Monomial Inequalities for Newton Coefficients and Determinantal Inequalities for p-Newton Matrices

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In memory of Julius Borcea

Abstract. We consider Newton matrices for which the Newton coefficients are positive. We show that one monomial in these coefficients dominates another for all such Newton matrices if and only if a certain generalized form of majorization occurs. As the Newton coefficients may be viewed as average values of principal minors of a given size, these monomial inequalities may be interpreted as determinantal inequalities in such familiar classes as the positive definite, totally positive, and M-matrices, etc.

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1. Introduction

Let A be an n-by-n real matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Denote the principal submatrix of A lying in rows and columns given by the index set $\alpha \subseteq N = \{1, 2, \ldots, n\}$ by $A[\alpha]$. Define the k-elementary symmetric function

$$S_k = S_k(A) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

and the k**-Newton coefficient**

$$c_k = c_k(A) = \frac{1}{\binom{n}{k}} S_k,$$

$k = 1, \ldots, n$, with $c_0 \equiv 1$. Of course, since $S_k(A) = \sum_{|\alpha|=k} \det A[\alpha]$, as well, $c_k(A)$ may be viewed as the average value of the k-by-k principal minors of A. The matrix A (or its spectrum $\lambda_1, \ldots, \lambda_n$) is called **Newton** if

$$c_{k-1}c_{k+1} \leq c_k^2, \quad k = 1, \ldots, n-1,$$

and these inequalities are referred to as the **Newton inequalities**. If, further, $c_k > 0$, $k = 1, \ldots, n$, A is called **p-Newton** and the sequence $c_0, c_1, \ldots, c_n$ is called p-Newton. It is known that if the eigenvalues are positive reals, if A is an M-matrix, or under further circumstances [2, 3, 4, 5, 8] that A is p-Newton.

The two sides of the Newton inequalities are particular monomials in the Newton coefficients $c_0, c_1, \ldots, c_n$. We henceforth assume that our matrix A is p-Newton and, implicitly, that A is n-by-n. For any nonnegative exponents $a_0, a_1, a_2, \ldots, a_n$, by a **monomial** in the c_i's, we mean an expression of the general form

$$c^a = c_0^{a_0} c_1^{a_1} \ldots c_n^{a_n}.$$

The general question that we wish to address here is for which pairs of monomials c^a and c^b do we have

$$c^a \leq c^b$$

for **all** p-Newton sequences

$$c : c_0, c_1, \ldots, c_n \text{ ?}$$

In this event, we say that the monomial c^b **dominates** the monomial c^a (with respect to p-Newton sequences). Since the c_k's may be viewed as average values of principal minors of a given size, we are motivated, in part, by the study of determinantal inequalities in p-Newton matrices. Since positive definite matrices, totally positive matrices, and M-matrices are p-Newton, as well as their inverse classes, this general class of determinantal inequalities includes inequalities common to these special classes.

Already in [4] and partly in [2] it was shown that in any p-Newton matrix, the inequalities

$$c_r c_s \leq c_p c_q$$

hold when $r < p \leq q < s$ and $p + q = r + s$. Of course, also the product of several such inequalities will give an inequality. Now, the special, 2-term monomial inequality above is the special case in which the subscripts appearing in the dominant monomial strictly majorize those appearing in the smaller, *i. e.*

$$r < p$$

and

$$p + q = r + s.$$

This led us to conjecture that, at least for monomials with positive integer exponents, majorization in the subscripts implies a monomial inequality for all p-Newton matrices. In fact, we will see that majorization in the subscripts is equivalent to a general inequality, in this case.

