , \quad \text{and} \quad \lambda_0 + \sum_{\lambda_i<0} \lambda_i \geq 0 \right\} \Rightarrow \sigma \in Spec\,.$$
- **Suleǐmanova-Perfect**, 1953: If $\sigma = \{\lambda_0, \lambda_{01}, \ldots, \lambda_{0t_0}\} \cup \{\lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}\} \cup \ldots \cup \{\lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}\}$:
$$\left.\lambda_0 \geq |\lambda|,\ \lambda \in \sigma\ , \quad \text{and} \quad \lambda_j + \sum_{\lambda_{ji}<0} \lambda_{ji} \geq 0\ ,\ 0 \leq j \leq r \right\} \Rightarrow \sigma \in Spec\,.$$
- **Perfect 1**, 1953: If $\sigma = \{\lambda_0\} \cup \{\lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}\} \cup \ldots \cup \{\lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}\} \cup \{\lambda_n\}$, with $\lambda_j \geq 0$ and $\lambda_{ji} \leq 0$ for $j = 1, \ldots, r$ e $i = 1, \ldots, t_j$ and $\lambda_n \leq 0$:
$$\left.\begin{array}{r} \lambda_0 \geq |\lambda|,\ \lambda \in \sigma\ ,\ \sum_{\lambda\in\sigma} \lambda \geq 0 \\ \lambda_j + \lambda_n \leq 0 \quad \text{and} \quad \lambda_j + \sum_{1\leq i\leq t_j} \lambda_{ji} \leq 0\ ,\ 1 \leq j \leq r \end{array}\right\} \Rightarrow \sigma \in Spec\,.$$
- **Perfect 2**, 1955: If $\sigma = \{\lambda_0, \lambda_1, \ldots, \lambda_r\} \cup \{\lambda_{01}, \ldots, \lambda_{0t_0}\} \cup \ldots \cup \{\lambda_{r1}, \ldots, \lambda_{rt_r}\}$, with $\{\lambda_0, \lambda_1, \ldots, \lambda_r\}$ the spectrum of a nonnegative matrix with diagonal $d_0, d_1, \ldots, d_r$ and $\lambda_{ji} \leq 0$ for $j = 0, \ldots, r$ and $i = 1, \ldots, t_j$:
$$\left.\lambda_0 \geq |\lambda|,\ \lambda \in \sigma\ ,\ \sum_{\lambda\in\sigma} \lambda \geq 0,\ d_j + \sum_{1\leq i\leq t_j} \lambda_{ji} \geq 0\ ,\ 0 \leq j \leq r \right\} \Rightarrow \sigma \in Spec\,.$$
When $\lambda_j \geq 0$, for $j = 0, \ldots, r$, we call it **Perfect 2^+** condition.
- **Ciarlet**, 1968: If $\sigma = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$:
$$\left.|\lambda_j| \leq \frac{\lambda_0}{n}\ ,\ j = 1, \ldots, n \right\} \Rightarrow \sigma \in Spec\,.$$
- **Kellogg**, 1971: If $\sigma = \{\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_n\}$, with $\lambda_0 \geq |\lambda|$ for $\lambda \in \sigma$, $M = \max\{j \in \{0, 1, \ldots, n\} \ :\ \lambda_j \geq 0\}$ and $K = \{i \in \{1, \ldots, \lfloor n/2 \rfloor\} \ :\ \lambda_i \geq 0,\ \lambda_i + \lambda_{n+1-i} < 0\}$:
$$\left.\begin{array}{r} \lambda_0 + \sum_{i\in K, i<s} (\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-s} \geq 0 \quad \forall s \in K \\ \lambda_0 + \sum_{i\in K} (\lambda_i + \lambda_{n+1-i}) + \sum_{j=M+1}^{n-M} \lambda_j \geq 0 \end{array}\right\} \Rightarrow \sigma \in Spec\,.$$
- **Salzmann**, 1972: If $\sigma = \{\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_n\}$:
$$\left.\sum_{\lambda\in\sigma} \lambda \geq 0 \quad \text{and} \quad \frac{\lambda_i + \lambda_{n-i}}{2} \leq \frac{1}{n+1} \sum_{0\leq j\leq n} \lambda_j\ ,\ 1 \leq i \leq \lfloor n/2 \rfloor \right\} \Rightarrow \sigma \in Spec\,.$$
- **Fiedler**, 1974: If $\sigma = \{\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_n\}$:
$$\left.\lambda_0 + \lambda_n + \sum_{\lambda\in\sigma} \lambda \geq \frac{1}{2} \sum_{1\leq i\leq n-1} |\lambda_i + \lambda_{n-i}| \right\} \Rightarrow \sigma \in Spec\,.$$

