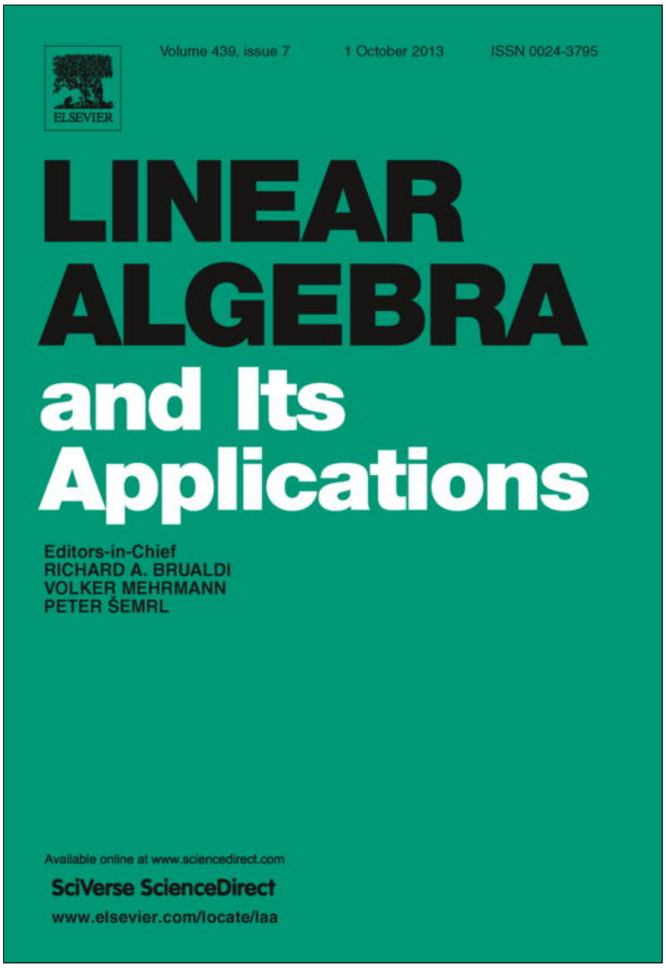
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LINEAR ALGEBRA and Its Applications

Inequalities for linear combinations of monomials in p-Newton sequences ☆

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The partial order on monomials that corresponds to domination when evaluated at positive Newton sequences is fully understood. Here we take up the corresponding partial order on linear combinations of monomials. In part using analysis based upon the cone structure of the exponents in p-Newton sequences, an array of conditions is given for this new partial order. It appears that a characterization in general will be difficult. Within the case in which all coefficients are 1, the situation in which, for general sequence length, there are two monomials, each of length two and nonnegative integer exponents, the partial order is fully characterized. The characterization is combinatorial, in terms of indices in the monomials, and, already here there is much more than termwise domination.

1. Introduction

Let A be an n-by-n real matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Denote the principal submatrix of A lying in the rows and columns indexed by $\alpha \subseteq N = \{1, \ldots, n\}$ by $A[\alpha]$. Define the *k-th elementary symmetric function* of $\lambda_1, \ldots, \lambda_n$ by

$$S_k = S_k(A) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

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and the *k-th Newton coefficient* by

$$c_k = c_k(A) = \binom{n}{k}^{-1} S_k,$$

$k = 1, \ldots, n$, with $S_0 = 1$. Of course, since

$$S_k(A) = \sum_{|\alpha|=k} \det A[\alpha],$$

as well, $c_k(A)$ may be viewed as the average value of the k-by-k principal minors of A. The matrix A, its spectrum $\lambda_1, \ldots, \lambda_n$, or the sequence $c_0, c_1, \ldots, c_n$ is called *Newton* if

$$c_{k-1}c_{k+1} \leqslant c_k^2, \quad k = 1, \ldots, n-1$$

and these inequalities are referred to as the *Newton inequalities* [8]. If, further, $c_k > 0$, $k = 1, \ldots, n$, each is called *p-Newton*. It is known that if the eigenvalues of A are real, A is Newton and that if the eigenvalues are positive, A is an $\mathcal{M}$-matrix or inverse $\mathcal{M}$-matrix or, under further circumstances [3,4], that A is p-Newton. Of course, this includes positive definite and totally positive matrices.

The two sides of the Newton inequalities are particular monomials in the Newton coefficients $c_0, c_1, \ldots, c_n$. We henceforth assume that our matrix A is p-Newton and that A is n-by-n. For any nonnegative exponents $a_0, a_1, \ldots, a_n$, by a *monomial* in the c_i's, we mean an expression of the form

$$c^a := c_0^{a_0} c_1^{a_1} \cdots c_n^{a_n}.$$

In [5], we addressed and fully answered the question for which pairs of monomials c^a and c^b do we have

$$c^a \leqslant c^b$$

for all p-Newton sequences

$$c: \quad c_0, c_1, \ldots, c_n?$$

In this event, we say that the monomial c^b *dominates* the monomial c^a (with respect to p-Newton sequences). The answer is a certain generalization of the sequence a being dominated by b. Since the c_k's may be viewed as average values of k-by-k principal minors, we were motivated, in part, by the study of determinantal inequalities in p-Newton matrices.

Given a monomial $c^a = c_0^{a_0} c_1^{a_1} \cdots c_n^{a_n}$, the *length* of c^a is the number $a_0 + a_1 + \cdots + a_n$, and the *weight* of c^a is the sum of the indices weighted by their exponents in the monomial.

Here, we consider (positive) linear combinations of monomials in $c_0, c_1, \ldots, c_n$. For (positive) coefficients $\alpha_1, \ldots, \alpha_h$ and (nonnegative) exponent sequences $a(j)$: $a_0(j), a_1(j), \ldots, a_n(j)$, $j = 1, \ldots, h$, define the linear combination

$$c_{\alpha,a} = \alpha_1 c^{a(1)} + \alpha_2 c^{a(2)} + \cdots + \alpha_h c^{a(h)}$$

of monomials $c^{a(j)}$, $j = 1, \ldots, h$. Our purpose is to raise the question of for which such pairs $c_{\alpha,a}$ and $c_{\beta,b}$ of linear combinations of monomials, we have

$$c_{\alpha,a} \leqslant c_{\beta,b}$$

for all p-Newton sequences c: $c_0, c_1, \ldots, c_n$. This question also has an interpretation in terms of principal minor determinantal inequalities common to p-Newton matrices.

Of course, when the number of monomials in each linear combination is the same and the coefficients are all 1, term-wise domination of the a monomials by the b monomials, under a 1–1 correspondence is sufficient, but we will see that it is not necessary, even for sums of two monomials. After further discussion of the single monomial case, we describe the cone of exponents for p-Newton sequences relative to a given base and describe further how our linear combinations lead to exponential polynomials. Then, we give necessary conditions for our general problem and give alternate versions of our general problem, both in terms of roots of exponential polynomials and in terms of an intriguing partial order on (non-square) row stochastic matrices. The important idea that index complementation preserves inequalities is identified here. While some information can be gained from these variations, both make it clear that our problem is difficult, even when all coefficients in the linear combinations are 1's. We hope that these variations will lead to further study by those who find interest in them. Finally, we discuss the case of $h=2$ monomials in greater detail when the coefficients are 1's. We completely solve our problem in this case, when each monomial has length two with nonnegative integer exponents, in terms of the indices of the Newton coefficients appearing in the monomials. Our result shows exactly how inequalities for linear combinations go beyond term-wise domination. We prove, also for this case, that increasing by 1 the highest index of each monomial of an inequality gives another valid inequality.

2. Single monomial inequalities

In [5] we studied for which pairs of single monomials c^a and c^b do we have

$$c^a \leqslant c^b$$

for all p-Newton sequences c: $c_0, c_1, \ldots, c_n$? That is, when the single monomial c^b dominates the single monomial c^a (with respect to p-Newton sequences). See also [6].

Already in [4] and partly in [1,2] it was shown that in any p-Newton matrix, the inequalities

$$c_r c_s \leqslant c_p c_q \tag{1}$$

hold when $r < p \leqslant q < s$ and $p+q = r+s$. Of course, also the product of several such inequalities will give an inequality. Now, the special, 2-term single monomial inequality above is the special case in which the subscripts appearing in the dominant single monomial strictly "dominate" those appearing in the smaller, *i.e.*

$$r < p$$

and

$$p+q = r+s.$$

This suggested "domination" in the subscripts, which turns out to be part of an answer. The rest is an interesting generalization of domination (also called majorization).

One list of integers $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$ is said to be *dominated* by another list $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k$ if

$$\begin{aligned}
&i_1 \leqslant j_1,\\
&i_1 + i_2 \leqslant j_1 + j_2,\\
&\quad\vdots\\
&i_1 + i_2 + \cdots + i_{k-1} \leqslant j_1 + j_2 + \cdots + j_{k-1}
\end{aligned}$$

and

$$i_1 + i_2 + \cdots + i_k = j_1 + j_2 + \cdots + j_k.$$

Note that this definition is the same for lists of real numbers, but only the integer case interests us from classical domination. In our setting, the i's and j's are subscripts that appear in the Newton coefficients in two single monomials, and each exponent of a c that appears is 1, with repeats allowed.