What, then, if the exponents are not integers? Clearly, there can still be monomial inequalities, but conventional majorization no longer makes sense. So, for this event, we define a more continuous version of majorization and show that it is equivalent to a general inequality. For sufficiency of generalized majorization, we note that it is the same as ordinary majorization in the integer case and then show that majorization is sufficient in the integer case by a known process called "pinching" [7], based upon the 2-term monomial inequalities above. The integer case is expanded to the rational exponent case by powering, and then the rational case to the general real case by a density argument based upon the fact that the relevant exponent vector pairs form a cone in the appropriate real space and thus that the rational points therein are dense. For necessity, we show that if $c^a \leq c^b$ for all p-Newton sequences, then

$$\sum_{j=0}^{n} j a_j = \sum_{j=0}^{n} j b_j$$

and that we may assume

$$\sum_{j=0}^{n} a_j = \sum_{j=0}^{n} b_j \equiv L \ .$$

Then, we show that if c^b dominates c^a, we must have that a is majorized by b in our generalized sense.

2. Majorization and Main Result

One list of integers $i_1 < i_2 < \ldots < i_k$ is said to be majorized by another list $j_i < j_2 < \ldots < j_k$ if

$$\begin{gathered} i_1 \leq j_1 \\ i_1 + i_2 \leq j_1 + j_2 \\ \vdots \\ i_1 + i_2 + \ldots + i_{k-1} \leq j_1 + j_2 + \ldots + j_{k-1} \end{gathered}$$

and

$$i_1 + i_2 + \ldots + i_k = j_1 + j_2 + \ldots + j_k \ .$$

Note that this definition is the same for lists of real numbers, but only the integer case interests us from classical majorization. In our setting, the i's and j's are subscripts that appear on the Newton coefficients in two monomials, and each exponent of a c that appears is 1, with repeats allowed.

What is the appropriate analog if the exponents are not integers? For the monomial c^a, we define a step function F_a as follows. For $0 \leq z < \sum_{j=0}^{n} a_j = L$,

$$F_a(z) = i$$

if and only if $a_i > 0$ and $\sum_{j<i} a_j \leq z < \sum_{j\leq i} a_i$. For $z \geq L$, $F_a(z) = 0$. Now, for the two exponent sequences in c^a and c^b, a and b, we may define (generalized) **majorization** as follows. We say $a \preceq b$ if

$$\int_0^x F_a(z)\,dz \leq \int_0^x F_b(z)\,dz$$

for $0 \leq x < \sum_{j=0}^n a_j = \sum_{j=0}^n b_j = L$, with equality for $x \geq L$. We note that $\int_0^L F_a(z)\,dz = \sum_{j=0}^n j a_j$, and that when the a_j's and b_j's are integers, the new notion of majorization coincides with the classical one.

With this definition of majorization in hand, our main result is then the following.

Theorem 1. *The monomial c^b dominates c^a with respect to p-Newton sequences if and only if $a \preceq b$.*

In the next section, we show the necessity of our notion of majorization by using carefully chosen p-Newton sequences and that, without loss of generality, we may assume that $\sum_{j=0}^n a_j = \sum_{j=0}^n b_j$. Then, in the following section, we show sufficiency by first considering the integer exponent case and then extending it to the rational case and then the general real case.

We close this section by noting that we might as well more generally consider monomials in which negative exponents appear or pairs of monomials in which the same term c_k appears in both with positive exponent. However, for convenience in our setting we may assume without loss of generality (by simple algebra) that all exponents are nonnegative and that a c_k appears with positive exponent in at most one of c^a and c^b.

3. The Necessity of Majorization

Here we show the necessity of majorization for monomial domination by designing appropiate p-Newton sequences. For a positive parameter r, define the sequence $Q_{n,i}(r)$ as

$$1, r, r^2, \ldots, r^i, r^i, \ldots, r^i,$$

i. e. this sequence of $n+1$ terms, beginning with term 0, starts as a geometric sequence with base r and then becomes constant starting with term i.

Proposition 3.1. *The sequence $Q_{n,n}(r)$ is p-Newton for any $r > 0$, while $Q_{n,i}(r)$ is p-Newton, $0 < i < n$, for any $r \geq 1$.*

Now, we establish the necessity of majorization by the following sequence of lemmas and a convention that they allow.

