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Submatrix monotonicity of the Perron root, II ☆

Susana Furtado [a], C.R. Johnson [b], C. Marijuán [c], M. Pisonero [d,*]

[a] *Faculdade de Economia, Universidade do Porto, Rua Dr. Roberto Frias, 4200-464 Porto, Portugal*
[b] *Dept. Mathematics, College of William and Mary, Williamsburg, VA, 23187 USA*
[c] *Dpto. Matemática Aplicada, E.T.S.I. Informática, Paseo de Belen 15, 47011-Valladolid, Spain*
[d] *Dpto. Matemática Aplicada, E.T.S. de Arquitectura, Avenida de Salamanca 18, 47014-Valladolid, Spain*

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ABSTRACT

The problem of comparing the Perron roots of two n-by-n nonnegative matrices, that differ only in a particular k-by-k principal submatrix, is considered. Several points of view are taken, under varying regularity conditions, and (at most) k polynomial conditions for the comparison are presented.

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* Corresponding author.
E-mail addresses: sbf@fep.up.pt (S. Furtado), crjohn@wm.edu (C.R. Johnson), marijuan@mat.uva.es (C. Marijuán), mpisoner@maf.uva.es (M. Pisonero).

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This work furthers ideas initiated in [2] by giving explicit conditions for a certain preorder defined there.

In general, it is difficult to compare the Perron roots of two given n-by-n nonnegative matrices G and H. Here, we study a special case in which G and H differ only in a certain principal submatrix, which, without loss of generality, we take to be the upper left k-by-k principal submatrix. Let A be k-by-k and

$$G(A) = \begin{pmatrix} A & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

be n-by-n, $0 < k \leq n$, in which G_{12}, G_{21} and G_{22} are given nonnegative matrices. Define $\rho_G(A) = \rho(G(A))$, the spectral radius of $G(A)$. When $G(A) \geq 0$, so that $\rho_G(A)$ is the Perron root of $G(A)$, we want to compare $\rho_G(A)$ and $\rho_G(B)$, and, in particular, to describe the set of $A \geq 0$ such that $\rho_G(A) < \rho_G(B)$ for a fixed $B \geq 0$. In this event, we write that $A \prec_G B$. Obviously, $\prec_G$ is a preorder on nonnegative k-by-k matrices. We write that $A \precsim_G B$ if $\rho_G(A) \leq \rho_G(B)$. Of course, if $k = 1$, by monotonicity of the Perron root, this partial order is just the total order on $\mathbb{R}_0^+$. In general, if, say A and B are nonnegative and $A \leq B$ in the entry-wise partial order, then $A \precsim_G B$, but not generally conversely.

If $G \geq 0$ is n-by-n, and $\rho > 0$ is given, by M-matrix theory [1,3], $\rho(G) < \rho$ if and only if $\rho I - G$ is an M-matrix. (When we say M-matrix, we mean a nonsingular M-matrix, otherwise we explicitly refer to a singular M-matrix.) Now, because of the determinantal characterization of M-matrices among the Z-matrices (nonpositive off-diagonal entries), $\rho(G) < \rho$ if and only if any nested sequence of n principal minors (PM's) of $\rho I - G$ is positive. And, if G is irreducible, $\rho(G) = \rho$ if and only if any nested sequence of n PM's of $\rho I - G$ has sign sequence $+, +, \ldots, +, 0$. Thus, given a fixed nonnegative k-by-k matrix B with $G(B)$ irreducible (a slightly weaker assumption could be made), the inequality $\rho_G(A) < \rho_G(B)$ may be checked via k polynomial inequalities in the entries of the k-by-k matrix $A \geq 0$. The polynomials may be taken to be the last k trailing PM's of $\rho I - G(A)$, for $\rho = \rho_G(B)$, as the first $n - k$ trailing PM's of $\rho I - G(A)$ are the same as those of $\rho I - G(B)$, which are positive. Thus, the set of nonnegative A's for which $A \prec_G B$ is a semi-algebraic set. However, according to the result of [2], $A \precsim_G B$ for all nonnegative G if and only if $A \leq B$ in the entry-wise partial order, i.e., the intersection of these complicated preorders is a simple one.