What, then, if the exponents are not integers? The integer case is expanded to the rational exponent case by powering, and then the rational case to the general real case by a density argument based upon the fact that the relevant exponent vector pairs form a cone in the appropriate real space and thus that the rational points therein are dense. In [5] it was proved that if $c^a \leqslant c^b$ for all p-Newton sequences, then

$$\sum_{j=0}^{n} ja_j = \sum_{j=0}^{n} jb_j$$

and that we may assume

$$\sum_{j=0}^{n} a_j = \sum_{j=0}^{n} b_j := L.$$

What is the appropriate analog if the exponents are not integers? For the monomial c^a, we define a step function F_a as follows. For $0 \leqslant z < \sum_{j=0}^{n} a_j = L$,

$$F_a(z) = i$$

if and only if $a_i > 0$ and $\sum_{j<i} a_j \leqslant z < \sum_{j\leqslant i} a_i$. For $z \geqslant L$, $F_a(z) = 0$. Now, for the two exponent sequences in c^a and c^b, a and b, we may define (generalized) *domination* as follows. We say $a \preceq b$ if

$$\int_0^x F_a(z)\,dz \leqslant \int_0^x F_b(z)\,dz$$

for $0 \leqslant x < \sum_{j=0}^{n} a_j = \sum_{j=0}^{n} b_j = L$, with equality for $x \geqslant L$. We note that $\int_0^L F_a(z)\,dz = \sum_{j=0}^{n} ja_j$, and that when the a_j's and b_j's are integers, the new notion of domination coincides with the classical one.

With this definition of domination in hand, in [5] we proved the next result:

Theorem 1. *The monomial c^b dominates c^a with respect to p-Newton sequences if and only if $a \preceq b$.*

To show the necessity of domination for single monomial domination we designed appropiate p-Newton sequences, see [5]. For a positive parameter r and a nonnegative integer i, define the sequence $Q_{n,i}(r)$ as

$$1, r, r^2, \ldots, r^i, r^i, \ldots, r^i,$$

i.e. this sequence of $n+1$ terms, beginning with term 0, starts as a geometric sequence with base r and then becomes constant starting with term i.

Proposition 2. *The sequences $Q_{n,n}(r)$ and $Q_{n,0}(r)$ are p-Newton for any $r > 0$, while $Q_{n,i}(r)$ is p-Newton, $0 < i < n$, for any $r \geqslant 1$.*

Remark 3. In this section (and this section only) we have used the term "domination" for what is often called (and was called in [5]) "majorization" [7]. Unfortunately, there is another opposing version of majorization which is convenient for us to use later in this work, and we reserve the term majorization for that. If, in our definition of domination, we instead write $i_1 \geqslant i_2 \geqslant \cdots \geqslant i_k$ and $j_1 \geqslant j_2 \geqslant \cdots \geqslant j_k$ (and no other change), we call the resulting relation *majorization*. Furthermore, in this form, if the final equality

$$i_1 + i_2 + \cdots + i_k = j_1 + j_2 + \cdots + j_k$$

is instead a weak inequality in the same direction as the others

$$i_1 + i_2 + \cdots + i_k \leqslant j_1 + j_2 + \cdots + j_k,$$

the resulting relation is called *weak majorization*. It is not so convenient to define weak majorization from domination, although majorization is simply the opposite of domination.

3. The exponential cone of p-Newton sequences

If we fix a number $r > 1$, each element c_i of a positive sequence c may be written as

$$c_i = r^{x_i}, \quad i = 0, 1, \dots, n.$$

The sequence c is then p-Newton if the exponent vector $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ satisfies

$$x_{k-1} + x_{k+1} \leqslant 2x_k, \quad k = 1, \dots, n-1, \tag{2}$$

or if for

$$d_k = (0, \dots, 0, -1, 2, -1, 0, \dots, 0), \tag{3}$$

with 2 in the k-th position, we have $x \cdot d_k \geqslant 0$ for $k = 1, \dots, n-1$. Since $x_0 \equiv 0$, these linear inequality constraints mean that the exponent vectors of p-Newton sequences form a cone $\mathcal{C}^{\uparrow}$ in $\mathbb{R}^n$.

We note that a p-Newton sequence is necessarily unimodal: it is either (weakly) increasing, (weakly) decreasing or (weakly) increasing and then decreasing. The exponents can become (arbitrarily) negative, but once they become negative, they stay negative and become more negative. When increasing, the sequence increases at most geometrically and, when decreasing, it decreases at least geometrically. In case the exponents do not become negative, the exponent vectors of the p-Newton sequences form a finitely generated subcone $\mathcal{C}^+$ of the nonnegative orthant in $\mathbb{R}^n$, as the addition of the constraint $x_n \geqslant 0$ (equivalent to $x \geqslant 0$) makes the cone simplicial. We give the generators of this cone below. For a particular sequence, $x_n \geqslant 0$ can be arranged. Since a (positive) multiple of a p-Newton sequence remains (quadratic homogeneity of the inequalities) p-Newton (though the normalization $c_0 = 1$ is lost) we may multiply by a sufficiently large scalar to achive $x_n \geqslant 0$. However, this cannot be arranged for the entire cone by a finite multiple, as x_n may be arbitrarily negative in the cone.

Theorem 4. *The generators of the cone $\mathcal{C}^+$ are $(1, 2, \dots, n)$, $(n-1, n-2, \dots, 1, 0)$ and*

$$\begin{aligned}&\big(n-1-j, 2(n-1-j), \dots, j(n-1-j), (j+1)\big(n-(j+1)\big),\\ &\quad (j+1)\big(n-(j+2)\big), \dots, (j+1)2, j+1, 0\big)\end{aligned}$$

for $j = 1, \dots, n-2$.

Proof. The generators of $\mathcal{C}^+$ are the solutions of the homogeneous linear systems in n unknowns and $n-1$ equations

$$S_{n-1} \equiv \begin{cases} -2x_1 + x_2 = 0, \\ x_1 - 2x_2 + x_3 = 0, \\ x_2 - 2x_3 + x_4 = 0, \\ \quad \vdots \\ x_{n-2} - 2x_{n-1} + x_n = 0, \end{cases} \qquad S_0 \equiv \begin{cases} x_1 - 2x_2 + x_3 = 0, \\ x_2 - 2x_3 + x_4 = 0, \\ \quad \vdots \\ x_{n-2} - 2x_{n-1} + x_n = 0, \\ x_n = 0, \end{cases}$$

$$S_j \equiv \begin{cases} -2x_1 + x_2 = 0, \\ x_1 - 2x_2 + x_3 = 0, \\ \quad \vdots \\ x_{j-1} - 2x_j + x_{j+1} = 0, \\ x_{j+1} - 2x_{j+2} + x_{j+3} = 0, \\ \quad \vdots \\ x_{n-2} - 2x_{n-1} + x_n = 0, \\ x_n = 0, \end{cases} \qquad j = 1, \ldots, n-2.$$

By induction on k for the solutions of S_{n-1} we have

$$x_k = kx_1, \quad k = 2, \ldots, n,$$

which gives the generator $(1, 2, \ldots, n)$.

By induction on k for the solutions of S_0 we have

$$x_{n-k} = kx_{n-1}, \quad k = 2, \ldots, n-1,$$

which gives the generator $(n-1, n-2, \ldots, 1, 0)$.

Let $j \in \{1, \ldots, n-2\}$, again by induction on k for the solutions of S_j we have

$$x_k = kx_1, \quad k = 2, \ldots, j+1$$

and

$$x_{n-k} = kx_{n-1}, \quad k = 2, \ldots, n-(j+1) \quad \Longleftrightarrow \quad x_k = (n-k)x_{n-1}, \quad k = j+1, \ldots, n-2.$$

Therefore

$$x_{j+1} = (j+1)x_1 = \bigl(n-(j+1)\bigr)x_{n-1} \quad \Longrightarrow \quad x_{n-1} = \frac{j+1}{n-1-j}x_1$$

and this gives the generator

$$\bigl(n-1-j, 2(n-1-j), \ldots, j(n-1-j), (j+1)\bigl(n-(j+1)\bigr), \\ (j+1)\bigl(n-(j+2)\bigr), \ldots, (j+1)2, j+1, 0\bigr). \qquad \square$$

Example 5. So according to Theorem 4, for $n = 6$ the generators of the cone $\mathcal{C}^+$ are

$$(1,2,3,4,5,6), \quad (5,4,3,2,1,0), \quad (4,8,6,4,2,0), \quad (3,6,9,6,3,0), \\ (2,4,6,8,4,0) \quad \text{and} \quad (1,2,3,4,5,0).$$

The study of an inequality over the generators of $\mathcal{C}^+$ does not guarantee the inequality over the whole cone $\mathcal{C}^+$.

Remark 6. The inequality $c_1^2c_2^2 \geqslant 3c_1c_2$ is not true for all p-Newton sequences.

$$c_1^2c_2^2 \geqslant 3c_1c_2 \quad \Longleftrightarrow \quad r^{2x_1+2x_2} \geqslant 3r^{x_1+x_2}.$$

Consider $r = 3$, so that the previous inequality becomes

$$3^{2x_1+2x_2} \geqslant 3 \cdot 3^{x_1+x_2} = 3^{x_1+x_2+1}. \tag{4}$$

For $n = 2$ the generators of the cone $\mathcal{C}^+$ are $(1,2)$ and $(1,0)$, and clearly both of them satisfy inequality (4). Note also that $(\frac{1}{3}, 0) \in \mathcal{C}^+$ and does not satisfy inequality (4).