- **Borobia**, 1995: Let $\sigma = \{\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_n\}$ and $M = \max\{j \in \{0, \ldots, n\} : \lambda_j \geq 0\}$. If there exists a partition $J_1 \cup \ldots \cup J_t$ of $\{\lambda_{M+1}, \ldots, \lambda_n\}$ such that $\{\lambda_0 \geq \ldots \geq \lambda_M \geq \sum_{\lambda \in J_1} \lambda \geq \ldots \geq \sum_{\lambda \in J_t} \lambda\}$ satisfies the Kellogg condition, then $\sigma \in Spec$.
- **Soto 2**, 2003: Let $\sigma = \{\lambda_{11} \geq \ldots \geq \lambda_{1t_1}\} \cup \ldots \cup \{\lambda_{r1} \geq \ldots \geq \lambda_{rt_r}\}$, with $\lambda_{11} \geq |\lambda|$ for $\lambda \in \sigma$ and $\lambda_{i1} \geq 0$ for $i = 1, \ldots, r$. Let $S_{ij} = \lambda_{ij} + \lambda_{i,t_i-j+1}$ for $j = 2, \ldots, \lfloor t_i/2 \rfloor$, $S_{(t_i+1)/2} = \min\{\lambda_{(t_i+1)/2}, 0\}$ if t_i is odd for $i = 1, \ldots, r$, and $T_i = \lambda_{i1} + \lambda_{it_i} + \sum_{S_{ij}<0} S_{ij}$ for $i = 1, \ldots, r$:

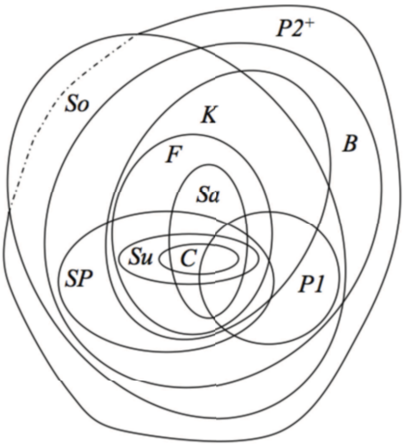
$$\left.\lambda_{11} \geq \max\{-\lambda_{1t_1} - \sum_{S_{1j}<0} S_{1j}\,,\ \max_{2\leq i\leq r}\{\lambda_{i1}\}\} - \sum_{T_i<0,\ 2\leq i\leq r} T_i \right\} \Rightarrow \sigma \in Spec\,.$$

- **Soto-Rojo**, 2006: If $\sigma = \{\lambda_{11} \geq \ldots \geq \lambda_{1t_1}\} \cup \ldots \cup \{\lambda_{r1} \geq \ldots \geq \lambda_{rt_r}\}$, with $\lambda_{11} \geq |\lambda|$ for $\lambda \in \sigma$, $\lambda_{i1} \geq 0$ for $i = 1, \ldots, r$ and $\{\lambda_{11}, \lambda_{21}, \ldots, \lambda_{r1}\}$ is the spectrum of a nonnegative natrix with diagonal $d_1, \ldots, d_r$, then

$$\left. d_i \geq \lambda_{i2} \quad \text{and} \quad \{d_i, \lambda_{i2}, \ldots, \lambda_{it_i}\} \in Spec\,,\ 1 \leq i \leq r \right\} \Rightarrow \sigma \in Spec\,.$$

If σ satisfies a sufficient condition X we say that σ is X-**realizable**. For example, if σ verifies the Fiedler condition we say that σ is Fiedler-realizable. If σ is given by a partition, σ_i denotes the i-th element of the partition.