We may record these observations as one solution to our problem. We denote by $q_i(A)$ the $(n - k + i)$th trailing PM of $\rho_G(B)I - G(A)$, viewed as a polynomial in the entries of $A \geq 0$.

Theorem 1. *Let A and B be k-by-k nonnegative matrices. Assume that $G(B)$ is irreducible. Then*

i) *$A \prec_G B$ if and only if $q_i(A) > 0$, for $i = 1, \ldots, k$;*

ii) $B \precsim_G A$ *if and only if there is* $i \in \{1, \ldots, k\}$ *such that* $q_i(A) \le 0$*;*
iii) *for irreducible* $G(A)$*, we have* $\rho_G(A) = \rho_G(B)$ *if and only if* $q_i(A) > 0$*, for* $i = 1, \ldots, k-1$*, and* $q_k(A) = 0$*.*

We note that the assumption in iii) that $G(A)$ is irreducible is necessary and is not implied by the irreducibility of $G(B)$, as the next example shows.

Example 2. Let $A = (a_{ij})$ be a 2-by-2 matrix and

$$G(A) = \begin{pmatrix} a_{11} & a_{12} & 3 \\ a_{21} & a_{22} & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

For

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix},$$

the matrix $G(A)$ is reducible and the matrix $G(B)$ is irreducible. We have $\rho_G(A) = \rho_G(B) = 3$. However,

$$q_1(A) = \det \begin{pmatrix} 0 & 0 \\ -1 & 3-1 \end{pmatrix} = 0.$$

If $G(0)$ is irreducible, so is $G(A)$ for any k-by-k matrix A. We then have Corollary 3 as a consequence of Theorem 1.

Corollary 3. *Suppose* $G(0) \ge 0$ *is irreducible and* $A, B \ge 0$ *are* k*-by-*k*. Then*

i) $A \prec_G B$ *if and only if* $q_i(A) > 0$*, for* $i = 1, \ldots, k$*;*
ii) $B \precsim_G A$ *if and only if there is* $i \in \{1, \ldots, k\}$ *such that* $q_i(A) \le 0$*;*
iii) $\rho_G(A) = \rho_G(B)$ *if and only if* $q_i(A) > 0$*, for* $i = 1, \ldots, k-1$*, and* $q_k(A) = 0$*.*

The next example illustrates this result.

Example 4. Consider the 3-by-3 matrix

$$G(A) = \begin{pmatrix} a_{11} & a_{12} & 1 \\ a_{21} & a_{22} & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

in which $A = (a_{ij}) \ge 0$ is a 2-by-2 matrix. Note that $G(0)$ is irreducible. Let

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{1}$$

Then, $\rho_G(B) = 3$. By Corollary 3, $\rho_G(A) < 3$ if and only if

$$\det \begin{pmatrix} 3 - a_{22} & -1 \\ -1 & 2 \end{pmatrix} > 0 \quad \text{and} \quad \det\bigl(3I - G(A)\bigr) > 0,$$

that is,

$$a_{22} < \frac{5}{2} \tag{2}$$

and

$$5a_{11} + a_{12} + a_{21} + 5a_{22} - 2a_{11}a_{22} + 2a_{12}a_{21} < 12. \tag{3}$$

Note that condition (2) is not implied by condition (3).

We now turn to an alternate approach to the problem.

Theorem 5. *Let A and B be k-by-k nonnegative matrices. Assume that $\rho = \rho_G(B)$. Then the following statements are equivalent:*

i) $A \prec_G B$,
ii) $\rho I - G(A)$ *is an M-matrix,*
iii) $\det(\rho I - G_{22}) \neq 0$ *and* $\rho I - (A + G_{12}(\rho I - G_{22})^{-1} G_{21})$ *is an M-matrix, and*
iv) $\det(\rho I - G_{22}) \neq 0$ *and* $\rho(A + G_{12}(\rho I - G_{22})^{-1} G_{21}) < \rho$.