4. General theory and necessary conditions

Using the exponents x, the monomial c^a may be written as

$$c^a = \left(r^{x_0}\right)^{a_0}\left(r^{x_1}\right)^{a_1}\cdots\left(r^{x_n}\right)^{a_n} = r^{a\cdot x}$$

and then the linear combination $c_{\alpha,a}$ may be written as

$$c_{\alpha,a} = \sum_{j=1}^{h} \alpha_j c^{a(j)},$$

subject, of course, to x being in our exponential cone $\mathcal{C}^{\uparrow}$ defined by the d_i's, see (3). Letting $p_j = a(j)\cdot x$, this means that $c_{\alpha,a}$ may be viewed as an exponential polynomial:

$$c_{\alpha,a} = \alpha_1 r^{p_1} + \alpha_2 r^{p_2} + \cdots + \alpha_h r^{p_h}.$$

Similarly,

$$c_{\beta,b} = \beta_1 r^{q_1} + \beta_2 r^{q_2} + \cdots + \beta_m r^{q_m}.$$

Then, in order to have

$$c_{\alpha,a} \leqslant c_{\beta,b} \tag{5}$$

for all p-Newton sequences, we must have that the polynomial

$$d(r) = c_{\beta,b} - c_{\alpha,a} \geqslant 0$$

for all $r > 1$ (and all x in our exponential cone $\mathcal{C}^{\uparrow}$). In particular, $d(r) \geqslant 0$, $r > 1$, must hold for each extremal in the cone. Each of these give necessary conditions on the coefficients α and β and exponents a and b for $c_{\alpha,a}$ to be dominated by $c_{\beta,b}$.

Theorem 7. *Let*

$$c_{\alpha,a} = \sum_{j=1}^{h} \alpha_j c^{a(j)} \quad \text{and} \quad c_{\beta,b} = \sum_{j=1}^{m} \beta_j c^{b(j)}$$

be two linear combinations of monomials and let

$$E_{\alpha,a}(r,x) = \sum_{j=1}^{h} \alpha_j r^{a(j)\cdot x} \quad \text{and} \quad E_{\beta,b}(r,x) = \sum_{j=1}^{m} \beta_j r^{b(j)\cdot x}$$

be the associated r exponential polynomials. Then $c_{\alpha,a} \leqslant c_{\beta,b}$ at all p-Newton sequences c if and only if $E_{\alpha,a}(r,x) \leqslant E_{\beta,b}(r,x)$ at all p-Newton exponential vectors x, and for all $r > 1$.

We record now the necessary conditions for domination of an α, a pair by a β, b pair that result from the extremals of the cone $\mathcal{C}^{+}$ of nonnegative x's.

Theorem 8. *If $c_{\alpha,a} \leqslant c_{\beta,b}$ for all p-Newton sequences c, that is, $c_{\beta,b}$ dominates $c_{\alpha,a}$, then*

1. $\sum_{i=1}^{h} \alpha_i r^{\sum_{j=1}^{n} j a_j(i)} \leqslant \sum_{i=1}^{m} \beta_i r^{\sum_{j=1}^{n} j b_j(i)}\ \forall r > 1$;
2. $\sum_{i=1}^{h} \alpha_i r^{\sum_{j=1}^{n-1} (n-j) a_j(i)} \leqslant \sum_{i=1}^{m} \beta_i r^{\sum_{j=1}^{n-1} (n-j) b_j(i)}\ \forall r > 1$;
3. $\sum_{i=1}^{h} \alpha_i r^{\sum_{j=1}^{t} (n-1-t) a_j(i) + (t+1)\sum_{j=t+1}^{n-1}(n-j) a_j(i)} \leqslant \sum_{i=1}^{m} \beta_i r^{\sum_{j=1}^{t} (n-1-t) b_j(i) + (t+1)\sum_{j=t+1}^{n-1}(n-j) b_j(i)}\ \forall r > 1$; *for $t = 1, \ldots, n-2$.*

Proof. By Theorem 7 we have

$$E_{\alpha,a}(r,x) \leqslant E_{\beta,b}(r,x)$$

at all p-Newton exponential vectors x. Now taking x as one of the generators of the cone $\mathcal{C}^+$ given in Theorem 4 we have:

$$\sum_{i=1}^{h} \alpha_i r^{\sum_{j=1}^{n} ja_j(i)} = E_{\alpha,a}\big(r,(1,2,\ldots,n)\big) \leqslant E_{\beta,b}\big(r,(1,2,\ldots,n)\big) = \sum_{i=1}^{m} \beta_i r^{\sum_{j=1}^{n} jb_j(i)}$$

and

$$\begin{aligned}\sum_{i=1}^{h} \alpha_i r^{\sum_{j=1}^{n-1}(n-j)a_j(i)} &= E_{\alpha,a}\big(r,(n-1,\ldots,1,0)\big) \leqslant E_{\beta,b}\big(r,(n-1,\ldots,1,0)\big)\\ &= \sum_{i=1}^{m} \beta_i r^{\sum_{j=1}^{n-1}(n-j)b_j(i)}\end{aligned}$$

which give conditions 1 and 2 from the theorem. Finally, let $t \in \{1,\ldots,n-2\}$ and let x be equal to

$$\begin{aligned}&\big(n-1-t, 2(n-1-t), \ldots, t(n-1-t), (t+1)\big(n-(t+1)\big),\\ &\quad (t+1)\big(n-(t+2)\big), \ldots, (t+1)2, t+1, 0\big).\end{aligned}$$

Then

$$\begin{aligned}&\sum_{i=1}^{h} \alpha_i r^{\sum_{j=1}^{t}(n-1-t)a_j(i)+(t+1)\sum_{j=t+1}^{n-1}(n-j)a_j(i)}\\ &\quad = E_{\alpha,a}(r,x) \leqslant E_{\beta,b}(r,x) = \sum_{i=1}^{m} \beta_i r^{\sum_{j=1}^{t}(n-1-t)b_j(i)+(t+1)\sum_{j=t+1}^{n-1}(n-j)b_j(i)}\end{aligned}$$

and condition 3 is proven. □

Another way to obtain necessary conditions is evaluating the inequality (5) at a particular p-Newton sequence. For example, $Q_{n,0}(r)$ gives the condition $d(1) \geqslant 0$ or

$$\sum_{i=1}^{h} \alpha_i \leqslant \sum_{i=1}^{m} \beta_i,$$

i.e. that the coefficient sums must follow the domination relation. In particular, in case all coefficients are 1, we must have $h \leqslant m$. Of course, this condition alone is not sufficient for domination.

Theorem 9. *If $c_{\alpha,a} \leqslant c_{\beta,b}$ for all p-Newton sequences c, that is, $c_{\beta,b}$ dominates $c_{\alpha,a}$, then*

1. $\sum_{i=1}^{h} \alpha_i \leqslant \sum_{i=1}^{m} \beta_i$;
2. $\sum_{i=1}^{h} \alpha_i r^{\sum_{j=1}^{n} a_j(i)} \leqslant \sum_{i=1}^{m} \beta_i r^{\sum_{j=1}^{n} b_j(i)}$ $\forall r > 1$;
3. $\sum_{i=1}^{h} \alpha_i r^{a_1(i)+2\sum_{j=2}^{n} a_j(i)} \leqslant \sum_{i=1}^{m} \beta_i r^{b_1(i)+2\sum_{j=2}^{n} b_j(i)}$ $\forall r > 1$;
4. $\sum_{i=1}^{h} \alpha_i r^{\sum_{j=1}^{t-1} ja_j(i)+t\sum_{j=t}^{n} a_j(i)} \leqslant \sum_{i=1}^{m} \beta_i r^{\sum_{j=1}^{t-1} jb_j(i)+t\sum_{j=t}^{n} b_j(i)}$ $\forall r > 1$; *for* $t = 0,1,\ldots,n$.

Proof. The conditions are obtained by evaluating the inequality $c_{\alpha,a} \leqslant c_{\beta,b}$ on the p-Newton sequence $Q_{n,t}(r)$, for $t = 0,1,\ldots,n$. Note that condition 4 for $t = 0$ gives condition 1, for $t = 1$ gives condition 2, and for $t = 2$ gives condition 3. □

Other p-Newton sequences that we call UD sequences, for “up and down”, can be useful as well.

Lemma 10. *Let c_i, for $i = 0, 1, \dots, n$, be defined by $c_i = r^{x_i}$, with*

$$x_i = \begin{cases} yi & \text{if } 0 \leqslant i \leqslant k; \\ u - zi & \text{if } k < i \leqslant n; \end{cases} \quad \text{for } y, z > 0,\ k \in \{0, 1, \dots, n-1\} \text{ and } u \in \mathbb{R}.$$

Then $P^u_{n,k,y,z}(r)$: $c_0, c_1, \dots, c_n = 1, r^y, r^{2y}, \dots, r^{ky}, r^{u-(k+1)z}, r^{u-(k+2)z}, \dots, r^{u-nz}$ is a p-Newton sequence if and only if

$$k(y+z) \leqslant u \leqslant (k+1)(y+z).$$

Proof. Clearly, the sequence is positive. Since the exponents are one arithmetic progression through k and another beginning at $k+1$, we need only check the Newton inequalities: $c_{k-1}c_{k+1} \leqslant c_k^2$ and $c_k c_{k+2} \leqslant c_{k+1}^2$. The former requires $u \leqslant (k+1)(y+z)$ and the latter $u \geqslant k(y+z)$. $\square$

An important source of getting a new inequality from one already known is by *index complementation*. By this we mean replacing c_k by c_{n-k} for $k = 0, 1, \dots, n$.