Lemma 1. *If c^b dominates c^a, then*

$$\sum_{j=0}^{n} ja_j = \sum_{j=0}^{n} jb_j.$$

Proof: From the previous proposition we know that $Q_{n,n}(r)$ is p-Newton for any $r > 0$, but c^b dominates c^a, so that

$$c^a(Q_{n,n}(r)) = r^{\sum_{j=1}^{n} ja_j} = r^{\sum_{j=0}^{n} ja_j} \leq c^b(Q_{n,n}(r)) = r^{\sum_{j=0}^{n} jb_j}.$$

This inequality gives us

$$\sum_{j=0}^{n} ja_j \leq \sum_{j=0}^{n} jb_j$$

when $r > 1$ and the reverse inequality when $r < 1$. This proves the lemma. ■

Lemma 2. *If c^b dominates c^a, then*

$$\sum_{j=1}^{n} a_j \leq \sum_{j=1}^{n} b_j.$$

Proof: Because c^b dominates c^a and $Q_{n,1}(r)$ is p-Newton for any $r \geq 1$ (Proposition 3.1), then

$$c^a(Q_{n,1}(r)) = r^{\sum_{j=1}^{n} a_j} \leq c^b(Q_{n,1}(r)) = r^{\sum_{j=1}^{n} b_j}.$$

But r may be chosen greater than 1, so the exponents must obey the inequality

$$\sum_{j=1}^{n} a_j \leq \sum_{j=1}^{n} b_j.$$ ■

Since changing the value of a_0 does not change the evaluation of the monomial c^a at any sequence, we may suppose, and henceforth do, that a consequence of domination is that

$$\sum_{j=0}^{n} a_j = \sum_{j=0}^{n} b_j \equiv L.$$

Lemma 3. *If c^b dominates c^a, then $a \preceq b$.*

Proof: The proof is by contradiction. Suppose that $c^b \geq c^a$, but there is a real value x, with $0 < x < L$, such that $\int_0^x F_a > \int_0^x F_b$. We can choose x so that

$$\max\left\{ \int_0^t F_a - \int_0^t F_b \quad | \quad 0 < t < L \right\} = \int_0^x F_a - \int_0^x F_b \ .$$

Observe that the maximizing x must occur in an interval in which $F_a(t) > F_b(t)$ and must furthermore be the last point in that interval, corresponding to a step in F_b.

Let $F_a(x) = i$. Then

$$x = \sum_{j=0}^{i} b_j \ .$$

Because $c^b \geq c^a$, we know by Lemma 1

$$\sum_{j=0}^{n} j a_j = \sum_{j=0}^{n} j b_j$$

and by convention we know

$$\sum_{j=0}^{n} a_j = \sum_{j=0}^{n} b_j = L.$$

So $\int_0^L F_a = \int_0^L F_b$. We split these integrals as follows:

$$\int_0^L F_b = \int_0^x F_b + \int_x^L i + \int_x^L (F_b - i) = \sum_{j=0}^{i} j b_j + \sum_{j=i+1}^{n} i b_j + \int_x^L (F_b - i) =$$

$$= \log_r(c^b(Q_{n,i}(r))) + \int_x^L (F_b - i)$$

Observe that if $\sum\limits_{j=0}^{i-1} a_j \leq m \leq \sum\limits_{j=0}^{i} a_j$ then

$$\int_0^m F_a + \int_m^L i = \sum_{j=0}^{i-1} j a_j + i\left(m - \sum_{j=0}^{i-1} a_j\right) + i\left(\sum_{j=0}^{n} a_j - m\right) = \sum_{j=0}^{i} j a_j + \sum_{j=i+1}^{n} i a_j = \log_r(c^a(Q_{n,i}(r))).$$

By definition, x falls within this range so

$$\int_0^L F_a = \int_0^x F_a + \int_x^L i + \int_x^L (F_a - i) = \log_r(c^a(Q_{n,i}(r))) + \int_x^L (F_a - i).$$