Proof. Since G_{22} is a principal submatrix of $G(B)$, $\rho(G_{22}) \leq \rho(G(B)) = \rho$; so, when $\rho I - G_{22}$ is nonsingular, it is an M-matrix. The equivalence of conditions i) and ii), and of conditions iii) and iv), follows from the comments before Theorem 1, given that $(\rho I - G_{22})^{-1} \geq 0$ because $\rho I - G_{22}$ is an M-matrix.

First, we show that condition ii) implies condition iii). Suppose that condition ii) holds. Then, since $\rho I - G(A)$ is an M-matrix, $\rho I - G_{22}$ (a principal submatrix) is an M-matrix and, so, its Schur complement in $\rho I - G(A)$ is also an M-matrix [4]. But this Schur complement is just

$$\rho I - \bigl(A + G_{12}(\rho I - G_{22})^{-1} G_{21}\bigr). \tag{4}$$

Since $\rho I - G_{22}$ is an M-matrix, it is invertible, so that iii) (and iv)) is proven.

Finally, we show that condition iii) implies condition ii). Suppose that condition iii) holds. Since $\rho I - G_{22}$ is invertible, it is an M-matrix as mentioned. So, in the Z-matrix $\rho I - G(A)$, we have that both $\rho I - G_{22}$ and its Schur complement (by condition iii)) are M-matrices. But, again using [4], a Z-matrix is an M-matrix if both a principal submatrix and its Schur complement are M-matrices, completing the proof of condition ii) and the theorem. □

We now need a purely algebraic observation.

Lemma 6. *Let*

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & G_{22} \end{pmatrix}$$

be an n-by-n matrix, in which g_{11} is a scalar, g_{12} is a row-vector and g_{21} is a column-vector. Then $\lambda \in \sigma(G)$ if and only if

$$g_{11} \det(G_{22} - \lambda I) = -\det \begin{pmatrix} -\lambda & g_{12} \\ g_{21} & G_{22} - \lambda I \end{pmatrix}.$$

So, if $\lambda \notin \sigma(G_{22})$, then $\lambda \in \sigma(G)$ if and only if

$$g_{11} = -\frac{\det\left(\begin{smallmatrix} -\lambda & g_{12} \\ g_{21} & G_{22}-\lambda I \end{smallmatrix}\right)}{\det(G_{22} - \lambda I)}.$$

Proof. The claim follows from the fact that $\lambda \in \sigma(G)$ if and only if $\det(G - \lambda I) = 0$, and

$$\det(G - \lambda I) = \det \begin{pmatrix} g_{11} & 0 \\ g_{21} & G_{22} - \lambda I \end{pmatrix} + \det \begin{pmatrix} -\lambda & g_{12} \\ g_{21} & G_{22} - \lambda I \end{pmatrix}$$
$$= g_{11} \det(G_{22} - \lambda I) + \det \begin{pmatrix} -\lambda & g_{12} \\ g_{21} & G_{22} - \lambda I \end{pmatrix}. \qquad \square$$

We may now characterize $A \geq 0$ such that $\rho_G(B) = \rho_G(A)$ for a fixed B, when $G(A)$ is irreducible.

In what follows $H(1)$ denotes the matrix obtained from H by deleting the first row and the first column. Also, E_{11} denotes the matrix of appropriate size with all entries equal to 0 except the one in position $(1, 1)$, which is 1.

Theorem 7. *Let $A = (a_{ij})$ and B be k-by-k matrices. Suppose that B is nonnegative and $G(A)$ is irreducible. Let $\rho = \rho_G(B)$, $H = G(A)$ and $H' = H - a_{11}E_{11}$. Then H is a nonnegative matrix with $\rho(H) = \rho$ if and only if*

i) $a_{ij} \geq 0$, *for* $i, j = 1, \ldots, k$ *and* $(i, j) \neq (1, 1)$,
ii) $\rho(H') \leq \rho$, *and*
iii) $\det(H(1) - \rho I) \neq 0$ *and* $a_{11} = -\frac{\det(H' - \rho I)}{\det(H(1) - \rho I)}$.