The following map shows the relations between the previous sufficient conditions and appears in [3]. The dotted line on the map expresses the fact that the authors did not know if Soto 2 implies (or not) Perfect 2^+. Now we know that Soto 2 implies Perfect 2^+ (see Theorem 3.1).



Suleĭmanova = Su
Suleĭmanova-Perfect = SP
Perfect 1 = $P1$
Perfect 2^+ = $P2^+$
Ciarlet = C
Kellogg = K
Salzmann = Sa
Fiedler = F
Borobia = B
Soto 2 = So
($S2$ in Section 3)

Soto-Rojo clearly extends Perfect 2^+, so it contains all the sufficient conditions in the previous map. In [3] we did not know if the inclusion of Perfect 2^+ in Soto-Rojo is strict, but now we know it is (see Theorem 3.1).

2 Negativity and Realizability Margin

Let X be a sufficient condition. Following [1], we define the **X-negativity** of a family $\sigma = \{\lambda_1, , \dots, \lambda_n\} \subset \mathbb{R}$ with $\lambda_1 \geq \lambda_j$ for $j = 2, \dots, n$, as

$$\mathcal{N}_X(\sigma) = \min\{\epsilon \geq 0 : \{\lambda_1 + \epsilon, \lambda_2, \dots, \lambda_n\} \text{ is } X\text{-realizable}\}$$

and as $+\infty$ if $\{\lambda_1 + \epsilon, \lambda_2, \dots, \lambda_n\}$ is not X-realizable for any $\epsilon \geq 0$. If σ is X-realizable, we define the **X-margin of realizability** of σ as

$$\mathcal{M}_X(\sigma) = \max\left\{\epsilon \geq 0 : \{\lambda_1 - \epsilon, \lambda_2, \dots, \lambda_n\}\ X\text{-realizable},\ \lambda_1 - \epsilon \geq |\lambda_j|,\ j \geq 2\right\}.$$

Note that the X-negativity of a family σ measures how far the family is from being X realizable. A similar interpretation can be given for the realizability margin of an X-realizable list. The following properties hold:

- ◇ σ is X-realizable if and only if $\mathcal{N}_X(\sigma) = 0$.
- ◇ If $X \subset Y$, then $\mathcal{N}_X(\sigma) \geq \mathcal{N}_Y(\sigma)$ and $\mathcal{M}_X(\sigma) \leq \mathcal{M}_Y(\sigma)$.
- ◇ $\mathcal{N}_X(\sigma) \geq \max\{0, -\sum_{j=1}^n \lambda_j, |\lambda_2| - \lambda_1, \dots, |\lambda_n| - \lambda_1\}$.
- ◇ $\mathcal{M}_X(\sigma) \leq \min\{\sum_{j=1}^n \lambda_j\ , \lambda_1 - |\lambda_2|\ , \dots, \lambda_1 - |\lambda_n|\}$.
- ◇ If σ is X-realizable with trace zero, then $\mathcal{M}_X(\sigma) = 0$.
- ◇ If σ is X-realizable with multiple spectral radius, then $\mathcal{M}_X(\sigma) = 0$.
- ◇ If σ is X-realizable with cyclicity index 2, then $\mathcal{M}_X(\sigma) = 0$.

For a real spectrum σ, with the notations of the sufficient conditions and of the map given in the previous section, we have:

$$\mathcal{N}_{\mathrm{Su}}(\sigma) = \max\{0, -\lambda_0 - \sum_{\lambda_i<0} \lambda_i\},\ \mathcal{M}_{\mathrm{Su}}(\sigma) = \min\{\lambda_0 + \sum_{\lambda_i<0} \lambda_i\ , \lambda_0 - |\lambda_1|, \dots, \lambda_0 - |\lambda_n|\},$$

$$\mathcal{N}_{\mathrm{SP}}(\sigma) = \min\{\mathcal{N}_{\mathrm{Su}}(\sigma_0) : \sigma = \bigcup_{i=0}^r \sigma_i\},\ \mathcal{M}_{\mathrm{SP}}(\sigma) = \min\{\sum_{\lambda\in\sigma} \lambda, \min\{\lambda_0 - |\lambda| : \lambda \in \sigma - \{\lambda_0\}\}, a\},$$