Lemma 11. *Index complementation in an inequality between two linear combinations of monomials yields another valid inequality.*

Proof. Since the required Newton inequalities are the same, reversal of a p-Newton sequence is a p-Newton sequence (though the normalization $c_0 = 1$ is lost). $\square$

Another source of additional inequalities would be the use of the next conjecture:

Conjecture. *An increase by* 1 *in the highest index of each monomial of an inequality yields another valid inequality.*

This conjecture will be partially proven for two monomials in Lemma 19. We suspect that it is also valid for general linear combination inequalities. It is in the case proven.

5. Equivalent statements of the problem

Our purpose here is to give two alternate formulations of our problem, one touched upon in the last section involving polynomials, and the other an interesting matrix formulation. Unfortunately, both suggest that a complete solution to our problem in terms of α, a and β, b is likely to be difficult.

First, from the prior section, see Theorem 7, we know:

"*$c_{\alpha,a} \leqslant c_{\beta,b}$ for all p-Newton sequences c if and only if $d(r) \geqslant 0$ for all $r > 1$ and all x in the full exponential cone $\mathcal{C}^{\uparrow}$*".

However, it seems problematic to give a sufficiently nice description of when the exponential polynomial $d(r) \geqslant 0$ for $r > 1$ in terms of the coefficients and exponents that result from a particular x. Even when the coefficients are all 1 and $h = m$, this seems unclear. This suggests the question: characterize $p_1, \dots, p_h, q_1, \dots, q_h$, with $q_1 \geqslant \dots \geqslant q_h$ and $p_1 \geqslant \dots \geqslant p_h$, so that

$$d(r) = r^{q_1} + \dots + r^{q_h} - r^{p_1} - \dots - r^{p_h} \geqslant 0$$

for all $r > 1$. For $h = 1$, this is clear, and for $h = 2$, this is done in the next section. However, for $h > 2$, we do not know a "nice" answer. It is clearly necessary that

$$q_1 \geqslant p_1$$

and

$$\sum_{i=1}^{h} q_i \geqslant \sum_{i=1}^{h} p_i$$

(and, in case $p_1 = q_1$, $q_2 \geqslant p_2$, etc.). This suggests weak majorization of the p's by the q's, which is easily shown to be sufficient. However, weak majorization is not necessary. A simple example is

Example 12. The polynomial

$$d(r) = r^{10} + r^6 + r^5 - r^9 - r^8 - r^4$$

satisfies $d(r) > 0$ for $r > 1$, as

$$d(r) = (r-1)^3 r^4 \left(r^3 + 2r^2 + 2r + 1\right).$$

But, as $10 + 6 < 9 + 8$, $(10, 6, 5)$ does not weakly majorize $(9, 8, 4)$.

Of course, as $d(r)$ is a real polynomial, it may be factored into linear and (irreducible) quadratic factors over the reals. Then, $d(r) > 0$ for $r > 1$ if and only if any linear term with a root bigger than 1 occurs with even multiplicity (assuming the quadratic factors are positive at $r = 1$). However, we do not know how to characterize this occurrence in terms of inequality relationships on the exponents for $h > 2$.

Next, we rewrite our linear combination of monomials in a novel form, using nonnegative (non-square) matrices that may be taken to be "row stochastic". Recall that for x in the exponential representation of c, we have that the i-th monomial in $c_{\alpha,a}$ is $r^{a(i)\cdot x}$. Define the h-by-$(n+1)$ nonnegative matrix

$$A = \begin{pmatrix} - & a(1) & - \\ - & a(2) & - \\ & \vdots & \\ - & a(h) & - \end{pmatrix},$$

so that $c_{\alpha,a} = \alpha^T r^{Ax}$, in which r^{Ax} is interpreted as the vector whose component in position i is $r^{(Ax)_i}$. Similarly, we may write $c_{\beta,b} = \beta^T r^{Bx}$. Now, we may define

$$A \leqslant_{\alpha,\beta} B$$

if and only if $\alpha^T r^{Ax} \leqslant \beta^T r^{Bx}$ for all x in our exponential cone. In case α and β are both e, the vector of 1's, we may simply write $A \leqslant B$. Since $x_0 = 0$, by convention, we may adjust the initial columns of A and B so that their row sums are a common constant, which may then be scaled by choice of r to be 1. Thus, in either case, we would have a new partial order on non-square, nonnegative matrices with row sums 1, which would be interesting to characterize. We comment that if x were to vary over some other cone, there would be different partial orders on such matrices that would result. How the partial order depends upon the cone, when it may be checked via a finite number of points from the cone, and how else it might be characterized all seem of general theoretical interest.

6. Two monomials versus two monomials

Here we consider likely the simplest case of our problem beyond the single monomial inequalities of Section 2: $h = m = 2$, with all coefficients 1 and just two Newton coefficients appearing, with exponent 1, in each monomial. Though the strategy developed here is helpful in this case, it remains difficult to give a simple answer.

We begin by proving a family of inequalities.

Lemma 13. *Let q be a nonnegative integer. Then*

$$c_q c_{q+1} + c_{q+2}^2 \geqslant c_q c_{q+2} + c_q c_{q+3}$$

holds for all p-Newton sequences.

Proof. Let us see first that the following statements are equivalent:

(i) $c_q c_{q+1} + c_{q+2}^2 \geqslant c_q c_{q+2} + c_q c_{q+3}$ holds for all p-Newton sequences; and
(ii) $c_{q+1} + c_{q+2}^2 \geqslant c_{q+2} + c_{q+3}$ holds for all p-Newton sequences with $c_q = 1$.

For (ii) $\Rightarrow$ (i) let c: $c_0, c_1, \ldots, c_n$ be a p-Newton sequence. Then $\frac{c}{c_q}$: $\frac{c_0}{c_q}, \frac{c_1}{c_q}, \ldots, \frac{c_n}{c_q}$ is also a p-Newton sequence and its q-coefficient is 1. By (ii) we have

$$\frac{c_{q+1}}{c_q} + \left(\frac{c_{q+2}}{c_q}\right)^2 \geqslant \frac{c_{q+2}}{c_q} + \frac{c_{q+3}}{c_q}$$

or equivalently $c_q c_{q+1} + c_{q+2}^2 \geqslant c_q c_{q+2} + c_q c_{q+3}$, which gives (i). The reverse implication is clear.

Now we will prove statement (ii). Let c be a p-Newton sequence with $c_q = 1$, then

$$c_{q+1}^2 \geqslant c_q c_{q+2} = c_{q+2} \quad \Longrightarrow \quad \begin{cases} c_{q+1} \geqslant \sqrt{c_{q+2}}, \\ \frac{1}{\sqrt{c_{q+2}}} \geqslant \frac{1}{c_{q+1}} \end{cases} \tag{6}$$

and

$$c_{q+2}^2 \geqslant c_{q+1} c_{q+3}. \tag{7}$$

If $c_{q+2} \geqslant 1$, then

$$c_{q+2}\big(c_{q+2} - \sqrt{c_{q+2}}\big) \geqslant c_{q+2} - \sqrt{c_{q+2}} \quad \Longleftrightarrow \quad \sqrt{c_{q+2}} + c_{q+2}^2 \geqslant c_{q+2} + c_{q+2}\sqrt{c_{q+2}} \tag{8}$$

and we have

$$\begin{aligned} c_{q+1} + c_{q+2}^2 &\overset{(6)}{\geqslant} \sqrt{c_{q+2}} + c_{q+2}^2 \overset{(8)}{\geqslant} c_{q+2} + c_{q+2}\sqrt{c_{q+2}} = c_{q+2} + \frac{c_{q+2}^2}{\sqrt{c_{q+2}}} \\ &\overset{(6)}{\geqslant} c_{q+2} + \frac{c_{q+2}^2}{c_{q+1}} \overset{(7)}{\geqslant} c_{q+2} + c_{q+3}. \end{aligned}$$

If $c_{q+2} < 1$, then

$$\begin{aligned} c_{q+2}^2 \overset{(7)}{\geqslant} c_{q+1} c_{q+3} \overset{(6)}{\geqslant} \sqrt{c_{q+2}}\, c_{q+3} \quad &\Longrightarrow \quad \sqrt{c_{q+2}} \geqslant c_{q+2}\sqrt{c_{q+2}} \geqslant c_{q+3} \\ &\Longrightarrow \quad \sqrt{c_{q+2}}\big(1 - \sqrt{c_{q+2}}\big) \geqslant c_{q+3}\big(1 - \sqrt{c_{q+2}}\big) \\ &\Longrightarrow \quad \sqrt{c_{q+2}} + \sqrt{c_{q+2}}\, c_{q+3} \geqslant c_{q+2} + c_{q+3} \end{aligned} \tag{9}$$

and we have

$$c_{q+1} + c_{q+2}^2 \overset{(7)}{\geqslant} c_{q+1} + c_{q+1} c_{q+3} \overset{(6)}{\geqslant} \sqrt{c_{q+2}} + \sqrt{c_{q+2}}\, c_{q+3} \overset{(9)}{\geqslant} c_{q+2} + c_{q+3}. \qquad \square$$