Proof. ($\Rightarrow$) Suppose that $H = G(A)$ is a nonnegative matrix with $\rho(H) = \rho$. Condition i) is obvious. Condition ii) holds because the Perron-root is a nondecreasing function of any entry of the matrix and $a_{11} \geq 0$. Condition iii) follows from Lemma 6, taking into account that ρ is an eigenvalue of H. Note that $\det(H(1) - \rho I) \neq 0$ as $\rho(H(1)) < \rho(H) = \rho$, where the inequality follows because H is irreducible [1, Chapter 2, Corollary 1.5]. Thus, ρ is not an eigenvalue of $H(1)$.

($\Leftarrow$) Suppose that conditions i), ii) and iii) are satisfied. Because of condition i), H is nonnegative if and only if $a_{11} \geq 0$. We have $\rho(H(1)) < \rho(H') \leq \rho$. Thus, $\rho I - H(1)$ and $\rho I - H'$ are M-matrices (possibly, the latter is singular) of sizes $(n-1)$-by-$(n-1)$ and n-by-n, respectively. Thus, either $a_{11} = 0$ or the signs of $\det(H' - \rho I)$ and $\det(H(1) - \rho I)$ are the signs of $(-1)^n$ and $(-1)^{n-1}$, respectively. In any case, we get $a_{11} \geq 0$. By condition iii) and Lemma 6, ρ is an eigenvalue of $G(A)$. Thus, $B \precsim_G A$. Suppose that $B \prec_G A$. Then, by decreasing the entry in position $(1,1)$ in $G(A)$, we would get a matrix $G(A')$ with an eigenvalue equal to ρ, a contradiction as, by Lemma 6, ρ is an eigenvalue of $G(A')$ if and only if the entry in position $(1,1)$ is the right hand side of condition iii). □

The next example illustrates the result for the same matrices used in Example 4. Note the differences in the conditions.

Example 8. Consider the 3-by-3 matrix

$$H = G(A) = \begin{pmatrix} a_{11} & a_{12} & 1 \\ a_{21} & a_{22} & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

in which $A = (a_{ij})$ is a 2-by-2 matrix, and let B be as in (1). Then, $\rho = \rho_G(B) = 3$. Let $H' = H - a_{11}E_{11}$. By Theorem 7, H is a nonnegative matrix such that $\rho(H) = 3$ if and only if $a_{12}, a_{21}, a_{22} \geq 0$,

$$\rho(H') \leq 3 \tag{5}$$

and

$$a_{11} = -\frac{\det(H' - 3I)}{\det(H(1) - 3I)}. \tag{6}$$

A calculation shows that (6) is equivalent to

$$a_{11} = -\frac{a_{12} + a_{21} + 5a_{22} + 2a_{12}a_{21} - 12}{5 - 2a_{22}}. \tag{7}$$

We now show that inequality (5) is equivalent to $\det(3I - H') \geq 0$, or, equivalently,

$$a_{12} + a_{21} + 5a_{22} + 2a_{12}a_{21} \leq 12. \tag{8}$$

If inequality (5) holds then $3I - H'$ is a possibly singular M-matrix and, therefore, $\det(3I - H') \geq 0$. Conversely, if $\det(3I - H') \geq 0$, from (8) we have $a_{22} < 3$. Therefore, $3I - H'$ is a 3-by-3 Z^+-matrix with nonnegative determinant, which implies that $3I - H'$ is a possibly singular M-matrix. Then, (5) holds. Thus, we conclude that H is a nonnegative matrix such that $\rho(H) = 3$ if and only if $a_{12}, a_{21}, a_{22} \geq 0$ and conditions (7) and (8) hold.

We may now characterize the $A \geq 0$ such that $\rho_G(A) \geq \rho_G(B)$ for a fixed B, when $G(A)$ is irreducible.