where $a = \max\{\mathcal{M}_{\mathrm{Su}}(\sigma_0) : \sigma = \bigcup_{i=0}^r \sigma_i \text{ is } SP\text{-realizable}\}$,

$$\mathcal{N}_{\mathrm{P1}}(\sigma) = \begin{cases} +\infty & \text{if } \forall \text{partition of } \sigma,\ \exists j \in \{1, \dots, r\} \text{ with } \begin{cases} \lambda_j + \lambda_n > 0 \text{ or} \\ \lambda_j + \sum\limits_{i=1}^{t_j} \lambda_{ji} > 0 \end{cases} \\ \max\{0, -\sum\limits_{\lambda\in\sigma} \lambda, \max\{|\lambda| - \lambda_0 : \lambda \in \sigma - \{\lambda_0\}\}\} & \text{otherwise,} \end{cases}$$

$$\mathcal{M}_{\mathrm{P1}}(\sigma) = \min\{\sum_{\lambda\in\sigma} \lambda\ , \min\{\lambda_0 - |\lambda|\ : \lambda \in \sigma - \{\lambda_0\}\}\},$$

$$\left.\begin{matrix} \mathcal{M}_{\mathrm{P2}}(\sigma) \\ \mathcal{M}_{\mathrm{P2^+}}(\sigma) \end{matrix}\right\} \geq \min\left\{\sum_{i=0}^r \min\{d_i + \sum_{j=1}^{t_i} \lambda_{ij}, m_i\}, \min\{\lambda_0 - |\lambda|\ : \lambda \in \sigma - \{\lambda_0\}\}\right\},$$

where m_i, $0 \leq i \leq r$, is the minimum element of the column $i+1$ of the considered nonnegative matrix with spectrum $\{\lambda_0, \dots, \lambda_r\}$ and arises from the use of Brauer's theorem.

$$\mathcal{N}_{\rm C}(\sigma) = n\max\{|\lambda_1|,\dots,|\lambda_n|\} - \lambda_0,\ \mathcal{M}_{\rm C}(\sigma) = \lambda_0 - n\max\{|\lambda_1|,\dots,|\lambda_n|\},$$

$$\mathcal{N}_{\rm K}(\sigma) = -\min\Bigg\{0, \min\{\lambda_0 + \sum_{i\in K,\, i<s}(\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-s} : k \in K\},\ \lambda_0 + \sum_{i\in K}(\lambda_i + \lambda_{n+1-i}) + \sum_{j=M+1}^{n-M}\lambda_j\Bigg\},$$

$$\mathcal{M}_{\rm K}(\sigma) = \min\Bigg\{\lambda_0 - \lambda_1, \lambda_0 + \lambda_n, \min\{\lambda_0 + \sum_{i\in K,\, i<s}(\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-s} : k \in K\},\ \lambda_0 + \sum_{i\in K}(\lambda_i + \lambda_{n+1-i}) + \sum_{j=M+1}^{n-M}\lambda_j\Bigg\},$$

$$\mathcal{N}_{\rm Sa}(\sigma) = -\min\{0, \sum_{j=0}^{n}\lambda_j, \min_{1\le i\le \lfloor n/2\rfloor}\{\sum_{j=0}^{n}\lambda_j - \frac{n+1}{2}(\lambda_i + \lambda_{n-i})\}\},$$

$$\mathcal{M}_{\rm Sa}(\sigma) = \min\{\lambda_0 - |\lambda_n|, \lambda_0 - |\lambda_1|, \sum_{j=0}^{n}\lambda_j, \min_{1\le i\le \lfloor n/2\rfloor}\{\sum_{j=0}^{n}\lambda_j - \frac{n+1}{2}(\lambda_i + \lambda_{n-i})\}\},$$

$$\mathcal{N}_{\rm F}(\sigma) = \max\{0, -\sum_{\lambda\in\sigma}\lambda, -\frac{1}{2}[\lambda_0 + \lambda_n + \sum_{\lambda\in\sigma}\lambda - \frac{1}{2}\sum_{i=1}^{n-1}|\lambda_i + \lambda_{n-i}|]\},$$

$$\mathcal{M}_{\rm F}(\sigma) = \min\{\sum_{\lambda\in\sigma}\lambda, \lambda_0 - |\lambda_1|, \dots, \lambda_0 - |\lambda_n|, \frac{1}{2}[\lambda_0 + \lambda_n + \sum_{\lambda\in\sigma}\lambda - \frac{1}{2}\sum_{i=1}^{n-1}|\lambda_i + \lambda_{n-i}|]\}.$$