Consider

$$c_{\alpha,a} = c_{i_{11}} c_{i_{12}} + c_{i_{21}} c_{i_{22}}$$

and

$$c_{\beta,b} = c_{j_{11}} c_{j_{12}} + c_{j_{21}} c_{j_{22}},$$

i.e.

$$\alpha = \beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and $a(1)$, resp. $a(2)$, $b(1)$, $b(2)$, the 0, 1 vectors with 1's in just positions i_{11} and i_{12}, resp. i_{21} and i_{22}, j_{11} and j_{12}, j_{21} and j_{22}. We may assume for convenience, without loss of generality, that

$$\begin{aligned} &i_{11} \leqslant i_{12}, \qquad i_{21} \leqslant i_{22}, \qquad i_{12} \leqslant i_{22}, \\ &j_{11} \leqslant j_{12}, \qquad j_{21} \leqslant j_{22}, \qquad j_{12} \leqslant j_{22}. \end{aligned} \tag{10}$$

Note that all monomials have length 2 and their weights are

$$w_{i_1} = i_{11} + i_{12}, \qquad w_{i_2} = i_{21} + i_{22}, \qquad w_{j_1} = j_{11} + j_{12} \quad \text{and} \quad w_{j_2} = j_{21} + j_{22}.$$

Using our exponential approach

$$c_{\alpha,a} = r^{x_{i_{11}} + x_{i_{12}}} + r^{x_{i_{21}} + x_{i_{22}}}$$

and

$$c_{\beta,b} = r^{x_{j_{11}} + x_{j_{12}}} + r^{x_{j_{21}} + x_{j_{22}}},$$

and we want the former polynomial to be at least the latter for all x in the exponential cone (and all $r > 1$). This necessitates

$$\max\{x_{i_{11}} + x_{i_{12}}, x_{i_{21}} + x_{i_{22}}\} \leqslant \max\{x_{j_{11}} + x_{j_{12}}, x_{j_{21}} + x_{j_{22}}\}$$

for all x in the cone and also that

$$x_{i_{11}} + x_{i_{12}} + x_{i_{21}} + x_{i_{22}} \leqslant x_{j_{11}} + x_{j_{12}} + x_{j_{21}} + x_{j_{22}}$$

by the necessary conditions from Section 4. We will show that these conditions are also sufficient.

Lemma 14. *Let $p \geqslant q$ and $s \geqslant t$ be nonnegative numbers. Then*

$$r^p + r^q - r^s - r^t \geqslant 0 \quad \forall r \geqslant 1$$

if and only if

$$p \geqslant s \quad \textit{and} \quad p + q \geqslant s + t,$$

i.e. (p, q) *weakly majorizes* (s, t).

Proof. For sufficiency, write

$$r^p + r^q - r^s - r^t = \left(r^q - r^{s+t-p}\right) + \left(r^{p-s} - 1\right)\left(r^s - r^{s+t-p}\right),$$

so that the exponential polynomial $r^p + r^q - r^s - r^t$ is a sum of two expressions, each of which is nonnegative for $r > 1$.

Necessity of $p \geqslant s$ follows from considering large r, while necessity of the second condition follows from differentiation and evaluation at $r = 1$, because the derivative must be nonnegative there. □

Let us now analyze the inequality $c_{j_{11}} c_{j_{12}} + c_{j_{21}} c_{j_{22}} \geqslant c_{i_{11}} c_{i_{12}} + c_{i_{21}} c_{i_{22}}$, with indices satisfying (10), in terms of indices. The following results give necessary conditions for domination, in terms of indices, in the case of two monomials versus two monomials.

Lemma 15. *If $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$, with indices satisfying (10), holds for all p-Newton sequences, then*:

1. $\max\{j_{11} + j_{12}, j_{21} + j_{22}\} \geqslant \max\{i_{11} + i_{12}, i_{21} + i_{22}\}$, *i.e. the largest weight monomial is on the larger side of the inequality.*
2. *The sum of weights on the left, $w_{j_1} + w_{j_2}$, is equal the sum on the right, $w_{i_1} + w_{i_2}$. That is $j_{11} + j_{12} + j_{21} + j_{22} = i_{11} + i_{12} + i_{21} + i_{22}$.*
3. $j_{22} \leqslant i_{22}$ *and* $\min\{j_{11}, j_{21}\} \geqslant \min\{i_{11}, i_{21}\}$.

Proof. Statements 1 and 2 from Theorem 8 applied to the inequality give:

$$r^{j_{11}+j_{12}} + r^{j_{21}+j_{22}} \geqslant r^{i_{11}+i_{12}} + r^{i_{21}+i_{22}} \tag{11}$$

and

$$r^{2n-(j_{11}+j_{12})} + r^{2n-(j_{21}+j_{22})} \geqslant r^{2n-(i_{11}+i_{12})} + r^{2n-(i_{21}+i_{22})}, \tag{12}$$

respectively. Now, Lemma 14 applied to (11) gives

$$\max\{j_{11} + j_{12}, j_{21} + j_{22}\} \geqslant \max\{i_{11} + i_{12}, i_{21} + i_{22}\}$$
$$j_{11} + j_{12} + j_{21} + j_{22} \geqslant i_{11} + i_{12} + i_{21} + i_{22}$$

and to (12) gives

$$4n - (j_{11} + j_{12} + j_{21} + j_{22}) \geqslant 4n - (i_{11} + i_{12} + i_{21} + i_{22})$$
$$\iff \quad j_{11} + j_{12} + j_{21} + j_{22} \leqslant i_{11} + i_{12} + i_{21} + i_{22},$$

which prove conditions 1 and 2.

3 We suppose that $j_{22} > i_{22}$. Because the truncation of a p-Newton sequence is p-Newton, we can assume $n = j_{22}$ and now apply statement 3 from Theorem 8 with $t = n - 2 = j_{22} - 2$ to the inequality (that is, the necessary condition obtained from the generator $(1, 2, \ldots, n-1, 0)$). We then have

$$r^{i_{11}+i_{12}} + r^{i_{21}+i_{22}} \leqslant r^{j_{11}+j_{12}} + r^{j_{21}+j_{22}} \leqslant r^{2j_{22}} + r^{2j_{22}} \leqslant r^0 + r^0.$$

By Lemma 14 and statement 2 from this lemma we have the contradiction

$$0 \geqslant i_{11} + i_{12} + i_{21} + i_{22} = j_{11} + j_{12} + j_{21} + j_{22} \geqslant j_{22} > i_{22} \geqslant 0.$$

This proves $j_{22} \leqslant i_{22}$. An index complementation argument, using Lemma 11, gives the condition about the minimum. □

Corollary 16. *If $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$, with indices satisfying (10), holds for all p-Newton sequences, then $\{w_{j_1}, w_{j_2}\}$ majorizes $\{w_{i_1}, w_{i_2}\}$.*

Proof. This follows from statements 1 and 2 of Lemma 15. □

We may now state the key portion of our final result that explains how inequalities for linear combinations involve more than term-wise domination.

Theorem 17. *If $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$, with indices satisfying (10), holds for all p-Newton sequences and $\{w_{j_1}, w_{j_2}\} \neq \{w_{i_1}, w_{i_2}\}$, then*

$$j_{12}, j_{22} \leqslant i_{12}, i_{22} \quad \textit{and} \quad j_{11}, j_{21} \geqslant i_{11}, i_{21}. \tag{13}$$

We call condition (13) DB, for “double between-ness”.

Proof. Note that the hypothesis about the weights in the theorem is preserved by index complementation. So, it is enough to prove $j_{12}, j_{22} \leqslant i_{12}, i_{22}$, because then $j_{11}, j_{21} \geqslant i_{11}, i_{21}$ follows by index complementation.

Lemma 15, statement 2, gives $\{w_{j_1}, w_{j_2}\} \neq \{w_{i_1}, w_{i_2}\}$ is equivalent to $\{w_{j_1}, w_{j_2}\} \cap \{w_{i_1}, w_{i_2}\} = \emptyset$, and $\{w_{j_1}, w_{j_2}\} \cap \{w_{i_1}, w_{i_2}\} \neq \emptyset$ if and only if $\{w_{j_1}, w_{j_2}\} = \{w_{i_1}, w_{i_2}\}$.