Corollary 9. *Let $A = (a_{ij})$ and B be k-by-k matrices. Suppose that B is nonnegative and $G(A)$ is irreducible. Let $\rho = \rho_G(B)$, $H = G(A)$ and $H' = H - a_{11}E_{11}$. Then H is a nonnegative matrix such that $\rho(H) \geq \rho$ if and only if*

i) $a_{ij} \geq 0$, *for* $i, j = 1, \ldots, k$ *and* $(i, j) \neq (1, 1)$, *and either*
ii') $\rho(H') > \rho$ *and*
iii') $a_{11} \geq 0$

or

ii") $\rho(H') \leq \rho$ *and*
iii") $\det(H(1) - \rho I) \neq 0$ *and* $a_{11} \geq -\frac{\det(H' - \rho I)}{\det(H(1) - \rho I)}$.

Proof. (⇒) Suppose that H is a nonnegative matrix such that $\rho(H) \geq \rho$. Conditions i) and iii') are obvious. Suppose that condition ii') does not hold. Then we want to show that condition iii") holds. By Theorem 7, $\rho_G(A') = \rho$, in which A' is obtained from A by replacing the entry in position $(1, 1)$ by a'_{11}, the right hand side of the inequality in condition iii"). Note that, since H is irreducible, so is $G(A')$. Then, by the strict monotonicity of the Perron-root, for $a_{11} < a'_{11}$ we have $\rho(H) < \rho$.

(⇐) Suppose that conditions i), ii') and iii') hold. Clearly, $H \geq 0$. By monotonicity, $\rho(H) \geq \rho(H') > \rho$. Now suppose that conditions i), ii") and iii") hold. By monotonicity, we have $\rho(H) \geq \rho_G(A')$, in which A' is obtained from A by replacing the entry in position $(1, 1)$ by the right hand side of condition iii"). By Theorem 7, $G(A') \geq 0$ and $\rho_G(A') = \rho$. Thus, $H \geq 0$ and $\rho(H) \geq \rho$. □

We close by noting that our approach also solves problems that are more general in terms of the placement of fixed entries. The entries in which the two n-by-n matrices $F, H \geq 0$ are allowed to differ may be assumed to be in any scattering of positions that avoid an $(n-k)$-by-$(n-k)$ principal submatrix. In fact, if F and H differ only in positions contained in a certain p-by-ℓ submatrix, not necessarily square, then, by permutation similarity, we can assume that, for $k = \min\{p, \ell\}$, F and H are matrices with the lower right $(n - k)$-by-$(n - k)$ principal submatrix in common.

The results given when A is k-by-k and principal remain valid in the general situation mentioned above, with obvious adaptations. The number of polynomials q_i that are necessary is n, less the size of the largest principal submatrix in which F and H are the same. They are polynomials whose variables are the entries in which F and H are allowed to differ, and the coefficients are polynomials in the common entries of F and H.

Example 10. Consider the 3-by-3 matrices

$$G(A) = \begin{pmatrix} a_{11} & a_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ a_{21} & a_{22} & g_{33} \end{pmatrix} \quad \text{and} \quad G(B) = \begin{pmatrix} b_{11} & b_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ b_{21} & b_{22} & g_{33} \end{pmatrix}$$

with $\rho = \rho(G(B))$. Then, we have the polynomials

$$q_1(A) = \det \begin{pmatrix} \rho - g_{22} & -g_{23} \\ -a_{22} & \rho - g_{33} \end{pmatrix} = -g_{23}a_{22} + (\rho - g_{22})(\rho - g_{33})$$

and

$$\begin{aligned} q_2(A) = \det\bigl(\rho I - G(A)\bigr) &= g_{23}a_{11}a_{22} - g_{23}a_{12}a_{21} - (\rho - g_{22})(\rho - g_{33})a_{11} \\ &\quad - g_{21}(\rho - g_{33})a_{12} - g_{13}(\rho - g_{22})a_{21} - (\rho g_{23} + g_{13}g_{21})a_{22} + \rho(\rho - g_{22})(\rho - g_{33}). \end{aligned}$$