Note that the negativity of all the considered sufficient conditions is finite, except for Perfect 1. For Borobia and Soto 2 we only have brute force procedures for the construction of all possible partitions of a spectrum under the corresponding constraints. To obtain the negativity and the realizability margin for Perfect 2 and Soto-Rojo implies, on the one hand, brute force for the construction of the partitions and, on the other hand, the determination of the diagonals and the minimum elements of the columns of the matrices that realize them. In relation with these facts, we can consider the following open problems:
<u>Problem 1</u>: Find the set of the diagonals of all nonnegative realizations of a real spectrum.
<u>Problem 2</u>: Find the set of realizable spectra with given diagonal.

These two problems can be as complex as the RNIEP itself; in fact, the first of them is solved for $n \le 3$, but for $n > 3$ we only know sufficient conditions. If the diagonal is null, the problem 2 is the RNIEP with trace zero, only solved for $n \le 5$. There are several other equally complex problems involved: those related with the determination of the minimum elements of the columns realizing the spectra and their relation with the diagonal elements, and the determination of the maximum diagonal element. See the following example.

Example 2.1 The spectrum $\sigma = \{18, 6, -6, -2, -2, -3, -3\}$ is P2-realizable: $\sigma_0 = \{18, 6, -6\}$ is the spectrum of

$$A = \begin{pmatrix} 12 & 3 & 3 \\ 0 & 6 & 12 \\ 12 & 6 & 0 \end{pmatrix} \text{ with } \begin{array}{l} m_0 = 0 \\ m_1 = 3 \\ m_2 = 0 \end{array} \text{ or } B = \begin{pmatrix} 12 & 5 & 1 \\ 6 & 6 & 6 \\ 6 & 12 & 0 \end{pmatrix} \text{ with } \begin{array}{l} m_0 = 6 \\ m_1 = 5 \\ m_2 = 0 \end{array} \text{ and } \begin{array}{l} d_0 = 12 \\ d_1 = 6 \\ d_2 = 0 \end{array}.$$

For $\{-2, -2, -3, -3\} = \sigma_1 \cup \sigma_2 \cup \sigma_3$ we have several possible partitions

Partition $\sigma_1 \cup \sigma_2 \cup \sigma_3$	bound $\mathcal{M}_{\mathrm{P2}}(\sigma)$ with A	bound $\mathcal{M}_{\mathrm{P2}}(\sigma)$ with B
$\{-2, -2\} \cup \{-3, -3\} \cup \emptyset$	0	6
$\{-3, -3\} \cup \{-2, -2\} \cup \emptyset$	2	8
$\{-2, -2, -3\} \cup \{-3\} \cup \emptyset$	3	8
$\{-2, -2, -3, -3\} \cup \emptyset \cup \emptyset$	3	7

The spectrum σ also is P2$^+$-realizable: $\sigma_0 = \{18, 6\}$ is the spectrum of

$$C_a = \begin{pmatrix} 18 & 0 \\ a & 6 \end{pmatrix}, \quad a \geq 0, \quad \text{and} \quad D = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}.$$

Depending on the partitions of $\{-2, -2, -3, -3, -6\} = \sigma_1 \cup \sigma_2$, we obtain different bounds for $\mathcal{M}_{\mathrm{P2}}(\sigma)$. We have that $\mathcal{M}_{\mathrm{P2}}(\sigma) = \mathcal{M}_{\mathrm{P2}^+}(\sigma) = 8$. Note that the knowledge of the diagonal does not guarantee an optimum bound.