Moreover $w_{j_1} \neq w_{j_2}$, since if $w_{j_1} = w_{j_2}$, this weight is maximum by Lemma 15, statement 1. So $w_{i_1}, w_{i_2} < w_{j_1} = w_{j_2}$, a contradiction with Lemma 15, statement 2. Thus we have

$$\min\{w_{j_1}, w_{j_2}\} < \min\{w_{i_1}, w_{i_2}\} \leqslant \max\{w_{i_1}, w_{i_2}\} < \max\{w_{j_1}, w_{j_2}\}.$$

To prove $j_{12}, j_{22} \leqslant i_{12}, i_{22}$ it is sufficient, by convention (10), to prove $j_{22} \leqslant i_{12}$. We consider the following cases:

1) Let $i_{11} \leqslant i_{21}$ and $j_{11} \leqslant j_{21}$. In this case, $w_{j_1} < w_{i_1} \leqslant w_{i_2} < w_{j_2}$.
Lemma 15, statement 3, implies $i_{11} \leqslant j_{11}$.
As $j_{11} + j_{12} < i_{11} + i_{12}$ and i_{11} is the minimum index, then $i_{12} < j_{12}$.
As $i_{21} + i_{22} < j_{21} + j_{22}$ and i_{22} is maximum, then $i_{21} < j_{21}$.
By reductio ad absurdum, we suppose $j_{22} > i_{12}$, and so $j_{11} < i_{21}$. Then the situation is

$$i_{11} \leqslant j_{11} \left\{ \begin{array}{l} \leqslant j_{12} < i_{12} < j_{22} \\ < i_{21} < j_{21} \leqslant j_{22} \end{array} \right\} \leqslant i_{22}.$$

Since $c_{j_{11}} c_{j_{12}} + c_{j_{21}} c_{j_{22}} \geqslant c_{i_{11}} c_{i_{12}} + c_{i_{21}} c_{i_{22}}$ holds for the UD sequences $P^u_{n,k,y,z}(r)$, in particular for $k = j_{22} - 1$, we have

$$r^{j_{11} y + j_{12} y} + r^{x + u - j_{22} z} \geqslant r^{i_{11} y + i_{12} y} + r^{i_{21} y + u - i_{22} z}, \tag{14}$$

with $(j_{22} - 1)(y + z) \leqslant u \leqslant j_{22}(y + z)$ and $y, z > 0$, in which $x = u - j_{22} z$ when $j_{21} = j_{22}$, and $x = j_{21} y$ when $j_{21} < j_{22}$.
If $j_{21} = j_{22}$, then $u \leqslant j_{22}(y + z) \iff j_{21} y \geqslant u - j_{22} z$, so it is sufficient to consider the case $x = j_{21} y$.
Since $(j_{11} + j_{12}) y < (i_{11} + i_{12}) y$ and $j_{21} y + u - j_{22} z > i_{21} y + u - i_{22} z$, by Lemma 14 applied to (14) we have

$$\begin{aligned} & j_{21} y + u - j_{22} z \geqslant (i_{11} + i_{12}) y \\ \iff \quad & u \geqslant (i_{11} + i_{12} - j_{21}) y + j_{22} z \\ \iff \quad & (i_{11} + i_{12} - j_{21}) y + j_{22} z \leqslant (j_{22} - 1)(y + z) \\ \iff \quad & z \leqslant (j_{21} + j_{22} - i_{11} - i_{12} - 1) y = (w_{j_2} - w_{i_1} - 1) y. \end{aligned}$$

As $w_{j_2} - w_{i_1} - 1 \geqslant 1$, the last inequality does not hold for all $y, z > 0$. In fact, for $k = j_{22} - 1$, $y = 1$, $z = w_{j_2} - w_{i_1}$ and $u = (j_{22} - 1)(y + z)$, the inequality $c_{j_{11}} c_{j_{12}} + c_{j_{21}} c_{j_{22}} \geqslant c_{i_{11}} c_{i_{12}} + c_{i_{21}} c_{i_{22}}$ does not hold for the UD sequence $P^u_{n,k,y,z}(r)$.

2) Let $i_{11} \leqslant i_{21}$ and $j_{11} > j_{21}$. In this case, $w_{i_1} \leqslant w_{i_2}$.
Lemma 15, statement 3, implies $i_{11} \leqslant j_{21}$.
If $w_{j_1} < w_{i_1} \leqslant w_{i_2} < w_{j_2}$, as $i_{21} + i_{22} < j_{21} + j_{22}$ and i_{22} is maximum, then $i_{21} < j_{21}$. So we have $j_{11}, j_{21} \geqslant i_{11}, i_{21}$.
If $w_{j_2} < w_{i_1} \leqslant w_{i_2} < w_{j_1}$, as $j_{21} + j_{22} < i_{11} + i_{12}$ and i_{11} is minimum, then $j_{22} < i_{12}$. So we have $j_{12}, j_{22} < i_{12}, i_{22}$.

3) Let $i_{11} > i_{21}$ and $j_{11} \leqslant j_{21}$. In this case, $w_{j_1} < w_{i_1}, w_{i_2} < w_{j_2}$.
Lemma 15, statement 3, implies $i_{21} \leqslant j_{11}$.

We suppose that $j_{22} > i_{12}$, and so $j_{11} < i_{11}$. Then the situation is

$$i_{21} \leqslant j_{11} \left\{ \begin{array}{l} < i_{11} \leqslant i_{12} < j_{22} \\ \leqslant j_{21}, j_{12} \leqslant j_{22} \end{array} \right\} \leqslant i_{22}$$

and, as in case 1), the inequality $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$ does not hold for the UD sequence $P^u_{n,k,y,z}(r)$ for $k = j_{22} - 1$.

4) Let $i_{11} > i_{21}$ and $j_{11} > j_{21}$.
Lemma 15, statement 3, implies $i_{21} \leqslant j_{11}$.
We suppose that $j_{22} > i_{22}$, and so $j_{21} < i_{11}$. Then the situation is

$$i_{21} \leqslant j_{21} \left\{ \begin{array}{l} < i_{11} \leqslant i_{12} < j_{22} \\ < j_{11} \leqslant j_{12} \leqslant j_{22} \end{array} \right\} \leqslant i_{22}.$$

If $w_{j_1} < w_{j_2}$, then we have $w_{j_1} < w_{i_1}, w_{i_2} < w_{j_2}$ and, as in case 1), the inequality $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$ does not hold for the UD sequences $P^u_{n,k,y,z}(r)$ for $k = j_{22} - 1$.
If $w_{j_2} < w_{j_1}$, then we have $w_{j_2} < w_{i_1}, w_{i_2} < w_{j_1}$. Since $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$ holds for the UD sequences $P^u_{n,k,y,z}(r)$, in particular for $k = j_{21}$, we have

$$r^{2u-(j_{11}+j_{12})z} + r^{j_{21}+u-j_{22}z} \geqslant r^{2u-(i_{11}+i_{12})z} + r^{i_{21}y+u-i_{22}z},$$

with $j_{21}(y+z) \leqslant u \leqslant (j_{21}+1)(y+z)$, $y, z > 0$.
Since $2u - (j_{11} + j_{12})z < 2u - (i_{11} + i_{12})z$ and $j_{21}y + u - j_{22}z \geqslant i_{21}y + u - i_{22}z$, we have

$$\begin{aligned} & j_{21}y + u - j_{22}z \geqslant 2u - (i_{11} + i_{12})z \\ \iff\quad & u \leqslant j_{21}y + (i_{11} + i_{12} - j_{22})z \\ \iff\quad & (j_{21}+1)(y+z) \leqslant j_{21}y + (i_{11} + i_{12} - j_{22})z \\ \iff\quad & y \leqslant (i_{11} + i_{12} - j_{21} - j_{22} - 1)z = (w_{i_1} - w_{j_2} - 1)z. \end{aligned}$$

As $w_{i_1} - w_{j_2} - 1 \geqslant 1$, the last inequality does not hold for all $y, z > 0$. □

Theorem 18. *If $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$, with indices satisfying (10), holds for all p-Newton sequences with $\{w_{j_1}, w_{j_2}\} = \{w_{i_1}, w_{i_2}\}$, then term-wise domination holds, i.e. $\{i_{11}, i_{12}\}$ majorizes one of $\{j_{11}, j_{12}\}$ or $\{j_{21}, j_{22}\}$ and $\{i_{21}, i_{22}\}$ majorizes the other.*

Proof. As i_{22} is the maximum index, then $\{i_{21}, i_{22}\}$ majorizes the couple of indices with equal weight. If $\min\{i_{11}, i_{21}\} = i_{11}$, then i_{11} is the minimum index among the indices of the other two couples with equal weight, so $\{i_{11}, i_{12}\}$ majorizes the other couple.

If $\min\{i_{11}, i_{21}\} = i_{21}$, then by Lemma 15, statement 3, we have $i_{21} \leqslant j_{11}, j_{21}$. With the convention (10), the general situation is

$$i_{21} \leqslant \left\{ \begin{array}{l} i_{11} \leqslant i_{12} \\ j_{11} \leqslant j_{12} \leqslant j_{22} \\ j_{21} \leqslant j_{22} \end{array} \right\} \leqslant i_{22}.$$

We consider the following cases:

a) Suppose $w_{j_1} = w_{i_1} \leqslant w_{i_2} = w_{j_2}$.
As i_{22} is the maximum index, then $\{i_{21}, i_{22}\}$ majorizes $\{j_{21}, j_{22}\}$.
If $\{i_{11}, i_{12}\}$ does not majorize $\{j_{11}, j_{12}\}$, then $j_{11} < i_{11} \leqslant i_{12} < j_{12}$.
Since $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$ holds for the sequences $Q_{n,i}(r)$, for $i = i_{11}$ we have

$$\begin{aligned} & r^{j_{11}+i_{11}} + r^{\min\{i_{11}, j_{21}\}+i_{11}} \geqslant r^{i_{11}+i_{11}} + r^{i_{21}+i_{11}} \\ \implies\quad & i_{11} \leqslant \min\{i_{11}, j_{21}\} \quad \implies \quad i_{11} \leqslant j_{21}. \end{aligned}$$