Since $\int_0^L F_a = \int_0^L F_b$ we know

$$\int_0^x F_a + \int_x^L i + \int_x^L (F_a - i) = \int_0^x F_b + \int_x^L i + \int_x^L (F_b - i).$$

Furthermore, since

$$\int_0^x F_a > \int_0^x F_b$$

we find

$$\int_x^L (F_a - i) < \int_x^L (F_b - i).$$

Substituting into the logarithms of the two monomials yields

$$\log_r(c^a(Q_{n,i}(r))) + \int_x^L (F_a - i) = \log_r(c^b(Q_{n,i}(r))) + \int_x^L (F_b - i).$$

So

$$\log_r(c^a(Q_{n,i}(r))) > \log_r(c^b(Q_{n,i}(r))).$$

So $c^a > c^b$ at $Q_{n,i}(r)$, which contradicts domination. Thus if $c^b \geq c^a$ then $\int_0^x F_a \leq \int_0^x F_b$ for all x. ■

4. Sufficiency

Our purpose here is to show that (generalized) majorization is sufficient for monomial domination, completing a proof of our main result. Again this proceeds in several steps. First we assume that the a's and b's are integers, so that majorization may be viewed in the classical sense.

Lemma 4. *Suppose that the exponents $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ are (nonnegative) integers and that $a \preceq b$. Then the monomial c^b dominates the monomial c^a.*

Proof: Our assumption is the same as that the sequence a' with a_0 0's, a_1 1's, ..., and a_n n's is majorized in the classical sense by the sequence b' with b_0 0's, b_1 1's, ..., and b_n n's. Because of this, we may transform a' into b' by a sequence of "pinches" [7]: replacements of two a' components $r < s$ by p and q with $r < p \leq q < s$ and $p + q = r + s$. In the monomial this amounts to replacing $c_r c_s$ by $c_p c_q$. Since it is known that $c_r c_s$ is dominated by $c_p c_q$ [4], the monomial resulting from this replacement can be no smaller on any p-Newton sequence. Since from a' we may arrive at b' by a finite sequence of pinches, it follows that the monomial c^b dominates c^a. ■

Lemma 5. *If the exponents $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ are (nonnegative) rational numbers and $a \preceq b$, then the monomial c^b dominates c^a.*

Proof: Because we only evaluate at p-Newton sequences, we have that c^b dominates c^a if and only if $(c^b)^m$ dominates $(c^a)^m$ for any positive number m. Choose m to be the least common multiple of all the denominators in the fractions $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$. Then $(c^a)^m$ and $(c^b)^m$ may be rearranged to be monomials with integer exponents. Since $a \preceq b$ if and only if $ma \preceq mb$, we may apply the result of the prior lemma to ma and mb to conclude that $(c^a)^m$ is dominated by $(c^b)^m$ and then conclude that c^a is dominated by c^b. ■

Lemma 6. *For nonnegative real exponents $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$, if $a \preceq b$, the monomial c^b dominates c^a.*

Proof: The set of vectors $(a, b) \in \mathbb{R}^{2n+2}$ for which $a \preceq b$ forms a cone of dimension $2n$, as it is orthogonal to both

$$(0, 1, 2, \ldots, n, 0, -1, -2, \ldots, -n)^T$$

and

$$(1, 1, \ldots, 1, -1, -1, \ldots, -1)^T.$$

It is easily checked that the set is closed under addition and positive scalar multiplication. By the prior lemma, for all rational points in this cone, we have that c^b dominates c^a. In addition, the rational points of such a finitely generated cone are dense in the cone. Now, suppose that there is a non-rational point in the cone for which $c^a > c^b$ on some p-Newton sequence. By density, and continuity of the values of the monomials, there would be a nearby rational point of the cone (a', b') for which $c^{a'} > c^{b'}$ on the same p-Newton sequence. But this contradiction completes the proof. ∎

We very much thank the referee for pointing out to us references [1] and [6] and the connection between them and our work. We would also like to mention that this work was done independently of prior work, with a different approach and a different proof. It takes some effort to deduce our results from the prior work.

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