3 New Sufficient Conditions and New Relations

In [4] we prove that S2 is strictly contained in P2$^+$ (see the dotted line in the diagram of Section 1) and we consider the following new sufficient conditions for $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ to be the spectrum of a nonnegative matrix:

- **Game**, 2008: If σ, with n elements, can be reached starting from the n spectra $\{0\}, \ldots, \{0\}$ successively applying, in any order and any number of times, either Rule 1, Rule 2 or Rule 3, then $\sigma \in \mathit{Spec}$, where
  Rule 1: If $\sigma = \{\lambda_1, \ldots, \lambda_n\} \in \mathit{Spec}$ with $\lambda_1 \geq |\lambda|$ for $\lambda \in \sigma$, then $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \ldots, \lambda_n\} \in \mathit{Spec}$, $\quad \forall \epsilon > 0$.
  Rule 2: If $\sigma = \{\lambda_1, \ldots, \lambda_n\} \in \mathit{Spec}$ with $\lambda_1 \geq |\lambda|$ for $\lambda \in \sigma$, then $\{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\} \in \mathit{Spec}$, $\quad \forall \epsilon > 0$.
  Rule 3: If $\sigma_1, \sigma_2 \in \mathit{Spec}$, then $\sigma_1 \cup \sigma_2 \in \mathit{Spec}$.

Note that the game-realizabiliy of a spectrum is not changed by the inclusion or exclusion of 0's in it.

In [2] this sufficient condition is given as the concept of C-realizability. To avoid misunderstandings with the notation and confusions with the names of other sufficient conditions, we have called it the game condition.

- **Soto 3 = S3**, 2013: If $\sigma = \{\lambda_{11} \geq \ldots \geq \lambda_{1t_1}\} \cup \ldots \cup \{\lambda_{r1} \geq \ldots \geq \lambda_{rt_r}\}$, with $\lambda_{11} \geq |\lambda|$ for $\lambda \in \sigma$, $\lambda_{i1} \geq 0$, for $i = 1, \ldots, r$, and $\{\lambda_{11}, \ldots, \lambda_{1t_1}\}$ is Soto 2 realizable:
$$\left.\lambda_{11} \geq \max\{\lambda_{11} - \mathcal{M}_{S2}(\sigma_1), \max_{2\leq i\leq r}\{\lambda_{i1}\}\} + \sum_{i=2}^{r} \mathcal{N}_{S2}(\sigma_i)\right\} \Rightarrow \sigma \in Spec\,.$$
- **Soto p = Sp**, 2013: If $\sigma = \{\lambda_{11} \geq \ldots \geq \lambda_{1t_1}\} \cup \ldots \cup \{\lambda_{r1} \geq \ldots \geq \lambda_{rt_r}\}$, with $\lambda_{11} \geq |\lambda|$ for $\lambda \in \sigma$, $\lambda_{i1} \geq 0$, for $i = 1, \ldots, r$, and $\{\lambda_{11}, \ldots, \lambda_{1t_1}\}$ is Soto $p-1$ realizable with $p \geq 3$:
$$\left.\lambda_{11} \geq \max\{\lambda_{11} - \mathcal{M}_{Sp-1}(\sigma_1), \max_{2\leq i\leq r}\{\lambda_{i1}\}\} + \sum_{i=2}^{r} \mathcal{N}_{Sp-1}(\sigma_i)\right\} \Rightarrow \sigma \in Spec\,.$$
In practice, it is not necessary to know $\mathcal{M}_{Sp-1}(\sigma_1)$ to use the Soto p condition. It is enough to know a nonnegative lower bound of it, see [4,5].

The next results appear in [4] and we give them without proof. After we summarize the theorem in a map and give examples to explain it.