Now the situation is $i_{21} \leqslant j_{11} < i_{11} \leqslant j_{21}, i_{12}$ and $i_{12} < j_{12} \leqslant j_{22} \leqslant i_{22}$.
Since $c_{j_{11}} c_{j_{12}} + c_{j_{21}} c_{j_{22}} \geqslant c_{i_{11}} c_{i_{12}} + c_{i_{21}} c_{i_{22}}$ holds for the UD sequences $P^u_{n,k,y,z}(r)$, for $k = j_{11}$ we have

$$r^{j_{11}y+u-j_{12}z} + r^{u-j_{21}z+u-j_{12}z} \geqslant r^{u-i_{11}z+u-i_{12}z} + r^{i_{21}y+u-i_{22}z},$$

with $j_{11}(y+z) \leqslant u \leqslant (j_{11}+1)(y+z)$, $y, z > 0$.
As $2u - (j_{21}+j_{22})z < 2u - (i_{11}+i_{12})z$ and $j_{11}y + u - j_{12}z \geqslant i_{21}y + u - i_{22}z$, then

$$\begin{aligned} & j_{11}y + u - j_{12}z \geqslant 2u - (i_{11}+i_{12})z \\ & \iff \quad u \leqslant j_{11}y + (i_{11}+i_{12}-j_{12})z \\ & \iff \quad (j_{11}+1)(y+z) \leqslant j_{11}y + (i_{11}+i_{12}-j_{12})z \\ & \iff \quad y \leqslant (i_{11}+i_{12}-j_{11}-j_{12}-1)z = -z \end{aligned}$$

against $y, z > 0$.

b) Suppose $w_{j_2} = w_{i_1} \leqslant w_{i_2} = w_{j_1}$.
As i_{22} is maximum, then $\{i_{21}, i_{22}\}$ majorizes $\{j_{11}, j_{12}\}$.
If $\{i_{11}, i_{12}\}$ does not majorize $\{j_{21}, j_{j2}\}$, then $j_{21} < i_{11} \leqslant i_{12} < j_{22}$.
Since $c_{j_{11}} c_{j_{12}} + c_{j_{21}} c_{j_{22}} \geqslant c_{i_{11}} c_{i_{12}} + c_{i_{21}} c_{i_{22}}$ holds for the sequences $Q_{n,i}(r)$, for $i = i_{11}$ we have

$$\begin{aligned} & r^{\min\{j_{11}, i_{11}\}+i_{11}} + r^{j_{21}+i_{11}} \geqslant r^{i_{11}+i_{11}} + r^{i_{21}+i_{11}} \\ & \quad \Longrightarrow \quad i_{11} \leqslant \min\{j_{11}, i_{11}\} \quad \Longrightarrow \quad i_{11} \leqslant j_{11}. \end{aligned}$$

Moreover $w_{i_1} = w_{j_2}$ and $j_{21} < i_{11}$ implies $i_{12} < j_{22}$. The situation is

$$i_{21} \leqslant j_{21} < i_{11} \leqslant j_{11} \leqslant j_{12} \leqslant j_{22} \leqslant i_{22} \quad \text{and} \quad i_{11} \leqslant i_{12} < j_{22} \leqslant i_{22}.$$

Since $c_{j_{11}} c_{j_{12}} + c_{j_{21}} c_{j_{22}} \geqslant c_{i_{11}} c_{i_{12}} + c_{i_{21}} c_{i_{22}}$ holds for the UD sequences $P^u_{n,k,y,z}(r)$, for $k = j_{21}$ we have

$$r^{2u-(j_{11}+j_{12})z} + r^{j_{21}y+u-j_{22}z} \geqslant r^{2u-(i_{11}+i_{12})z} + r^{i_{21}y+u-i_{22}z},$$

with $j_{21}(y+z) \leqslant u \leqslant (j_{21}+1)(y+z)$, $y, z > 0$.
As $2u - (j_{11}+j_{12})z \leqslant 2u - (i_{11}+i_{12})z$ and $j_{21}y + u - j_{22}z \geqslant i_{21}y + u - i_{22}z$, then

$$\begin{aligned} & j_{21}y + u - j_{22}z \geqslant 2u - (i_{11}+i_{12})z \\ & \iff \quad u \leqslant j_{21}y + (i_{11}+i_{12}-j_{22})z \\ & \iff \quad (j_{21}+1)(y+z) \leqslant j_{21}y + (i_{11}+i_{12}-j_{22})z \\ & \iff \quad y \leqslant (i_{11}+i_{12}-j_{21}-j_{22}-1)z = -z \end{aligned}$$

against $y, z > 0$.

The cases $w_{j_1} = w_{i_2} < w_{i_1} = w_{j_2}$ and $w_{j_2} = w_{i_2} < w_{i_1} = w_{j_1}$ can be reduced by index complementation to the cases b) and a) respectively, which completes the proof. □

Before proving that all these conditions are also sufficient for domination, we will prove that increasing by 1 the highest index of each monomial of an inequality gives a correct (new) inequality.

Lemma 19. *If $c_{j_{11}} c_{j_{12}} + c_{j_{21}} c_{j_{22}} \geqslant c_{i_{11}} c_{i_{12}} + c_{i_{21}} c_{i_{22}}$, with indices satisfying (10), holds for all p-Newton sequences, then*

$$c_{j_{11}} c_{j_{12}+1} + c_{j_{21}} c_{j_{22}+1} \geqslant c_{i_{11}} c_{i_{12}+1} + c_{i_{21}} c_{i_{22}+1} \tag{15}$$

and

$$c_{j_{11}-1}c_{j_{12}} + c_{j_{21}-1}c_{j_{22}} \geqslant c_{i_{11}-1}c_{i_{12}} + c_{i_{21}-1}c_{i_{22}} \tag{16}$$

also hold for all p-Newton sequences.

Intuitively, we can say that "spreading of indices" preserves inequalities.

Proof. The second claim (16) follows from reversal of the first. To prove the first, we show that the change in the left hand side of (15) is no more than the change in the right hand side, so that the inequality is preserved. In case $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$ holds by domination, the operation preserves majorization, so that (15) holds also by domination. Thus we may assume DB (13) holds for $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$; this will be used. Now

$$\begin{aligned}
&c_{j_{11}}c_{j_{12}+1} - c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}+1} - c_{j_{21}}c_{j_{22}} \\
&\quad = c_{j_{11}}c_{j_{12}}\left(\frac{c_{j_{12}+1}}{c_{j_{12}}} - 1\right) + c_{j_{21}}c_{j_{22}+1} - c_{j_{21}}c_{j_{22}} \\
&\quad \overset{(1)}{\geqslant} c_{j_{11}}c_{j_{12}}\left(\frac{c_{j_{22}+1}}{c_{j_{22}}} - 1\right) + c_{j_{21}}c_{j_{22}+1} - c_{j_{21}}c_{j_{22}} = (c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}})\left(\frac{c_{j_{22}+1}}{c_{j_{22}}} - 1\right) \\
&\quad \geqslant (c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}})\left(\frac{c_{j_{22}+1}}{c_{j_{22}}} - 1\right) = c_{i_{11}}c_{i_{12}}\left(\frac{c_{j_{22}+1}}{c_{j_{22}}} - 1\right) + c_{i_{21}}c_{i_{22}}\left(\frac{c_{j_{22}+1}}{c_{j_{22}}} - 1\right) \\
&\quad \overset{(13)\&(1)}{\geqslant} c_{i_{11}}c_{i_{12}}\left(\frac{c_{i_{12}+1}}{c_{i_{12}}} - 1\right) + c_{i_{21}}c_{i_{22}}\left(\frac{c_{i_{22}+1}}{c_{i_{22}}} - 1\right) \\
&\quad = c_{i_{11}}c_{i_{12}+1} - c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}+1} - c_{i_{21}}c_{i_{22}}. \qquad \square
\end{aligned}$$

We finish this section with the characterization, in terms of indices, of the inequalities $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$ for p-Newton sequences. The sufficiency of this characterization will be proved by induction on n, the highest index of the Newton sequence. The case $n = 3$ will initiate the induction and we verify it by inventory of all inequalities in this case. These inequalities, listed in increasing order with respect to their weight and in two columns, having on the right the inequality obtained by index complementation from the one on the left, are:

	weight $= 2$	weight $= 10$	
1	$c_0c_0 + c_1c_1 \geqslant c_0c_0 + c_0c_2$	$c_2c_2 + c_3c_3 \geqslant c_1c_3 + c_3c_3$	$\bar{1}$
2	$c_0c_0 + c_1c_1 \geqslant c_0c_1 + c_0c_1$	$c_2c_2 + c_3c_3 \geqslant c_2c_3 + c_2c_3$	$\bar{2}$
	weight $= 3$	weight $= 9$	
3	$c_0c_0 + c_1c_2 \geqslant c_0c_0 + c_0c_3$	$c_1c_2 + c_3c_3 \geqslant c_0c_3 + c_3c_3$	$\bar{3}$
4	$c_0c_1 + c_1c_1 \geqslant c_0c_1 + c_0c_2$	$c_2c_2 + c_2c_3 \geqslant c_1c_3 + c_2c_3$	$\bar{4}$
	weight $= 4$	weight $= 8$	
5	$c_0c_0 + c_2c_2 \geqslant c_0c_0 + c_1c_3$	$c_1c_1 + c_3c_3 \geqslant c_0c_2 + c_3c_3$	$\bar{5}$
6	$c_0c_0 + c_2c_2 \geqslant c_0c_2 + c_0c_2$	$c_1c_1 + c_3c_3 \geqslant c_1c_3 + c_1c_3$	$\bar{6}$
1^+	$c_0c_1 + c_1c_2 \geqslant c_0c_1 + c_0c_3$	$c_1c_2 + c_2c_3 \geqslant c_0c_3 + c_2c_3$	$\overline{1^+}$
2^+	$c_0c_1 + c_1c_2 \geqslant c_0c_2 + c_0c_2$	$c_1c_2 + c_2c_3 \geqslant c_1c_3 + c_1c_3$	$\overline{2^+}$
7	$c_1c_1 + c_0c_2 \geqslant c_0c_2 + c_0c_2$	$c_2c_2 + c_1c_3 \geqslant c_1c_3 + c_1c_3$	$\bar{7}$
8	$c_1c_1 + c_1c_1 \geqslant c_0c_2 + c_0c_2$	$c_2c_2 + c_2c_2 \geqslant c_1c_3 + c_1c_3$	$\bar{8}$
9	$c_1c_1 + c_1c_1 \geqslant c_1c_1 + c_0c_2$	$c_2c_2 + c_2c_2 \geqslant c_2c_2 + c_1c_3$	$\bar{9}$

	weight $=5$	weight $=7$	
10	$c_0c_1+c_2c_2 \geqslant c_0c_1+c_1c_3$	$c_1c_1+c_2c_3 \geqslant c_0c_2+c_2c_3$	$\overline{10}$
11	$c_0c_1+c_2c_2 \geqslant c_0c_2+c_0c_3$	$c_1c_1+c_2c_3 \geqslant c_0c_3+c_1c_3$	$\overline{11}$
4^+	$c_0c_2+c_1c_2 \geqslant c_0c_2+c_0c_3$	$c_1c_2+c_1c_3 \geqslant c_0c_3+c_1c_3$	$\overline{4^+}$
12	$c_1c_1+c_0c_3 \geqslant c_0c_2+c_0c_3$	$c_2c_2+c_0c_3 \geqslant c_0c_3+c_1c_3$	$\overline{12}$
13	$c_1c_1+c_1c_2 \geqslant c_0c_2+c_0c_3$	$c_1c_2+c_2c_2 \geqslant c_0c_3+c_1c_3$	$\overline{13}$
14	$c_1c_1+c_1c_2 \geqslant c_0c_2+c_1c_2$	$c_1c_2+c_2c_2 \geqslant c_1c_2+c_1c_3$	$\overline{14}$
15	$c_1c_1+c_1c_2 \geqslant c_1c_1+c_0c_3$	$c_1c_2+c_2c_2 \geqslant c_2c_2+c_0c_3$	$\overline{15}$
	weight $=6$	weight $=6$	
16	$c_0c_0+c_3c_3 \geqslant c_0c_3+c_0c_3$		
6^+	$c_0c_1+c_2c_3 \geqslant c_0c_3+c_0c_3$		
17	$c_0c_2+c_2c_2 \geqslant c_0c_2+c_1c_3$	$c_1c_1+c_1c_3 \geqslant c_0c_2+c_1c_3$	$\overline{17}$
2^{++}	$c_0c_2+c_1c_3 \geqslant c_0c_3+c_0c_3$		
18	$c_1c_1+c_2c_2 \geqslant c_0c_2+c_1c_3$		
19	$c_0c_2+c_2c_2 \geqslant c_0c_3+c_0c_3$	$c_1c_1+c_1c_3 \geqslant c_0c_3+c_0c_3$	$\overline{19}$
20	$c_1c_1+c_2c_2 \geqslant c_0c_2+c_2c_2$	$c_1c_1+c_2c_2 \geqslant c_1c_1+c_1c_3$	$\overline{20}$
7^+	$c_1c_2+c_0c_3 \geqslant c_0c_3+c_0c_3$		
21	$c_1c_1+c_2c_2 \geqslant c_0c_3+c_0c_3$		
8^+	$c_1c_2+c_1c_2 \geqslant c_0c_3+c_0c_3$		
22	$c_1c_1+c_2c_2 \geqslant c_1c_2+c_0c_3$		
9^+	$c_1c_2+c_1c_2 \geqslant c_1c_2+c_0c_3$		
23	$c_1c_1+c_2c_2 \geqslant c_1c_2+c_1c_2$		

Inequalities 1, 4, 5, 7, 8, 9, 10, 12, 14, 17, 18 and 20 are obtained directly from the Newton inequalities. Inequalities 2, 6, 16 and 23 are obtained from $(c_i - c_j)^2 \geqslant 0$. Inequalities 3, 13 and 15 are obtained from the generalization (1) of the Newton inequalities. Inequality 11 is due to Lemma 13 with $q = 0$.

The inequalities j^+ of the previous table are obtained from increasing by one the highest index of each monomial of the inequalities j, and j^{++} is the application of this procedure twice. By Lemma 19 the inequalities of this type in the table are satisfied.

Inequality 19 is due to

$$c_0c_2 + c_2c_2 \geqslant c_0c_2 + c_1c_3 \overset{2^{++}}{\geqslant} c_0c_3 + c_0 + c_0c_3.$$

Inequality 21 is obtained, see (1), from $(c_1 - c_2)^2 + 2(c_1c_2 - c_0c_3) \geqslant 0$ and inequality 22 from $(c_1 - c_2)^2 + c_1c_2 - c_0c_3 \geqslant 0$.

Finally, $\bar{j}$ is the inequality obtained by index complementation in the inequality j. By Lemma 11 the inequalities in the right column from the table hold because of those on the left.

Theorem 20. *The inequality $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$, with indices satisfying* (10)*, holds for all p-Newton sequences if and only if the following conditions are satisfied*:

1. $w_{j_1} + w_{j_2} = w_{i_1} + w_{i_2}$;
2. $\max\{w_{j_1}, w_{j_2}\} \geqslant \max\{w_{i_1}, w_{i_2}\}$ *and*
3. *either* a) *term-wise domination holds or* b) *DB holds.*

An alternate, somewhat more succinct statement of this main result is the following:

Theorem 21. *The inequality $c_{j_{11}}c_{j_{12}} + c_{j_{21}}c_{j_{22}} \geqslant c_{i_{11}}c_{i_{12}} + c_{i_{21}}c_{i_{22}}$, with indices satisfying* (10)*, holds for all p-Newton sequences if and only if the following conditions are satisfied*:

1. *the weights $\{w_{j_1}, w_{j_2}\}$ majorize the weights $\{w_{i_1}, w_{i_2}\}$, and*
2. *either* a) *term-wise domination* (*majorization of indices*) *holds for a matching of left hand monomials with right hand monomials* (*in case the weight pairs are equal*) *or* b) *DB holds* (*in case the weight pairs are different*)*.*

Proof. (Of the sufficiency of DB.) The proof is by induction on n, the highest index of a coefficient in the p-Newton sequence. The induction is initiated for the case $n = 3$, which was done by inventory, as was already mentioned. To go from n to $n+1$, we may consider only proposed inequalities for $n+1$ that both meet the necessary conditions and in which both indices 0 and $n+1$ appear. Otherwise, perhaps using translation and/or completation, the proposed inequality is valid by direct application of the induction hypothesis. Of course, we also assume DB.

To complete a proof of Theorem 20 or Theorem 21, it suffices to show that DB is sufficient when the weight pairs are not the same. (Necessity of the conditions has already been shown and term-wise domination is obviously sufficient.)

Now, the only way that the monomials c_0^2 and c_{n+1}^2 both appear in the proposed inequality is the obvious inequality

$$c_0^2 + c_{n+1}^2 \geqslant 2c_o c_{n+1}$$

($[c_0 - c_{n+1}]^2 \geqslant 0$). If c_0^2 alone appears, it will no longer after an application of index complementation. Thus, we may and do henceforth assume that 0 is not the largest index in any monomial appearing in the propose inequality.

In the proposed inequality decrease the largest index of each monomial by 1. If n is the largest index that now appears, we have a valid inequality for n, by the induction hypothesis, as DB is preserved. To this inequality apply (15), which, by Lemma 19, is then an inequality. Either the proposed inequality will have been returned, and thus now verified, or there was a monomial in the proposed inequality of the form $c_k c_k$, with $0 < k < n+1$, which is now $c_{k-1}c_{k+1}$. Barring trivialities, this term could only have been on the larger side of the inequality because of the DB condition. Then, by majorization, replacing this term by the original $c_k c_k$ can only increase the larger side, assuring a valid inequality and completing a proof in this case.

If $n+1$ still appears, there must have been a term $c_{n+1}c_{n+1}$, on the larger side of the proposed inequality only (barring trivialities). In this event, apply the operation of "decreasing the largest index in each monomial" twice. The result still satisfies DB and is now a valid inequality for n. Now, we apply (15) twice. Instead of returning the monomial $c_{n+1}c_{n+1}$, we will, instead, have returned $c_n c_{n+2}$, on the larger side. Since this monomial is again dominated by $c_{n+1}c_{n+1}$, we may replace it, returning the proposed inequality as verified.

Since necessity has been proved, in Lemma 15 and Theorems 17 and 18, this completes the induction and the proof of the theorems. □

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