Note that $q_1(A)$ and $q_2(A)$ are polynomials in a_{11}, a_{12}, a_{21} and a_{22}, i.e., in the entries in which $G(A)$ and $G(B)$ are allowed to differ. Note also that the coefficients of $q_1(A)$ and $q_2(A)$, as polynomials in the entries of A, are themselves polynomials in g_{13}, g_{21}, g_{22}, g_{23} and g_{33}, i.e., in the common entries of $G(A)$ and $G(B)$.

In particular, for

$$G(A) = \begin{pmatrix} a_{11} & a_{12} & 1 \\ 1 & 1 & 1 \\ a_{21} & a_{22} & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

we have $\rho = \rho(G(B)) = 3$,

$$q_1(A) = \det \begin{pmatrix} 2 & -1 \\ -a_{22} & 2 \end{pmatrix} = -a_{22} + 4$$

and

$$q_2(A) = \det\bigl(3I - G(A)\bigr) = a_{11}a_{22} - a_{12}a_{21} - 4a_{11} - 2a_{12} - 2a_{21} - 4a_{22} + 12.$$

We have $\rho(G(A)) < \rho$ if and only if

$$a_{22} < 4$$

and

$$-a_{11}a_{22} + a_{12}a_{21} + 4a_{11} + 2a_{12} + 2a_{21} + 4a_{22} < 12.$$

Note that these conditions are different from the conditions (2) and (3) obtained in Example 4.

In the next example the entries in which the matrices are allowed to differ do not form a submatrix.

Example 11. Consider the 3-by-3 matrices

$$G_a = \begin{pmatrix} a_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad G_b = \begin{pmatrix} b_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & b_{33} \end{pmatrix}$$

with $\rho = \rho(G_b)$. The matrices G_a and G_b are simultaneously permutation similar to

$$G'_a = \left(\begin{array}{cc|c} a_{11} & g_{13} & g_{12} \\ g_{31} & a_{33} & g_{32} \\ \hline g_{21} & g_{23} & g_{22} \end{array}\right) \quad \text{and} \quad G'_b = \left(\begin{array}{cc|c} b_{11} & g_{13} & g_{12} \\ g_{31} & b_{33} & g_{32} \\ \hline g_{21} & g_{23} & g_{22} \end{array}\right).$$

Then, we have the polynomials

$$q_1^a = \det\begin{pmatrix} \rho - a_{33} & -g_{32} \\ -g_{23} & \rho - g_{22} \end{pmatrix} = -g_{23}g_{32} + (\rho - a_{33})(\rho - g_{22})$$

and

$$\begin{aligned} q_2^a = \det(\rho I - G_a) &= (\rho - a_{11})(\rho - a_{33})(\rho - g_{22}) - g_{32}g_{23}(\rho - a_{11}) \\ &\quad - g_{31}g_{13}(\rho - g_{22}) - g_{31}g_{12}g_{23} - g_{21}g_{13}g_{32} - g_{21}g_{12}(\rho - a_{33}). \end{aligned}$$

In particular, for

$$G_a = \begin{pmatrix} a_{11} & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & a_{33} \end{pmatrix} \quad \text{and} \quad b_{11} = b_{33} = 1,$$

we have $\rho = \rho(G_b) = 3$,

$$q_1^a = \det\begin{pmatrix} 3 - a_{33} & -1 \\ -1 & 2 \end{pmatrix} = -2a_{33} + 5$$

and

$$q_2^a = \det(3I - G_a) = 2a_{11}a_{33} - 5a_{33} - 5a_{11} + 8.$$

We have $\rho(G_a) < \rho$ if and only if

$$a_{33} < 5/2$$

and

$$5a_{33} + 5a_{11} - 2a_{11}a_{33} < 8.$$

We note that our results may be interpreted as comparing $\rho_G(A)$ to a reference value ρ, irrespective of B.

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