Theorem 3.1 (i) *Game implies Perfect* 2^+ *and the inclusion is strict.*

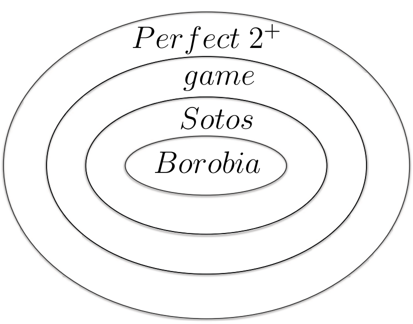
(ii) *The inclusion of Perfect* 2^+ *in Soto-Rojo is strict.*

(iii) *Soto* p *is strictly contained in Soto* $p+1$*, for* $p \geq 3$*.*

(iv) *Kellogg and Borobia are independent of Soto* p*, for* $p \geq 3$*.*

(v) *Soto* p *implies game, for* $p \geq 3$*, and the inclusion is strict.*

(vi) *If* σ *is Borobia realizable, then* σ *is Soto* p *realizable for some* p*.*



$Sotos = \bigcup_{p\geq 2} Soto\ p$

$\overline{X}$ = Condition X is not satisfied

$Sotos \cap \overline{Borobia} : \{3, 3, 1, 1, -2, -2, -2, -2\}$

$game \cap \overline{Sotos}$: ?

$Perfect\ 2^+ \cap \overline{game} : \{6, 1, 1, -4, -4\}$

The sufficient condition game is hardly algorithmizable and we do not have explicit expressions for $\mathcal{N}_{\text{game}}(\sigma)$ and $\mathcal{M}_{\text{game}}(\sigma)$. A necessary condition for game is that, in the ordered spectrum, the positive elements weakly majorize the negative elements. This fact provides a lower bound for the margin of realizability of a spectrum σ game-realizable that, without loss of generality, can assume the form $\sigma = \{\lambda_1 \geq \ldots \geq \lambda_q \geq 0 \geq -\mu_q \geq \ldots \geq -\mu_1\}$:

$$\mathcal{M}_{\text{game}}(\sigma) \geq \min\{\lambda_1 - \lambda_2, \sum_{\lambda\in\sigma} \lambda\, , \min_{1\leq k\leq q}\{\sum_{i=1}^{k}(\lambda_i - \mu_i)\}\}.$$

To calculate the negativity and the realizabilily margin with respect to Soto p, again, we only have brute force to construct all the possible partitions.

Example 3.2 The spectrum $\sigma = \{6, 3, 3, -5, -5\}$ is only realizable by Perfect 2^+ (so also by Perfect 2 and Soto-Rojo) and not by any of the other sufficient conditions included in Perfect 2^+. For this spectrum we have

$$\mathcal{N}_{\text{Su}}(\sigma) = 4\, , \quad \mathcal{N}_{\text{SP}}(\sigma) = 4\, , \quad \mathcal{N}_{\text{P1}}(\sigma) = +\infty\, , \quad \mathcal{N}_{\text{C}}(\sigma) = 14\, ,$$
$$\mathcal{N}_{\text{Sa}}(\sigma) = 13\, , \quad \mathcal{N}_{\text{F}}(\sigma) = 1\, , \quad \mathcal{N}_{\text{game}}(\sigma) = 1\, , \quad \mathcal{M}_{\text{P}2^+}(\sigma) = 0\, .$$

Since Kellogg, Borobia and Soto p are between Fiedler and game, we have that all the negativities with respect to them are 1.

Example 3.3 Let $\sigma = \{12, 6, 1, 1, 1, 1, -2, -3, -3, -4, -4\}$. We obtain

$$\mathcal{N}_{\text{C}}(\sigma) = 48\, , \quad \mathcal{N}_{\text{Su}}(\sigma) = 4\, , \quad \mathcal{N}_{\text{P1}}(\sigma) = +\infty\, , \quad \mathcal{N}_{\text{Sa}}(\sigma) = 5\, ,$$
$$\mathcal{M}_{\text{SP}}(\sigma) = 2\, , \quad \mathcal{M}_{\text{F}}(\sigma) = 3\, , \quad \mathcal{M}_{\text{K}}(\sigma) = 3\, , \quad \mathcal{M}_{\text{B}}(\sigma) = 4\, ,$$
$$\mathcal{M}_{\text{Sp}}(\sigma) = 5\, , \quad p \geq 2\, , \quad \mathcal{M}_{\text{game}}(\sigma) = 5\, , \quad \mathcal{M}_{\text{P}2^+}(\sigma) = 6\, .$$

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